Magnetohydrodynamics of superfluid solutions

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The three-velocity magnetohydrodynamic equations for a two-condensate solution are established by specified thermodynamic functions and condensate densities, and are supplemented by Maxwell's equations. In particular, these equations yield the drag current and the magnetic vortex lattice of the neutral component. The magnetic-field flux through a neutral vortex is greater than the usual flux, the excess being due to the drag. The velocities of sound waves in the system are calculated.

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Systems in which there simultaneously exist condensates of two types, and therefore superfluid motions of two types, are of interest for both their thermodynamic properties. A good example of such a system is a solution of He^3 atoms in liquid He^4 at a temperature below the point at which the Fermi component undergoes a phase transition to the superfluid state. The properties of such a solution are described by the three-velocity hydrodynamic equations for the case of two superfluid velocities and one normal one. These equations have been derived by Andreev and Bashkin[1] with allowance for dragging of the He^3 component by the He^4 atoms. Starting from Khalatnikov's analysis of the conservation laws,[2] these authors showed that each of the superfluid motions is accompanied by transport of both the other components of the solution. Galasiewicz,[3] and Volovik, Mineev, and Khalatnikov[4] have investigated certain properties of such a solution.

The "prephase" of neutron stars[5] is another example of a system having two superfluid condensates that has recently attracted the interest of investigators. In this phase of the star, the proton density is about one percent of the neutron density ($N_n=10^{38}$ cm^{-3}), the protons' charge being neutralized by electrons ($N_p=N_e$). Cooper pairs of neutrons and protons are formed as a result of the strong nuclear interaction.[6,7] We note that there are no neutron-proton pairs because of the large difference between their chemical potentials. The interaction of the protons with the neutrons, however, converts them into quasiparticles with an effective mass m^*, so the proton and neutron condensates are actually coupled. The authors, generalizing Gor'kov's technique[9] for a two-component superconducting Fermi liquid, have previously shown that there is a current due to dragging of the protons by the neutrons.[7,8]

Let us look at one more example of such a system: a solution of protons and neutrons in a heavy-metal matrix.[10] It is known that such a solution manifests the properties of a quantum crystal, owing to the large mass difference between the impurity particles and the matrix atoms[11] Under quite definite conditions, the neutron-proton liquid can undergo a phase transition to a superconducting state.[12]

In this paper we propose a set of magnetohydrodynamic equations for the two-condensate superconductive systems mentioned above. The present work differs from that of Andreev and Bashkin,[1] who derived equations for three-velocity hydrodynamics, in that here we allow the impurity component to be charged and include a third, normal, component that ensures the local neutrality of the system. Study of the resulting equations will make it possible, in particular, to obtain the form of the drag current due to the coupling of the condensates and to investigate the properties of the neutron and proton vortex filaments. These equations will also make it possible to determine the propagation velocity of excitations in a superconductive solution.

1. It follows from the analysis of the conservation laws carried through in Refs. 1 and 2 that the complete set of three-velocity magnetohydrodynamic equations in the absence of dissipation has the form

$$\begin{gathered}
\dot{\rho}_1+\operatorname{div}(\rho_1\mathbf{v}_n+\mathbf{p}_1)=0,\quad \dot{\rho}_2+\operatorname{div}(\rho_2\mathbf{v}_n+\mathbf{p}_2)=0,\\
\dot{\mathbf{j}}_i+\partial\Pi_{ik}/\partial x_k=0,\quad \dot{S}+\operatorname{div}(S\mathbf{v}_n)=0,\\
\dot{\rho}_e+\operatorname{div}(\rho_e\mathbf{v}_n)=0,\quad \dot{\mathbf{v}}_e+\frac{\nabla P_e}{\rho_e}=-\frac{e}{m_e}\mathbf{E}-\frac{e}{c}[\mathbf{v}_e\mathbf{H}],\\
\dot{\mathbf{v}}_1+\nabla\left(\mu_1-\frac{1}{2}\mathbf{v}_n^2+\mathbf{v}_n\mathbf{v}_1\right)=\frac{e}{m}\mathbf{E}+\frac{e}{c}[\mathbf{v}_1\mathbf{H}],\\
\dot{\mathbf{v}}_2+\nabla(\mu_2-{}^1/{}_2\mathbf{v}_n^2+\mathbf{v}_n\mathbf{v}_2)=0,\\
\operatorname{rot}\mathbf{v}_1=\frac{\pi\hbar}{m}\mathbf{i}\delta(\mathbf{r}-\mathbf{r}_k)-\frac{e}{mc}\mathbf{H},\quad \operatorname{rot}\mathbf{v}_2=\frac{\pi\hbar}{m}\mathbf{i}\delta(\mathbf{r}-\mathbf{r}_i).
\end{gathered}\tag{1}$$

