Correlation functions of one-dimensional systems

V. Ya. Krivnov and A. A. Ovchinnikov

L. Ya. Karpov Scientific Research Institute of Physics and Chemistry
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The correlation functions are found for a one-dimensional Fermi gas with strong repulsion between the fermions. At zero temperature, the correlators fall off at large distances according to a power law with the exponents simply expressed in terms of the velocity of sound. At $T \neq 0$, they fall off exponentially; the correlation radius $\propto T^{-1}$. The results are compared with those found for a model with a linear spectrum.

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A problem of great interest in the theory of one-dimensional systems is the finding of various types of correlation functions. In particular, their behavior at large distances determines the transition temperature in actual quasi-one-dimensional systems.

So far, exact expressions for the correction functions have been found only in a few special models. Even Bethe's method, which makes it possible to calculate exactly the energy of the ground state and the spectrum of excitations for a δ-function potential, proves inadequate for finding the correlation functions.

On the other hand, in papers of Efetov and Larkin[1] and of Luther, Emery, and Peschel[2-4] it has been suggested that the behavior of the correlation functions of one-dimensional Fermi systems at large distances is determined by the long-wavelength gapless excitations. A characteristic feature of correlation functions found on the basis of this hypothesis is their power-law decrease at large distances when $T=0$ and the continuous variation of the corresponding critical exponents with the interaction constant g. It has also been shown[1] that the correlation functions calculated in this way coincide with those found for a model with a linear spectrum (the linear model), where the corresponding calculations can be carried out exactly.[2-4] It must be noted, however, that the hypothesis regarding the decisive role of low-lying excitations in shaping the features of the correlation functions does not have a rigorous basis.

In estimating the validity of this hypothesis, one must take into account the fact that the asymptotic behavior of the pair correlation function at large distances may be determined by singularities of the structure factor $S(k)$ with $k=2k_F$ ($4k_F$, $6k_F$, etc.) and may therefore not be due to excitations of the acoustical type.

As regards the approach that uses linearization of the real quadratic fermion spectrum, its most serious shortcoming is the absence of the necessary symmetry of the total wave function of the system. If such a linearization is correct at all, it is probably so only for small values of g. This is indicated in particular by the fact that the single-particle density matrix of the linear model agrees with the density matrix of a slightly nonideal Fermi gas with a quadratic spectrum, as found by us[5] only qualitatively.

The present paper will consider another limiting case, namely strong repulsion between fermions. In this limit, which is in itself of interest, the correlation functions can be found exactly and compared with those obtained for the model with a linear spectrum.

We consider a system of Fermi particles (for simplicity spinless) with the Hamiltonian

$$\hat{H}=-\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}+\frac{g}{2}\sum_{i\neq j}^{N}V(x_i-x_j)\quad(\hbar=2m=1). \tag{1}$$

We shall also assume that the interaction potential $V(x)$ is long-range. For example, we may choose for $V(x)$ potentials of the form

$$V(x)=e^{-\gamma|x|},\quad V(x)=|x|^{-n},$$

etc. Although the whole subsequent treatment is independent of the specific form of the interaction potential, special attention will be given to the potential

$$V(x)=x^{-2}, \tag{2}$$

because in this case the energy of the ground state, the excitation spectrum, and all the thermodynamic characteristics are known exactly.[6-9] For this potential, the wave function of the ground state is also known for arbitrary g; in the sector $x_1<\ldots<x_N$ it has the form

$$\psi(x_1,\ldots,x_N)=\psi_0^{\alpha}(x_1,\ldots,x_N),\quad \alpha(\alpha-1)=g/2, \tag{3}$$

where ψ_0 is the wave function of the ground state of (1) for $g=0$, which is (N odd)

$$\psi_0(x_1,\ldots,x_N)=\prod_{i>k}^{N}\sin\frac{\pi}{L}(x_i-x_k) \tag{4}$$

(L is the length of the system).

