

# Heavy nuclei in a superstrong magnetic field

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The production of positrons by the Coulomb field of heavy nuclei in a superstrong magnetic field is considered. It is shown that in fields  $B > B_0 = m^2 c^3 e^{-1} \hbar^{-1} G$  ( $B/B_0 > Z^3 \alpha^2$ )  $Z$  positrons with polarization along the vector  $\mathbf{B}$  are produced in the field of a nucleus with charge  $Z > Z_{cr}$  ( $Z_{cr}$  is the critical charge of the nucleus) as well as a vacuum electron shell having a small magnetic moment. The vacuum charge distribution in superheavy nuclei ( $Z\alpha \gg 1$ ) located in a superstrong magnetic field  $Z^3 \alpha^2 \gg B/B_0 \gg 1$  is found.

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In superstrong magnetic fields with induction  $B \sim 10^{12}$  G or higher, existence of which on the surface of neutron stars has been proved in practice, electrons move in the plane transverse to  $\mathbf{B}$  in a region with characteristic linear dimensions  $\Delta R_{\perp} \sim \gamma^{-1/2} = \lambda_c (B_0/B)^{1/2}$ , where  $\lambda_c = \hbar/mc$  is the electron Compton wavelength and  $B_0 = m^2 c^3 e^{-1} \hbar^{-1}$  is the strength of the so-called critical field.<sup>1</sup>

As a consequence of this, in a strong magnetic field the effective interaction of electrons with heavy nuclei becomes, as it were, stronger than in the absence of the field, since the electrons are more strongly attracted to the nucleus. If  $B/B_0 \gg Z$ , where  $B_0 = B_0 \alpha^2 = 2.35 \cdot 10^9$  G, solutions of the Dirac equation in magnetic and Coulomb fields can be found by using the approximation of a strong magnetic field,<sup>2</sup> i.e., for motion in the plane transverse to  $\mathbf{B}$  the Coulomb field can be taken into account by perturbation theory, assuming that the nature of the energy spectrum of the electron does not change by any substantial amount in the magnetic field.

## 1. CREATION OF POSITRONS BY THE COULOMB FIELD OF HEAVY NUCLEI IN A SUPERSTRONG MAGNETIC FIELD (SINGLE-PARTICLE APPROXIMATION)

Let us consider the creation of positrons by the Coulomb field of a heavy nucleus in a superstrong magnetic field. This phenomenon has been the subject of many studies (see for example Refs. 3-8), where, in particular, the situation in a strong magnetic field has been discussed.<sup>7</sup>

Consider the Dirac equation in a field

$$A_0 = -Ze/r, \quad A_\varphi = A_z = 0, \quad A_r = B_0/2, \quad (1)$$

assuming that  $B > B_0$ . Since in this case the motion of the electron in a plane perpendicular to the vector  $\mathbf{B}$  is determined by the magnetic field, and the Coulomb field can be considered as a perturbation, a solution of the Dirac equation in the field (1) will be sought in the form

$$\psi = \exp \left\{ -i \frac{eEt}{\hbar} + i \left( l - \frac{1}{2} \right) \varphi \right\} \left( \frac{\gamma}{\pi} \right)^{1/2} \begin{pmatrix} C_1 I_{n-1, s}(x) \chi_1(z) e^{-i\varphi/2} \\ i C_2 I_{n, s}(x) \chi_2(z) e^{i\varphi/2} \\ C_3 I_{n-1, s}(x) \chi_3(z) e^{-i\varphi/2} \\ i C_4 I_{n, s}(x) \chi_4(z) e^{i\varphi/2} \end{pmatrix}, \quad (2)$$

i.e., the radial functions describing the motion of the electron in a plane transverse to  $\mathbf{B}$  will be taken from the problem of motion of an electron in a magnetic

field.<sup>1</sup> In Eq. (2)  $I_{n, s}(x)$  is a generalized Laguerre polynomial with argument  $x = \gamma \rho^2 = eH\rho^2/2c\hbar$ :

$$I_{n, s}(x) = (n!s!)^{-1/2} \exp \left( -\frac{x}{2} \right) x^{(n-1)/2} Q_s^{n-s}(x), \\ Q_s^{n-s}(x) = e^x x^{-l} \frac{d^s}{dx^s} (x^{l+s} e^{-x}); \quad l+s=n, \quad n=0, 1, 2, \dots, \quad s=0, 1, 2, \dots \quad (3)$$

Here  $E$  is the energy of the electron and the quantum numbers  $n$ ,  $s$ , and  $l$  have the following meaning:  $n$  numbers the electron energy levels in a magnetic field (Landau levels),  $s$  determines the distance from the origin of the coordinate system to the "center of orbit" of the electron motion in the magnetic field, and  $l$  is the eigenvalue of the operator of projection of the orbital angular momentum onto the direction of the field. Any level with a fixed value of  $n$  is degenerate with infinite multiplicity if the electron moves only in a uniform and constant magnetic field. We note that according to Eq. (3) we have  $l = -s$  in the state with  $n = 0$ .