Here ρ_1, ρ_2, and ρ_e are the mass densities of the solute, the solvent, and the electrons, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_n$ are the velocities of the two superfluid motions and the normal motion; m_e and $\mathbf{v}_e$ are the electron mass and velocity ($\mathbf{v}_e=\mathbf{v}_n$ under certain quite definite conditions—see below); S and $\mathbf{j}$ are the entropy and momentum per unit volume; and μ_1, μ_2, p_1, and p_2 are the chemical potentials and relative momenta of the superfluid solvent and the solution. The fact that $m_e^*\sim m_e\ll m$ (m_e^* is the effective electron mass) is taken into account in Eqs. (1).

The momentum flux tensor has the form

$$\begin{aligned}
\Pi_{ik}={}&(\rho_1+\rho_2)v_{ni}v_{nk}+(p_{1i}+p_{2i})v_{nk}+(p_{1k}+p_{2k})v_{ni}+p_{1k}(v_{1i}-v_{ni})\\
&+p_{2k}(v_{2i}-v_{ni})+P\delta_{ik}-\frac{1}{4\pi}\left(H_iH_k-\frac{1}{2}\delta_{ik}H_iH_k\right),
\end{aligned}$$

where $P=-\varepsilon+\mu_1\rho_1+\mu_2\rho_2+\mu_e\rho_e+TS$ is the pressure. The electric and magnetic fields are determined by Maxwell's equations, which supplement Eqs. (1):

$$\begin{gathered}
\operatorname{rot}\mathbf{H}=\frac{4\pi}{c}\mathbf{j}_c+\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t},\quad \operatorname{rot}\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t},\\
\operatorname{div}\mathbf{H}=0,\quad \operatorname{div}\mathbf{E}=4\pi\rho_c,
\end{gathered}\tag{2}$$

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where ρ_c and $\mathbf{j}_c$ are the electric charge and current densities.

By employing the method proposed in Ref. 1, making use of the fundamental conservation laws, and taking account of the magnetic field and the laws of thermodynamics, one can easily show that the quantities $\rho_{11}^{(s)}$, $\rho_{12}^{(s)}$, and $\rho_{22}^{(s)}$, which characterize the superconducting condensates, remain in force:

$$\rho_{11}^{(s)}=\frac{m^2}{m^*}N_s,\quad \rho_{12}^{(s)}=\rho_{21}^{(s)}=\frac{m(m^*-m)}{m^*}N_s,$$
$$\rho_{22}^{(s)}=\rho_2-(m^*-m)N_n-\frac{m(m^*-m)}{m^*}N_s. \tag{3}$$

Here N_s and N_n are the densities of the superfluid and normal particles of the impurity component of the system, and m and m^* are the "bare" and effective masses, respectively, of an impurity particle.