The possibility of exact treatment of a system with the Hamiltonian (1) for $g\to\infty$ and for sufficiently low temperatures is due to a well known fact: the formation of a Wigner crystal with a lattice constant[1)] $a=\rho^{-1}=L/N$. On making the change of variables

$$x_n=na+\xi_n$$

and expanding the potential energy in (1) through terms quadratic in ξ_n, we get

$$\hat{H}=-\sum_{n=1}^{N}\frac{\partial^2}{\partial\xi_n^2}+\frac{g}{2}\sum_{n\neq m}V((n-m)a)+\frac{g}{4}\sum_{n\neq m}V''((n-m)a)(\xi_n-\xi_m)^2, \tag{5}$$

which is the Hamiltonian of a system of harmonic oscillators with long-range action. Vanishing of the terms containing the first derivative of $V(x)$ occurs, strictly speaking, only for a potential periodic with period L. In other words, instead of the original potential $V(x)$ it

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is necessary to consider a potential

$$\tilde{V}(x)=\sum_{n=-\infty}^{\infty}V(x+nL),$$

but in the limit as $L\to\infty$ the two potentials coincide.

The Hamiltonian (5) reduces in the usual way to a system of noninteracting phonos, whose spectrum has the form

$$\omega(q)=2\left[g\sum_{n=1}^{\infty}V''(na)(1-\cos qna)\right]^{1/2},\quad -\pi<qa<\pi \tag{6}$$

therefore the energy of the elementary excitations of a Fermi gas when $g\to\infty$ is $\omega(q)$. These excitations have acoustical character for $q\to 0$ and are periodic with period $2k_F$ ($k_F=\pi\rho$, where ρ is the particle density), as is characteristic of one-dimensional Fermi systems. In particular, for the interaction potential (2)

$$\omega(q)=(g/2)^{1/2}(2\pi\rho|q|-q^2) \tag{7}$$

for $g\gg 1$, this agrees with the exact expression[8,9]

$$\omega(q)=\alpha(2\pi\rho|q|-q^2). \tag{8}$$

The wave function of the ground state of (5) has the form

$$\psi=C\exp\left(-\frac{1}{4}\sum_{q\neq 0}\omega(q)\xi_q\xi_{-q}\right), \tag{9}$$

where $\omega(q)$ is defined by the expression (6), and where ξ_q is the Fourier transform of ξ_n. For the potential (2) when $g\gg 1$ ($\alpha\gg 1$), the function ψ can be obtained directly from (3) by taking into account that according to (4), $\psi_0(x_1,\ldots,x_N)$ attains its maximum in the sector $x_1<\ldots<x_N$ when

$$x_n=na\quad (n=1,\ldots,N).$$

Then ψ has the form (9), with $\omega(q)$ defined by (8).

We turn now to the calculation of the pair correlation function $G(R,t)$:

$$G(R,t)=\langle\rho(0,0)\rho(R,t)\rangle, \tag{10}$$

where $\rho(R,t)$ is the time-dependent density operator. The averaging in (10) reduces to the calculation of Gaussian integrals and leads to the following expression for $G(R,t)$:

$$G(R,t)=\rho^2 2^{-1}\pi^{-1/2}\sum_{m=-\infty}^{\infty}[\varphi(m,t)]^{-1/2}\exp[-(R\rho-m)^2/4\varphi(m,t)],$$

$$\varphi(m,t)=\rho^2\pi^{-1}\left\{\int_0^{\pi}[1-\cos qm\cos(\omega(q\rho)t)]\omega^{-1}(q\rho)\operatorname{cth}[\beta\omega(q\rho)/2]dq\right. \tag{11}$$

$$\left.-i\int_0^{\pi}\cos qm\sin(\omega(q\rho)t)\omega^{-1}(q\rho)dq\right\},\quad \beta=(kT)^{-1}.$$

The expression (11) is correct for $c\rho\gg kT$, where the velocity of sound according to (6) is

$$c=(2g)^{1/2}a\left[\sum_{n=1}^{\infty}V''(na)n^2\right]^{1/2}. \tag{12}$$

When $|R+ct|\gg a$, the function $G(R,t)$ has the form

$$G(R,t)-\rho^2=h(R,t)=2\rho^2\sum_{s=1}^{\infty}\cos(2k_FsR)\exp[-4\pi^2s^2\varphi(R\rho,t)], \tag{13}$$

where

$$\varphi(x,t)=\rho(2\pi c)^{-1}[F(x+c\rho t)+F(c\rho t-x)],$$
$$F(y)=\ln\{\beta c\rho\operatorname{sh}(|y|\pi/\beta c\rho)\}-{}^1/_2 i\pi\operatorname{sign}y. \tag{14}$$