Let us consider the state corresponding to the lowest energy level of the electron in a magnetic field, i.e., the state with  $n = 0$ . We also set  $s = 0$ . Then  $I_{-1, 0}(x) = 0$ , and for  $C_2 \chi_2(z) \equiv g(z)$  and  $C_4 \chi_4(z) \equiv f(z)$  we obtain, after substituting  $\psi$  into the Dirac equation in the field (1),

$$\left[ \frac{\partial g}{\partial z} - (\varepsilon K + k_0 - V(r)) \right] I_{0,0}(x) = 0, \\ \left[ \frac{\partial f}{\partial z} + (\varepsilon K - k_0 - V(r)) \right] I_{0,0}(x) = 0, \quad (4)$$

where  $K = E(c\hbar)^{-1}$ ,  $k_0 = mc\hbar^{-1}$ ,  $\varepsilon = +1$ ,  $V = -Z\alpha/r$ . Multiplying these equations by  $I_{0,0}(x)$  and integrating them over  $x$  with inclusion of the normalization

$$\int I_{0,0}^2(x) dx = 1,$$

we obtain the following system:

$$dg/dz - (\varepsilon K + k_0 - \bar{V}(z)) f = 0, \quad df/dz + (\varepsilon K - k_0 - \bar{V}(z)) g = 0, \quad (5)$$

$$\bar{V}(z) = - \int_0^z \frac{Z\alpha e^{-\rho}}{(\rho/\gamma + z^2)^{3/2}} d\rho. \quad (6)$$

In this way we have obtained the one-dimensional Dirac equation for an electron in a field  $\bar{V}(z)$ , and the constant  $E$  has the meaning of the particle energy. Thus, according to our assumption, the motion of the electron in the direction of  $\mathbf{B}$  is described by the system (5) with  $\bar{V}(z)$  determined by Eq. (6).

We shall show that in a strong magnetic field the "effective Coulomb potential"  $\bar{V}(z)$  is cut off at distances  $z$  of the order  $\gamma^{-1/2}$ . Calculating the integral (6)

for  $\bar{V}(z)$ , we find  $\bar{V}(z) = -Z\alpha(\pi\gamma)^{1/2}e^{-\pi z^2}(1 - \Phi[(\gamma z^2)^{1/2}])$ , where  $\Phi(y)$  is the error function. Using the representation of  $\Phi(y)$  in the form

$$\Phi(y) = 2\pi^{-1/2} \exp(-y^2) \sum_{k=0}^{\infty} \frac{2^k y^{2k+1}}{(2k+1)!},$$

we find that for  $z=0$  we have  $\bar{V}(0) = -Z\alpha(\pi\gamma)^{1/2}$ . We note also that for  $(\gamma z^2)^{1/2} \gg 1$  the "effective potential" can be approximated by a one-dimensional Coulomb potential, since for  $z \gg \gamma^{-1/2}$

$$1 - \Phi((\gamma z^2)^{1/2}) = \pi^{-1} e^{-\gamma z^2} \sum_{k=0}^{n-1} (-1)^k \frac{\Gamma(k+1/2)}{(\gamma z^2)^{k+1/2}} + \frac{e^{-\gamma z^2}}{\pi} R_n.$$

Since the magnetic field effectively cuts off the interaction of the electron with the Coulomb field at distances  $z \sim \gamma^{-1/2}$ , is convenient from the very beginning to choose the effective potential in the form<sup>9</sup>

$$\bar{V}(z) = -Z\alpha/(|z| + a_H), \quad (7)$$

where  $a_H \sim \gamma^{-1/2}$ .

The system (5) with  $\bar{V}(z)$  chosen in the form (7) can be solved exactly.<sup>9</sup> In the region  $z > 0$  these solutions have the form

$$\begin{aligned} g(z) &= \frac{A}{[2t(1-\varepsilon K_0)]^{1/2}} \left\{ \frac{\xi}{(1-K_0^2)^{1/2}} W_{t, -it}(t) + W_{t, it}(t) \right\}, \\ f(z) &= \frac{A}{[2t(1+\varepsilon K_0)]^{1/2}} \left\{ \frac{\xi}{(1-K_0^2)^{1/2}} W_{t, it}(t) - W_{t, -it}(t) \right\}, \quad (8) \\ \xi_{\pm} &= \xi \operatorname{ctg} \bar{\varepsilon} \pm 1/2, \quad \xi = Z\alpha, \quad K_0 = K/k_0, \quad t = 2(a_H + z)(k_0^2 - K^2)^{1/2}, \\ &\operatorname{ctg} \bar{\varepsilon} = \varepsilon K_0(1 - K_0^2)^{-1/2}. \end{aligned}$$

From the condition of matching of the solutions at the point  $z=0$  [we note that  $\bar{V}(z)$  is an even function of  $z$ ] it follows that two classes of solution are possible: 1) even  $g$  and odd  $f$ , and 2) the opposite case. There are no solutions of the first class in an ordinary three-dimensional Coulomb field.<sup>9</sup>

Let us consider solutions of the first class, among which there is a solution corresponding to the ground state. The energy spectrum in this case is found from the condition  $f(z=0) = 0$  or

$$\frac{\xi}{(1-K_0^2)^{1/2}} W_{t, -it}(t_0) = W_{t, it}(t_0), \quad (9)$$

where the argument of the Whittaker function  $W_{\lambda, \mu}$  at the point  $z=0$  satisfies the inequality

$$t_0 = 2(1-K_0^2)^{1/2} a_H / \lambda_c \ll 1.$$

Using the well known expansion for  $W_{\lambda, \mu}$  near zero

$$W_{\lambda, \mu} |_{t \rightarrow 0} = t^{1/2} \left\{ \frac{\Gamma(2\mu)}{\Gamma(1/2 + \mu - \lambda)} t^{-\mu} + \frac{\Gamma(-2\mu)}{\Gamma(1/2 - \mu - \lambda)} t^{\mu} \right\},$$