First let us assume that $\mathbf{v}_2=\mathbf{v}_n=0$. Then, according to the BCS theory, the mass flux density of the first component will be

$$\mathbf{j}_1=\frac{m}{e}\left(\frac{e\hbar}{2m^*}\nabla\varphi_1-\frac{e^2N_s}{m^*c}\mathbf{A}\right), \tag{4}$$

while the total mass flux density of the system will be

$$\mathbf{j}=\mathbf{j}_1+\mathbf{j}_2=\frac{m}{e}\left(\frac{e\hbar}{2m}\nabla\varphi_1-\frac{e^2N_s}{mc}\mathbf{A}\right).$$

Hence the mass flux density of the principal component of the system will be

$$\mathbf{j}_2=m\left(1-\frac{m}{m^*}\right)\left(\frac{\hbar}{2m}\nabla\varphi_1-\frac{e}{mc}\mathbf{A}\right)N_s.$$

If we introduce the notation

$$\mathbf{v}_1=\frac{\hbar}{2m}\nabla\varphi_1-\frac{e}{mc}\mathbf{A},$$

we can write

$$\mathbf{j}_2=\rho_{12}^{(s)}\mathbf{v}_2,\quad \rho_{12}^{(s)}=\frac{m(m^*-m)}{m^*}N_s. \tag{5}$$

Next, let us assume that $\mathbf{v}_1=\mathbf{v}_n=0$. Then, in the linear approximation in $\mathbf{v}_2$, the energy of a quasiparticle of the impurity component takes the form

$$\varepsilon=\frac{1}{2m^*}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}'\right)^2,$$
$$\mathbf{A}'=\mathbf{A}-\frac{c}{e}(m^*-m)\mathbf{v}_2.$$

Then the flux is determined from the London formula

$$\mathbf{j}_e=-\frac{e^2N_s}{m^*c}\mathbf{A}'$$

by substituting m for e:

$$\mathbf{j}_1=\frac{m(m^*-m)}{m^*}N_s\mathbf{v}_2-\frac{m_e}{m^*c}\mathbf{A}. \tag{6}$$

The operator for the total mass flux has the form

$$\mathbf{j}=\rho_2\mathbf{v}_2+\sum_{n_\mathbf{p}}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)n_\mathbf{p}.$$

On converting this mass flux operator to the form

$$\mathbf{j}=\rho_2\mathbf{v}_2+\sum_{n_\mathbf{p}}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}'\right)n_\mathbf{p}-(m^*-m)N\mathbf{v}_2$$

and making use of the expression

$$\mathbf{j}_e=\frac{e}{m^*}\sum_{n_\mathbf{p}}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}'\right)n_\mathbf{p},$$

for the current operator, we obtain the following expression for the total mass flux:

$$\mathbf{j}=\rho_2\mathbf{v}_2-(m^*-m)N_n\mathbf{v}_2-\frac{e}{c}\mathbf{A}N_s, \tag{7}$$

where $N_n=N-N_s$. Then from Eqs. (7) and (6) we can obtain

$$\mathbf{j}_2=\rho_{22}^{(s)}\mathbf{v}_2-\frac{e}{mc}\rho_{12}^{(s)}\mathbf{A},$$
$$\rho_{22}^{(s)}=\rho_2-(m^*-m)N_n-\frac{m(m^*-m)}{m^*}N_s. \tag{8}$$

Finally, let us assume that $\mathbf{v}_1=\mathbf{v}_2=\mathbf{A}=0$. Then the problem reduces to the case treated in Ref. 1. In this case the mass flux has the form

$$\mathbf{j}_1=(\rho_1-\rho_{11}^{(s)}-\rho_{12}^{(s)})\mathbf{v}_n=\rho_1^{(n)}\mathbf{v}_n,$$
$$\mathbf{j}_2=(\rho_2-\rho_{22}^{(s)}-\rho_{21}^{(s)})\mathbf{v}_n=\rho_2^{(n)}\mathbf{v}_n. \tag{9}$$

According to formulas (4)–(9) we can write

$$\mathbf{j}_1=\rho_{11}^{(s)}\mathbf{v}_1+\rho_{12}^{(s)}\mathbf{v}_2+\rho_1^{(n)}\mathbf{v}_n,$$
$$\mathbf{j}_2=\rho_{22}^{(s)}\mathbf{v}_2+\rho_{21}^{(s)}\mathbf{v}_1+\rho_2^{(n)}\mathbf{v}_n. \tag{10}$$