Substituting (14) in (13), we get

$$h(R,t)=2\rho^2\sum_{s=1}^{\infty}\cos(2k_FsR)\exp[i\pi^2\rho s^2c^{-1}(\operatorname{sign}(R+ct)$$
$$+\operatorname{sign}(ct-R))]\{(\beta c\rho)^2\operatorname{sh}(|R+ct|\pi/\beta c)\operatorname{sh}(|R-ct|\pi/\beta c)\}^{-\mu s^2}, \tag{15}$$
$$\mu=2\pi\rho/c. \tag{16}$$

The asymptotic behavior of this correlator is determined by the first term of the sum in (15). When $t=0$,

$$h(R,0)=2\rho^2\cos(2k_FR)[\beta c\rho\operatorname{sh}(R\pi/\beta c)]^{-2\mu},\quad R\to\infty. \tag{17}$$

As follows from (17), the function $h(R,0)$ falls off at $T=0$ in power-law fashion, with an exponent that depends on g; at $T\neq 0$, it falls off exponentially, with correlation radius $\propto T^{-1}$. In connection with formula (17), we note the following fact. The correlator $h(R,0)$ at $T=0$ was calculated by Sutherland[6] for the model with the potential (2), for two special values of g: $g=0$ (the trivial case) and $g=4$. As $R\to\infty$ we have

$$h(R,0)\sim R^{-2},\quad h(R,0)\sim R^{-4}$$

for $g=0$ and for $g=4$, respectively. If we use the fact that in this model the exact expression for c is according to (8)

$$c=2\alpha\pi\rho, \tag{18}$$

and if we substitute (18) in (16), we find that formula (17) is also correct at $T=0$ for these special values of g.

It is interesting to compare (15) with the expression for $h(R,t)$ obtained by Luther and Peschel[3,4] for the linear model. Functionally these expressions coincide [in Refs. 3 and 4, only the leading term of the asymptotic expression, corresponding to $s=1$ in (15), was found]. The corresponding exponent, however, differs from μ and does not coincide with it as $g\to\infty$. If we suppose, in light of the fact mentioned above, that (15) and (16) are correct for all $g>0$, then as $g\to 0$ the exponents coincide to terms of the first order in g.

Together with the correlator $G(R,t)$, an important role is played by its Fourier transform, the dynamical form factor $S(k,\omega)$, which determines the scattering probability of neutron and x-ray beams. It has the form

$$S(k,\omega)=N(2\pi)^{-1}\int_{-\infty}^{\infty}dt\exp(-i\omega t)\sum_{m=-\infty}^{\infty}\exp[-ikma-(ka)^2\varphi(m,t)]. \tag{19}$$

Replacing $\varphi(m,t)$ by its asymptotic expression (14), we get at $T=0$ as $k\to 0$

$$S(k,\omega)=N\pi^{-1}(c\rho)^{-3}k^4[(\omega^2-c^2k^2)/(c\rho)^2]^{\lambda}\theta(\omega-c|k|),$$
$$\lambda=-1+k^2/2\pi\rho c \tag{20}$$

and as $k\to 2k_Fn$, $n=\pm 1,\ \pm 2,\ \ldots$

$$S(k,\omega)=16N(\pi/c)^3n^4\rho[(\omega^2-c^2(k-2k_Fn)^2)/(c\rho)^2]^{\mu n^2-1}\theta(\omega-c|k-2k_Fn|). \tag{20'}$$

According to (20) and (20′), $S(k,\omega)$ has a power-law divergence near the absorption threshold $\omega=c|k-2k_Fn|$ for all n satisfying the inequality

$$c>2\pi\rho n^2. \tag{21}$$

At $T\neq 0$, these singularities are smoothed out; and in particular, for $k = 2k_F n$ ($n \neq 0$) and $\omega = 0$

$$S(2k_F n, 0)/N = (\beta c\rho)^{2(1-\mu n^2)} c/4\pi^5\rho^3 n^4.$$

In the linear model,[3,4] $S(k, \omega)$ also has a power-law singularity of the form (20)–(20′), but only for $n = \pm 1$, and with an exponent that does not coincide with $(-1 + \mu)$.