where  $\Gamma(y)$  is the gamma function, it is easy to obtain from Eq. (9) the equation for determination of the energy spectrum of electron states ( $\varepsilon = +1$ ) in implicit form:

$$\begin{aligned} \xi \ln(2\Lambda a_H / \lambda_c) + \operatorname{arctg} \frac{\Lambda}{1-K_0} + \arg \Gamma\left(-\frac{\xi K_0}{\Lambda} + i\xi\right) \\ - \arg \Gamma(1+2i\xi) = \frac{\pi}{2} + \pi m, \quad m = -1, 0, 1, \dots \end{aligned} \quad (10)$$

where the lowest state corresponds to  $m = -1$ . Here

$$\Lambda = (1-K_0^2)^{1/2}, \quad \lambda = K_0 \xi \Lambda^{-1-1/2}.$$

The value of the nuclear charge  $Z(\xi)$  at which the energy level of the electronic ground state  $K_0$  crosses the boundary of the lower energy continuum (i.e.,  $K_0$  is equal to  $-1$ ) is usually called the critical charge. Differentiating Eq. (10), it is easy to show that the two derivatives are

$$\left. \frac{\partial K_0}{\partial \xi} \right|_{\xi \rightarrow \xi_{cr}} < 0, \quad \left. \frac{\partial K_0}{\partial \ln B/B_0} \right|_{K_0 \rightarrow -1+0} < 0,$$

i.e., with increase of the charge and of the magnetic field strength the ground-state energy level drops and crosses the limit of the lower energy continuum.

Taking the limit  $\xi \rightarrow \xi_{cr}$ ,  $K_0 \rightarrow -1$  in Eq. (10), we find the value of the magnetic field at which the intersection of the lower continuum with the energy level  $K = -k_0$  occurs, as a function of  $\xi_{cr}$ :

$$B = 4\xi_{cr}^2 B_0 \exp\left[\frac{\pi - 2 \arg \Gamma(1+2i\xi_{cr})}{\xi_{cr}}\right],$$

which agrees with the conclusions of Ref. 7. The threshold probability of pair production by the Coulomb field of a superheavy nucleus in a strong magnetic field  $w(\xi, \xi_{cr})$  can be found analytically, continuing  $K_0(\xi)$  into the region  $\xi > \xi_{cr}$ , as was done, for example, in Ref. 7. As a result we obtain

$$\begin{aligned} w(\xi, \xi_{cr}) &= \frac{6ck_0 \xi_{cr} \operatorname{sh}(2\pi\xi_{cr})}{1+2\xi_{cr}^2} \exp\left(-\frac{2\pi\xi_{cr}}{\delta}\right), \\ \delta &= 2\xi_{cr} \left(-\frac{3\pi s(\xi, \xi_{cr})}{1+2\xi_{cr}^2}\right)^{1/2}, \quad (11) \end{aligned}$$

$$s(\xi, \xi_{cr}) = -\frac{1}{2} \left(\frac{1}{\xi_{cr}} - \frac{1}{\xi}\right) + \frac{1}{\pi} \left[\ln \frac{\xi}{\xi_{cr}} - \frac{1}{\xi_{cr}} \Gamma(1+2i\xi_{cr}) - \frac{1}{\xi} \Gamma(1+2i\xi)\right].$$

Thus, the threshold probability of pair production by the Coulomb field of a nucleus in superstrong magnetic field depends only weakly on the choice of  $\bar{V}(z)$  near the point  $z=0$ . This is confirmed by comparison of Eq. (11) with the result obtained by Oraevskii *et al.*,<sup>7</sup> in which the function  $\bar{V}(z)$  near  $z \approx 0$  was different.

## 2. WAVE FUNCTION OF THE GROUND STATE OF THE DISCRETE SPECTRUM AS $\xi \rightarrow \xi_{cr}$ AND AT $K_0 = -1+0$

The wave function of the ground state of the discrete spectrum for  $\xi \rightarrow \xi_{cr}$  and  $K_0 = -1+0$ , i.e., in the subcritical region of values of  $\xi$ , in the field configuration considered can be written in terms of a MacDonal function. For this purpose we shall use the well known relation (see page 257 of Ref. 10)

$$W_{\lambda, \mu}(x) = e^{-x/2} x^{\mu+1/2} \psi(1/2 - \lambda + \mu, 2\mu + 1, x), \quad (12)$$

where  $\psi$  is the confluent hypergeometric function and the formula for the limiting transition (page 253 of Ref. 10) is

$$\lim_{a \rightarrow \infty} |\psi(a, c, x) \Gamma(a)| = (2xa)^{-a} K_{2a}((8xa)^{1/2}). \quad (13)$$

We note that  $a \rightarrow \infty$ , since  $a \sim -\lambda \rightarrow \infty$ .