In closing this section, we consider the correlator $S(k)$, the so-called structure factor, defined by the relation

$$S(k) = N^{-1}\int_{-\infty}^{\infty} S(k, \omega)\, d\omega. \qquad (22)$$

Substituting (19) in (22), we find that for $T = 0$

$$S(k) = |k|/c, \quad k \to 0,$$
$$S(k) = 4\pi^2 n^2 \rho c^{-1} |(k - 2k_F n)/\rho|^{2\mu n^2 - 1}, \quad k \to 2k_F n \quad (n \neq 0). \qquad (23)$$

According to (23), $S(k)$ diverges in power-law fashion when $k \to 2k_F n$ ($n \neq 0$) for n satisfying the inequality (21). It is easy to see that the divergence of $S(k)$ as $k \to \pm 2k_F$ corresponds to the asymptotic expression $h(R, 0)$ [see (17)]. The singularities of $S(k)$ at $k = \pm 4k_F$, $\pm 6k_F$, etc. correspond to subsequent terms of the expansion ($s = 2$, 3, etc.) in (15).

For $T \neq 0$, the singularities (23) are smoothed out, and and $S(k)$ is finite at all k:

$$S(0) = 2/\beta c^2, \quad S(2k_F n) = \operatorname{cth}(2\pi^2 n^2/\beta c^2).$$

Figures 1 and 2 show the function $S(k)$ calculated by formulas (22) and (19) for the potential (2), for certain values of α and T.

Together with the correlation functions considered above, an important role is played by the single-particle density matrix, which at $T = 0$ is

$$\rho(x - x') = \rho \int \psi^*(x, x_2, \ldots, x_N)\psi(x', x_2, \ldots, x_N)\, dx_2 \ldots dx_N. \qquad (24)$$

Unfortunately, the integral (24) has not been sucessfully calculated even in the limit $g \to \infty$, i.e., when ψ is a function of the form (9). On the other hand, the correlator $\rho(R)$ was calculated by us earlier[5] for a slightly nonideal ($g \to 0$) one-dimensional Fermi gas with an arbitrary interaction potential. The corresponding calculations[5] were based on the following representation for the wave function of the ground state of a Fermi gas:

$$\psi(x_1, \ldots, x_N) = \exp\Big[\sum_{i \neq j} S(x_i - x_j)\Big]\psi_0, \qquad (25)$$

where ψ_0 is defined according to (4). It was shown earlier[5] that the asymptotic behavior of $\rho(R)$ as $R \to \infty$

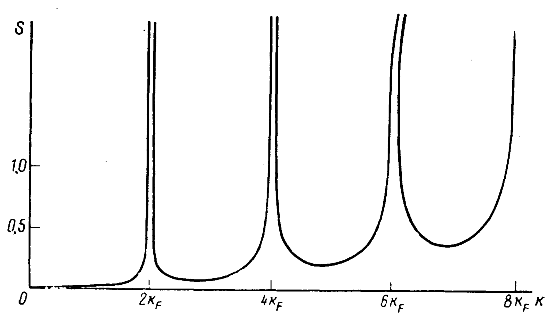

FIG. 1. The function $S(k)$ for $\alpha = 100$ and $T = 0$.

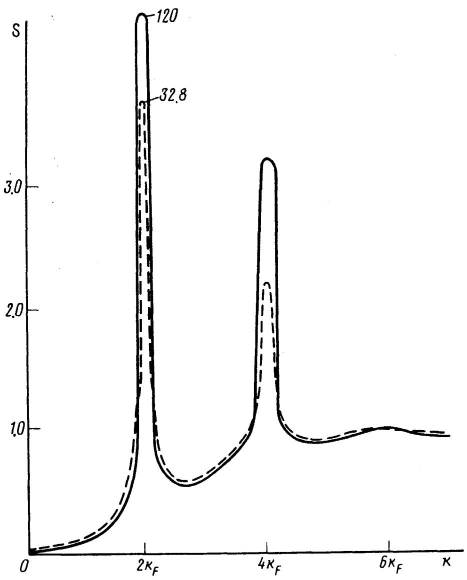

FIG. 2. The function $S(k)$ for $\alpha = 10$. Solid curve, $kT/\rho^2 = 3/5$; dashed curve, $kT/\rho^2 = 3$. The numbers in the figure indicate the heights of the peaks of $S(k)$ at $k = 2k_F$.

is determined by the behavior of the Fourier transform of $S(x)$, the function $\sigma(k)$, as $k \to 0$. To the first order in g, we have[5]

$$\sigma(k) = \pi(c_0 - c)/2c_0|k|, \quad k \to 0, \qquad (26)$$

where c and c_0 are the velocities of sound to terms $\sim g$ and at $g = 0$, respectively. Calculation of $\sigma(k)$ to higher orders in g involves significant difficulties. There is, however, a basis for supposing that formula (26) is correct to all orders of perturbation theory, if the term c in (26) is the exact velocity of sound. Thus, for example, it is easy to test the correctness of (26) for the model with the interaction potential (2). In fact, it follows from (3) that $\sigma(k)$ for this model is

$$\sigma(k) = \pi(1 - \alpha)/2|k|. \qquad (27)$$

Since for this model, according to (18), $c = 2\alpha k_F$, the expression (26) coincides with (27) for all g. The following fact may also serve as indirect corroboration of the validity of (26). We consider the correlator $S(k)$ for $T = 0$ and $k \to 0$. To calculate it, we may use a diagram technique developed earlier.[5] Separating in the diagram of the expansion for $S(k)$, by analogy with Ref. 5, the principal terms as $k \to 0$, we get

$$S(k) = |k|\,[1 - 2\sigma(k)|k|/\pi]^{-1}(2k_F)^{-1}. \qquad (28)$$

Substitution of (26) in (28) gives

$$S(k) = |k|/c, \quad k \to 0. \qquad (29)$$

The expression (29) agrees with the well-known formula of Feynman, coincides with the exact answer for weak ($g \to 0$, Ref. 5) and strong [$g \to \infty$, cf. (23)] interactions of general form, and does so also for the Sutherland model[6-8] with the potential (2) with $g = -\frac{1}{2}$ and with $g = 4$, for which $S(k)$ has been found exactly.[6]

We turn now to calculation of the correlator $\rho(R)$. As follows from Ref. 5, the nonoscillatory part of $\rho(R)$ as $R \to \infty$ is given by the expression

$$\rho(R) \sim R^{-1} e^{-F(R)},$$

$$F(R) = 2\pi^{-2} \int_0^{2k_F} (1 - \cos qR)\, q\sigma^2(q)\, [1 - 2q\sigma(q)/\pi]^{-1} dq. \qquad (30)$$

Substituting (26) in (30), we get

$$\rho(R) \sim R^{-\varkappa}, \quad \varkappa(g) = (c_0{}^2 + c^2)/2cc_0, \qquad (31)$$

$\varkappa = 1$ if $g = 0$ and $\varkappa = c/2c_0$ as $g \to \infty$. In particular, $\varkappa(\infty) = g^{1/2}/2$ for the potential.

In conclusion, we make two remarks.

1. We have considered the correlation functions of a spinless Fermi gas. It is clear, however, that the considerations regarding crystallization of a Fermi gas at large values of g hold also for a Fermi gas with spin. In particular, the energy of density excitations is given by formula (7). The presence of spin leads to the appearance of a new spin-wave branch of the spectrum. For $g \to \infty$, however, the energies of these excitations are exponential small[10] and can be neglected.

2. Although the asymptotic behaviors of the pair-correlation functions of a spinless Fermi gas were calculated by us for $g \ll 1$, there is a basis for expecting that the corresponding expressions are correct for all $g > 0$. Here the critical exponents are expressed according to (16) in a simple way in terms of a renormalized velocity of sound. There asymptotic expressions coincide functionally with those found within the framework of the linear model. The corresponding critical exponents, however, are different for $g \to \infty$ but they coincide for $g \ll 1$.

In a similar way, the critical exponent (31), characterizing the powerlaw decrease of $\rho(R)$, agrees with the exponent found in the linear model[11] only accurate to terms $\sim g^2$. This fact enables us to conclude that linearization of the spectrum is correct only when $g \ll 1$. Although this last remark is quite trivial (and has been made repeatedly), there are a significant number of papers in which the results of the linear model are extrapolated into the range of intermediate values of g.

[1)] The picture of a Wigner crystal with weak oscillations is applicable if the inequality $E_M \gg E_0$ is satisfied (E_M is the Madelung energy, E_0 the energy of the zero-point oscillations). In particular, for power-law interaction potentials this inequality is satisfied at large g. For an exponential potential, the corresponding criterion has the form $g \gg x_0^{-2} \exp(a/x_0)$ for $x_0 \ll a$, $g \gg a x_0^{-3}$ for $a \ll x_0$ ($x_0 = \gamma^{-1}$ is the interaction radius).

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