To find the explicit form of the normalized wave function of the ground state it is convenient to represent the solution (8) in the following approximate form:

$$g = \frac{A}{t^{1/2}} \frac{\xi}{\Lambda} W_{\lambda, it}(\bar{t}), \quad f = \frac{A}{t^{1/2}} \frac{1-i\xi}{\Lambda} W_{\lambda, it}(\bar{t}). \quad (14)$$

Here  $\lambda = \xi/\Lambda - \frac{1}{2}$  and  $t = 2\Lambda(z + a_H)/\lambda_c$ . The normaliza-

tion coefficient  $A$  is found from the condition

$$\int (|f|^2 + |g|^2) dz = 1. \quad (15)$$

Recognizing (see page 872 of Ref. 11) that

$$\int_0^\infty \{W_{\lambda,\mu}(t)\}^2 \frac{dt}{t} = \frac{\pi}{\sin 2\pi\mu} \frac{\psi(1/2 + \mu - \lambda) - \psi(1/2 - \mu - \lambda)}{\Gamma(1/2 + \mu - \lambda)\Gamma(1/2 - \mu - \lambda)},$$

and using the well known representation for the difference of the  $\psi$  functions (see page 958 of Ref. 11)

$$\psi(x+iy) - \psi(x-iy) = \sum_{k=0}^{\infty} \frac{2iy}{y^2 + (x+k)^2},$$

and also calculating this sum by means of the Euler summation formula

$$\sum_{k=0}^{\infty} \frac{2iy}{(x+k)^2 + y^2} \Big|_{x \rightarrow \infty} \approx \frac{2iy}{x},$$

from Eq. (15) for determination of  $|A|^2$  we obtain

$$\lambda_c |A|^2 \frac{\pi}{\text{sh } 2\pi\xi_{cr}} \frac{2\xi_{cr}^2 + 1}{\Lambda^2} \left| \Gamma\left(\frac{\xi_{cr}}{\Lambda} + 1 + i\xi_{cr}\right) \right|^{-2} = 1$$

or

$$|A| = \left| \Gamma\left(\frac{\xi_{cr}}{\Lambda} + 1 + i\xi_{cr}\right) \right| \left[ \frac{\Lambda^2 \text{sh } 2\pi\xi_{cr}}{\pi\lambda_c(1+2\xi_{cr}^2)} \right]^{1/2}. \quad (16)$$

With inclusion of Eqs. (12)–(15) and also (16) for the normalized wave function of the ground state for  $\xi \rightarrow \xi_{cr}$  and  $K_0 \rightarrow -1$  we obtain

$$g = D\xi_{cr}K_\nu(x), \quad f = D(1+\xi_{cr}^2)^{1/2}K_\nu(x),$$

$$D = \left[ \frac{\text{sh } 2\pi\xi_{cr}}{\pi\lambda_c(1+2\xi_{cr}^2)} \right]^{1/2}, \quad \nu = 2i\xi_{cr}, \quad x = 8^{1/2}\xi_{cr} \frac{z+a_H}{\lambda_c}. \quad (17)$$

### 3. MANY-ELECTRON CASE

Quantitative estimates of the production of positrons by the field of a superheavy nucleus in the presence of a strong magnetic field, obtained in Ref. 7, showed that the effect of the magnetic field on the energy spectrum of an electron in the field of a heavy nucleus leads to a substantial lowering of the energy of the ground state of the electron for a given nuclear charge  $Z(\xi)$ . This means that in superstrong magnetic fields the critical charge of the nucleus decreases in comparison with the case  $B=0$ . A second important difference of the effect considered by us from the creation of positrons by the Coulomb field of nuclei in the absence of a magnetic field is the following. In a Coulomb field for  $Z > Z_{cr}$  but  $Z < Z_{cr}^{(2)}$ , where  $Z_{cr}^{(2)}$  is the charge of the nucleus for which the  $2s$  level crosses the boundary of the lower continuum, two positrons with antiparallel spins are created. If the effect is studied in the framework of second quantization, then from the point of view of this theory as  $Z$  goes through  $Z_{cr}$  there is a decay of the neutral vacuum, as a result of which two positrons are formed (if the ground-state level for  $Z < Z_{cr}$  was not filled with electrons) and a charged vacuum arises, the total charge of which is  $-2e$  (see for example Refs. 4, 5, and 8). In a superstrong magnetic field the spin of an electron located in the Landau level  $n=0$  can be oriented only opposite to the vector  $B$ . Therefore, if the ground state was not filled with electrons before the transition through  $Z_{cr}$  (from  $Z < Z_{cr}$  to  $Z > Z_{cr}$ ), positrons should be created with polarization along the vector  $B$ , and in the case when this state is filled with electrons, no positrons will be created, but both in

that case and in the other case there is a rearrangement of the vacuum, as a result of which the charged vacuum arises. Here the charge density of the new vacuum for  $0 < Z - Z_{cr} \ll Z_{cr}$  as a function of the coordinates has approximately the same distribution as the probability density of various values of the coordinates of the electron located in the ground state for  $Z < Z_{cr}$ .

The distribution of the vacuum charge near supercritical nuclei ( $Z\alpha \gg 1$ ) was found in Ref. 5, where it was shown in particular that for  $Z \gg Z_{cr}$  the total charge of the vacuum shell becomes equal to  $-Ze$ . For  $Z \gg \alpha^{-3/2}$  (supercharged nuclei) the vacuum electrons, penetrating inside the nucleus, compensate its charge practically completely. According to Ref. 5, supercharged nuclei consist of an electrically neutral plasma with equal concentrations of electrons, protons, and neutrons (it is assumed that the mass number  $A$  is equal to  $2Z$ ). The question of the vacuum shell of a heavy nucleus in the presence of a superstrong magnetic field, and also several other aspects of this problem, will be discussed below.

Let us consider qualitatively the question of filling of electron shells of an atom located in a strong magnetic field  $B \gg B'_0 Z$ . We shall take into account that the state with  $n=0$  in a magnetic field is degenerate in  $s$ , and the wave function of an electron in this level is proportional to  $I_{0,s}(\gamma\rho^2)$ , where  $s=0,1,2,\dots$ . We recall that in the absence of a Coulomb field the level  $n=0$  is degenerate with an infinite multiplicity. However, if the electron is in magnetic and Coulomb fields the value of  $s_{\max}$  can be found by using physical prerequisites. For this we will take into account that in the state  $n=0, s \neq 0$  the probability density of various values of the coordinates of an electron in the plane transverse to  $B$  has the form

$$I_{0,s}^2(x) = x^s e^{-x/s!}, \quad x = \gamma\rho^2,$$

$I_{0,s}^2(x)$  has a maximum for  $x=s$ , and the spatial width of the distribution for each  $s$  is approximately the same and is determined by the function  $I_{0,s}^2(x) = e^{-x}$ . Therefore in a crude approximation we can modify the expression for the average potential energy of the electron in the state  $s$ , introducing into this expression an explicit dependence on  $s$ . Then assuming that  $Z \gg 1$ , let us consider qualitatively how the electron shells are filled for  $Z < Z_{cr}$ . We shall take it into account that in each state characterized by numbers  $m$  and  $s$  there can be only one electron. As was shown above, the area of the region of motion of an electron in a strong magnetic field is equal in order of magnitude to the quantity  $\Delta R_1^2 \sim (B'_0/B)a_B^2$ . The quantity  $\Delta R_1^2$  must be compared with  $r_B^2 = a_B^2 Z^{-2}$ , where  $r_B$  is the radius of the Bohr orbit of a hydrogen-like atom of charge  $Z$ . Thus, if  $B/B'_0 \ll Z^3$ , then

$$Z\Delta R_1^2 \gg r_B^2. \quad (18)$$

Here  $Z\Delta R_1^2$  is the area of the region in which  $Z$  electrons could be located. Therefore it is evident that for  $B/B'_0 \ll Z^3$  the electrons will fill only the lowest  $s$ -states, and in each  $s$ -state there can be several electrons, i.e., the electrons will be distributed over the  $m$ -levels.

It is evident that the inequality (18) is satisfied if the bulk of the electrons are concentrated in a region with linear dimension

$$l \sim (Z^4 B_0 / B)^{1/2} a_B. \quad (19)$$

This qualitative conclusion agrees with the result obtained in Ref. 2. Hence we can also obtain an estimate of the value of  $s_{\max}$ :

$$s_{\max} \sim \gamma l^2 = (BZ^2 / B_0)^{1/2},$$

i.e.,  $s_{\max} \ll Z$ .

However, if  $B/B_0 \gg Z^3$ , then  $Z \Delta R_1^2 \ll r_B^2$ , i.e., in this case all electrons can be located in the ground-state  $m$ -level, and in each  $s$ -state there will be only one electron.

For  $B/B_0 \ll Z^3$  the distribution of the vacuum charge near superheavy nuclei located in a superstrong magnetic field, like the charge distribution in a neutral atom in which both the vacuum and external electron shells are completely filled, can be found from the relativistic Thomas-Fermi equation (see Sec. 4). However, this equation is not suitable for analysis of the situation with  $B/B_0 \gg Z^3$ . As a complete trial function describing the distribution of vacuum charge, in the latter case we can choose a function in the form of the product of single-particle functions

$$\psi_i = e^{i\epsilon_i} \left( \frac{\gamma}{\pi} \right)^{1/2} I_{0,s}(x_i) W_{\lambda_0,\mu}(z_i)$$

[ $W_{\lambda_0,\mu}(z)$  is a function corresponding to the ground state of longitudinal motion for a given  $s$ ], which is antisymmetrized in  $s$ . A trial function is constructed similarly for the ground state of a neutral heavy atom located in a strong magnetic field (see Ref. 12).

We note that for  $B \gg B_0$  the lowest energy level  $K$  crosses the lower continuum boundary for such  $Z_{cr}$  that  $B/B_0 \gg Z_{cr}^3$ . Therefore in the approximation considered here if  $Z > Z_{cr}$  the neutral vacuum should decay into a supercharged vacuum having a total charge  $-Ze$ , and  $Z$  positrons should be formed, which penetrate through the potential barrier and go off to infinity. The vacuum charge in this case will be localized in a region

$$\pi \Delta R_1^2 \sim \pi Z \frac{B_0}{B} \lambda_c^2, \quad \Delta z \sim \frac{\lambda_c}{Z\alpha},$$

and the number of electrons located in a sphere with a radius equal to the radius of the nucleus is

$$N_e \approx \frac{4}{\pi} Z^3 \alpha \left( \frac{\lambda_\pi}{\lambda_c} \right)^3 \frac{B}{B_0},$$

where  $\lambda_\pi = \hbar/m_\pi c$  is the pion Compton wavelength. The vacuum shell has a small magnetic moment which for  $Z \sim Z_{cr}$  is equal to

$$\mu_V = -Z \frac{\partial E}{\partial B} = \frac{3\xi_{cr} Z}{1+2\xi_{cr}^2} \frac{mc^2}{B},$$

in which  $\mu_V \ll \mu$ ,  $Z = Ze\hbar/2mc$ . We note that our discussion is applicable for the condition  $B \ll B_\pi = m_\pi^2 c^3 / |e|\hbar$ , since in the fields  $B_\pi$  it is necessary to take into account the influence of the field  $B$  on the structure of the nucleus, although probably in weaker fields  $B \sim 12\pi\alpha^{-1}B_0$  the contributions of characteristic nonlinear

quantum effects may become important: vacuum polarization and other effects which must be taken into account. Therefore the value of  $B$  most likely must be restricted to a value  $12\pi\alpha^{-1}B_0$ .

If  $Z \gg Z_{cr}$ , the entire discussion can be carried out for very large  $Z$  such that  $B/B_0 \ll Z^3$  in the framework of a spherically symmetric relativistic model of the Thomas-Fermi type.

#### 4. DISTRIBUTION OF VACUUM CHARGE IN SUPERHEAVY NUCLEI ACCORDING TO THE THOMAS-FERMI MODEL

The distribution of vacuum charge in superheavy nuclei located in a strong magnetic field for  $B/B_0 > Z^3$  can be discussed qualitatively, proceeding from a modified relativistic Thomas-Fermi equation. We shall give the derivation of this equation, using the method set forth in Ref. 5. For this purpose it is necessary to find a solution of the system (5) in the quasiclassical approximation. Setting in (5)

$$g = a(x) \exp[is(x)], \quad f = b(x) \exp[is(x)],$$

where  $x = k_0 z$ , and assuming that  $a(x)$  and  $b(x)$  are slowly varying functions of  $x$ , i.e., neglecting the derivatives  $a'(x)$  and  $b'(x)$ , we obtain<sup>5</sup>

$$g = A \left[ \frac{K_0 - \mathcal{V} + 1}{p(z)} \right]^{1/2} \sin \Phi(z), \quad f = A \left[ \frac{K_0 - \mathcal{V} - 1}{p(z)} \right]^{1/2} \cos \Phi(z),$$

$$\Phi(z) = \int_0^z p(x) dx + \frac{\pi}{4}, \quad z > 0.$$

The normalization coefficient  $A$  is found from the condition

$$\int_0^{z_t} (|g|^2 + |f|^2) dz = |A|^2 \int_0^{z_t} \frac{K_0 - \mathcal{V}}{p(z)} dz = \frac{1}{2},$$

from which

$$|A| = 2^{-1/2} \left\{ \int_0^{z_t} \frac{K_0 - \mathcal{V}}{p(z)} dz \right\}^{-1/2}. \quad (20)$$

Here  $z_t$  is the turning point in the region  $z > 0$  for the effective equation

$$u'' + k^2(z)u = 0,$$

where

$$k^2 = 2(\omega - U), \quad \omega = (K_0^2 - 1)/2$$

$$U = \mathcal{V} K_0 - \frac{1}{2} \mathcal{V}^2 + \frac{1}{4} \left\{ \frac{\mathcal{V}''}{1 + K_0 - \mathcal{V}} + \frac{3}{2} \left( \frac{\mathcal{V}'}{K_0 + 1 - \mathcal{V}} \right)^2 \right\},$$

i.e.,  $z_t$  is the positive root of the equation  $k^2(z) = 0$ .

If the barrier penetrability is exponentially small, the energy levels  $K_0^{(m)}$  can be determined from the Bohr-Sommerfeld quantization condition:

$$\int_0^{z_t} p(z) dz = (m + 1/2) \frac{\pi}{2}. \quad (21)$$

Differentiating Eq. (21) with respect to  $m$  and comparing the result with Eq. (20), we find

$$A = (\pi^{-1} \partial K_0 / \partial m)^{-1}. \quad (22)$$

Then, as shown in Ref. 5, for  $w(\xi, \xi_{cr}) \ll 1$  for description of the vacuum electrons one can use the single-particle approximation, and the vacuum electron den-

sity can be found by direct means, summing over single-particle states localized in the region  $2\Delta z$  lying between the turning points:

$$n_s = - \sum_m |\psi_m|^2 = - \sum_m |A|^2 \frac{K_0 - \bar{V}}{p(z)} = - \frac{1}{\pi} \sum_m \frac{\partial K_0}{\partial m} \frac{K_0 - \bar{V}}{p(z)}. \quad (23)$$

We recall that at  $B/B'_0 > Z^2$  in each state  $n=0, m, s$  there can be only one electron, with a spin oriented opposite to the magnetic field direction. Therefore the right-hand side of Eq. (23) does not contain the factor 2. Going over from summation over  $m$  to integration in Eq. (23), we obtain

$$\rho = - \frac{1}{\pi} \int dK_0 \frac{K_0 - \bar{V}}{[(K_0 - \bar{V})^2 - 1]^{1/2}} = - \frac{1}{\pi} \int \frac{x dx}{(x^2 - 1)^{1/2}}.$$

If we are interested in the distribution of the electrons in the vacuum shell of the nucleus, integration over the energy  $K_0$  must be carried out in the interval (see Ref. 5)

$$-1 > K_0 > 1 + \bar{V}.$$

In the case of a neutral atom in which both the external and vacuum shells are filled, the upper limit is equal to 1. Consequently we obtain  $n_s = -(2\bar{V} + \bar{V}^2)^{1/2} / 2\pi$  for a superheavy nucleus with a filled vacuum shell and  $n_s = -(2\bar{V} + \bar{V}^2)^{1/2} / 2\pi$  for an atom in which all shells are filled. Here we have taken into account that the characteristic dimensions of the region of motion of the electrons along the magnetic field are  $2\Delta z$ .

We note that, as was done for the state  $s=0$ , quasi-classical functions can be constructed also for other  $s$ -states. Here, if  $B/B'_0 \ll Z^3$ , then as was shown above the wave function of the electrons located in neighboring  $s$ -levels will overlap to a significant degree. Therefore it will be more advantageous to the electrons to fill the lowest  $s$ -levels (where  $s \ll Z$ ) and, this means, the rather high  $m$ -levels, since such a distribution will correspond to the minimum energy of the system. In this case the function  $n_s$  can be considered to be a function of all variables ( $\rho, z$ ), and we obtain the true electron density by multiplying  $n_s$  by the normalization coefficient of the total electron wave function  $\gamma\pi^{-1}$ .

We shall write the Poisson equation in the form

$$\Delta\varphi = -4\pi e(n_p - n_s). \quad (24)$$

Here  $e > 0$ ,  $\varphi = -\bar{V}e^{-1}$  is the electrostatic potential and  $n_p$  is the density of protons. For the function  $\bar{V}$  we have the equation

$$\Delta\bar{V} = -4\pi e^2(n_s - n_p). \quad (25)$$

We shall take  $n_p$  in the form<sup>5</sup>:

$$n_p(r) = n_p \theta(R-r).$$

Obviously the vacuum shell has a finite radius  $r_v$ , since  $n_s(r)$  is different from zero only in the region of space where  $\bar{V}^2 > 2\bar{V}$ . Therefore the boundary conditions for solutions of Eq. (25) in the case of a supernucleus with a filled vacuum shell has the form<sup>5</sup>

$$\bar{V}(0) = \text{const}, \quad \bar{V}(r_v) = -r_v \bar{V}'(r_v) = 2, \quad \bar{V}(r_v) = -Z_0 e^2 / r_v,$$

where  $Z_0$  is the combined charge of the nucleus and the

vacuum shell for an external observer. In the variables  $r = k_0 r$ , Eq. (25) has the form

$$\Delta V = \lambda^2 [\mp 2V + V^2]^{1/2} + 4\pi\alpha n_p, \quad (26)$$

$$\lambda^2 = \frac{\alpha^3 B}{\pi B'_0 r}, \quad n_p = \frac{Z}{A} n_0 = \frac{Z}{A} \left( \frac{3}{4\pi r_0^3} \right),$$

where  $R = A^{1/3} r_0$  is the nuclear radius,  $r_0 = 1.1 \text{ F}$ , and  $A$  is the mass number. In the relativistic limit  $V^2 \ll 2V$  Eq. (26) coincides with Eq. (9) of Ref. 2, which was obtained by a different method for description of the potential of a neutral atom with  $Z\alpha < 1$  located in a strong magnetic field. In this case the value of the ratio  $B/B'_0$  for which Eq. (26) is valid is also bounded from below:  $B/B'_0 > Z^4/3$ . This condition arises from the requirement that there be no electrons in the first excited Landau level in the main region of the potential. The condition that there be no vacuum electrons with a high binding energy ( $-K_{\text{coul}} \gg mc/\hbar$ ) in the first excited Landau level has the form  $(1/2)(-K_{\text{coul}}/\hbar mc) \times (B_0/B)^{1/2} \ll 1$ .

Going over to a new function  $V_0 = V \mp 1$ , we obtain for it the simpler relation

$$\Delta V_0 = \lambda^2 (V_0^2 - 1)^{1/2} + 4\pi\alpha n_p. \quad (27)$$

Let us find the charge distribution inside the vacuum shell for  $Z\alpha \gg 1$ . We shall consider the region of values  $r \ll Z_0 \alpha \lambda_c / 2$ . Then Eq. (26) is greatly simplified and takes the form

$$\Delta V_0 - \lambda^2 V_0 = B\theta(R-r), \quad B = 4\pi\alpha n_p. \quad (28)$$

The spherically symmetric solution of this equation which (together with its first derivative) satisfies the condition of continuity for  $r=R$  and is finite for  $r=0$  has the form

$$V_0 = \frac{B}{\lambda^2 r} \left\{ [(1+\lambda R)e^{-\lambda R} \text{sh } \lambda r - \lambda r] \theta(R-r) + e^{-\lambda(r-R)} \left( 1 - \frac{1+\lambda R}{2} e^{-\lambda R} \right) \theta(r-R) \right\}.$$

The number of vacuum-shell electrons located inside a sphere of radius  $r$  is  $N_e(r) = Z - Q(r)$ , where  $Q(r)$  can be found from Gauss's theorem,  $V'(r) = eE = Q(r)e^2/r^2$ . We note that  $\lambda \ll (Z\alpha)^{3/2}$ , and therefore the interval of values of  $r$  in which the obtained solution is valid has the form

$$(Z\alpha)^{-3/2} \ll r \lambda_c^{-1} \ll Z_0 \alpha,$$

i.e., for  $Z\alpha \gg 1$ ,  $Z_0 \sim Z$  this interval turns out to be rather broad.

Let us find  $Q(r)$ . Differentiating  $V_0$  with respect to  $r$ , we obtain

$$V'(r) = \frac{B}{\lambda^2} \left\{ \frac{1+\lambda R}{r} e^{-\lambda R} \left( \text{ch } \lambda r - \frac{\text{sh } \lambda r}{\lambda r} \right) \theta(R-r) - \frac{e^{-\lambda(r-R)}}{r} \left( 1 + \frac{1}{\lambda r} \right) \left( 1 - \frac{1+\lambda R}{2} e^{-\lambda R} \right) \theta(r-R) \right\}.$$

From this we obtain for  $Q(r)$

$$Q(R) = \frac{3Z}{(\lambda R)^2} (1+\lambda R) e^{-\lambda R} \left( \text{ch } \lambda R - \frac{\text{sh } \lambda R}{\lambda R} \right).$$

If  $\lambda R \ll 1$ , then

$$Q(R) = Z [1 - 1/2(\lambda R)^2] \sim Z, \quad N_e(R) \approx 1/2 Z (\lambda R)^2 \ll Z.$$

In the second case  $\lambda R \gg 1$  we have

$$Q(R) \approx 3Z/2(\lambda R) \ll Z.$$

Therefore the total negative charge of electrons located inside the nucleus for  $\lambda R \gg 1$  is of the order of  $Z$ .

As in the absence of a magnetic field, in the case considered here (for  $\lambda R \gg 1, Z\alpha \gg 1$ ) inside the superheavy nucleus there will be formed an electrically neutral plasma and the screening of the nuclear charge by the electrons of the plasma will become important. Since inside the nucleus the total charge is close to zero, there will be no electric field in this region. However, the electric field strength is large for  $r=R$ , i.e., at the nuclear boundary, and the value of this field is inversely proportional to  $B^{1/2}$ :

$$E_R = 2\pi^{1/2} \frac{n_p}{\alpha^{1/2}} \left( \frac{B_0'}{B} \right)^{1/2} \frac{c\hbar}{e} \lambda_c,$$

where for  $A=2Z$

$$n_p \approx 1/4 (m_\pi c/\hbar)^3 = 1/4 \lambda_\pi^{-3}.$$

Here we must have in mind that  $Z^{-3/2} \ll (B_0'/B)^{1/2} \ll \alpha$ ,  $Z\alpha \gg 1$ . For an estimate let us set  $(B_0'/B)^{1/2} = \alpha^{3/2}$ . Then  $E_R = (\pi^{3/2}/2)\alpha^2(\lambda_c/\lambda_\pi)^3 B_0$ . In this case we obtain essentially the maximum possible value of  $E_R$  at the nuclear boundary (see Ref. 5). We note that without taking into account the vacuum charge which screens the nuclear field, the field  $E_R$  will approach infinity with increase of  $Z$  as  $Z^{1/3}$ :  $E_R^0 = Z^{1/3} e(\pi/3)^{2/3} \lambda_\pi^{-2}$ , in which  $Z^{1/3} \gg (\lambda_c/\lambda_\pi)$ .

As is well known, the effect of diamagnetism in atoms leads to appearance of a magnetic field induced by the external field in the atom, and the strength of this field at the center can be estimated from the formula (see Ref. 13, p. 535)  $B_{\text{ind}} = -(e^2/3mc^2)\varphi_e(0)B$ , where  $\varphi_e(0)$  is the electrostatic potential created by the electrons of the atom at its center. The validity of this formula for the case of a neutral heavy atom located in a strong magnetic field  $B \gg B_0$  has not been proved by us, but if we nevertheless use it for estimates, we obtain

$$B_{\text{ind}} = -\frac{2^{1/2}}{3\pi^{1/2}} \alpha^2 \left( \frac{B}{B_0'} Z^2 \right)^{1/2} B.$$

[Here we have taken the value of  $\varphi_e(0)$  from Ref. 2.] At the limit of applicability of the nonrelativistic equation (26), i.e., for  $B=B_0'(Z^3)$  (here we also assume that  $Z\alpha < 1$ ), the following estimate is obtained:  $B_{\text{ind}} = -[(Z\alpha)^2/3\pi^{12/5}]2^{1/5}B$ , i.e., the value of the field induced at the center of the atom, obtained from Eq. (26), is approximately  $Z^{2/3}$  times greater ( $Z \gg 1$ ) than the value of this field obtained on the basis of the Thomas-Fermi equation.

## CONCLUSIONS

1. In the superstrong magnetic field of strength  $B/B_0 \gg Z^3\alpha^2$  in the field of heavy nuclei with charge  $Z > Z_{\text{cr}}$  there should be created  $Z$  positrons with polarization along the vector  $B$ . Here there is formed simultaneously an electron vacuum shell which has a small magnetic moment and a charge  $-Ze$ .

2. The distribution of charge in the vacuum shell of superheavy nuclei ( $Z\alpha \gg 1$ ) located in a strong magnetic field of strength  $Z^3\alpha^2 > B/B_0 \gg 1$  can be studied on the basis of a modified relativistic equation of the Thomas-Fermi type. We have found a spherically symmetric solution of this equation which describes the distribution of the vacuum charge in the region  $r \ll Z\alpha\lambda_0$ , where  $\lambda_c = \hbar/mc$  is the electron Compton wavelength. It is shown that for  $(\alpha B/\pi B_0)^{1/2}R \gg 1$ , where  $R$  is the nuclear radius, the total negative charge of the vacuum electrons inside the nucleus becomes of the order  $-Ze$ . As in the absence of a magnetic field (for  $Z\alpha^{1/2} \gg 1$ ), in the case considered by us the electric field in the internal region of the nucleus is close to zero, and the strength of the electric field at the nuclear boundary, with allowance for screening of the nuclear charge by the vacuum electrons, is equal to a finite value and does not approach infinity (with increase of  $Z$ ). The value of the electric field strength at the nuclear boundary agrees in order of magnitude with the value of  $E_R$  found previously in the absence of a magnetic field.<sup>5</sup>

3. In the superstrong fields  $B \gg B_0$  the critical charge of the nucleus is significantly less than in the absence of a field. Therefore in fields  $B \gg B_0$  nuclei with relatively small charge  $Z$  ( $Z \ll Z_{\text{cr}}^{B_0}$ ) should become unstable.

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