2. Let us consider the rotation of the solution at a constant angular velocity $\mathbf{\Omega}$. In the approximation in which the system may be regarded as locally neutral, the electric field vanishes in the solution and the magnetic field is determined by the equation

$$\operatorname{rot}\mathbf{H}=\frac{4\pi}{c}\mathbf{j}_c, \tag{11}$$

in which

$$\mathbf{j}_c=\frac{em}{m^*}N_s\mathbf{v}_1+e\left(1-\frac{m}{m^*}\right)N_s\mathbf{v}_2-eN_s\mathbf{v}_n.$$

Taking the curl of both sides of Eq. (11) and using the last two of Eqs. (1), we obtain

$$\mathbf{H}'+\lambda^2\operatorname{rot}\operatorname{rot}\mathbf{H}=\Phi_0\mathbf{i}\sum_i\delta(\mathbf{r}-\mathbf{r}_i)+\Phi_1\mathbf{i}\sum_k\delta(\mathbf{r}-\mathbf{r}_k),$$
$$\lambda^2=\frac{m^*c^2}{4\pi e^2N_s},\quad \Phi_0=\frac{\pi\hbar c}{|e|},\quad \Phi_1=\frac{m^*-m}{m}\Phi_0,$$
$$\mathbf{H}'=\mathbf{H}+\frac{2m^*c}{e}\mathbf{\Omega}. \tag{12}$$

We have actually derived the London equations for a superfluid solution. It is evident that, because of the electric drag current, there appears not only the usual vortex lattice system of the charged component of the solution, but also a magnetic vortex lattice that coincides with the quantum lattice of the neutral component. We note that the conditions for the appearance of a lattice of the neutral component (when the solution is only rotated!) are not so stringent as those for the appearance of a lattice of the charged component of the solution. This means that under certain quite definite conditions there can exist only a lattice of the neutral component, which, however, will have a magnetic structure. The presence of such a structure facilitates the problem of detecting the lattice experimentally. We also note that the magnetic field flux through the neutral vortex will be larger by a factor of $(m^* - m)/m$ than the usual flux, which amounts to $\Phi_0 = 2\times10^{-7} \mathrm{G\cdot cm^2}$.

In the absence of vortex filaments, the rotation of the superfluid charged component gives rise to a static magnetic field of strength $2m^* c\Omega/|e|$.[13]

3. Let us consider what sort of low-amplitude waves can propagate in a superfluid solution. The linearized magnetohydrodynamic equations with the magnetic field neglected have the form

$$\frac{\partial^2\rho}{\partial t^2}-\Delta P=0,\quad \frac{\partial^2\rho_1}{\partial t^2}-\frac{\rho_1^{(n)2}+\rho_{11}^{(s)}\rho^{(n)}}{\rho_1\rho^{(n)}}(\Delta P_1+\Delta P_e)=0,$$

$$\frac{m^2}{e^2\rho_1\rho^{(n)}}\frac{\partial \mathbf{j}_n}{\partial t}=\mathbf{E}+\frac{m}{e\rho_1}\nabla P_1, \tag{13}$$

where $\mathbf{j}_n = (e/m)\rho_1^{(n)}(\mathbf{v}_n - \mathbf{v}_e)$. In the last of Eqs. (13), which expresses Ohm's law for a superfluid plasma, we have omitted terms in $\rho_e/\rho_1 \sim m_e/m$, since this ratio is always very small. The first of Eqs. (13) represents longitudinal waves that propagate with the velocity

$$u_1^2=(\partial P/\partial\rho)_s, \tag{14}$$

as a result of oscillations of the density of the solution. The last two of Eqs. (13) represent oscillations of the solute and the electrons. As in the case of a normal plasma, one obtains two longitudinal waves: a plasma wave in which the ions are stationary but the electrons oscillate, and ionic waves (second sound), in which the ions and electrons move almost together, i.e. what propagate are fluctuations of the concentration of the solution.

To find the propagation velocities of these waves we write down the last two of Eqs. (13) with allowance for violation of the local neutrality of the solution:

$$\frac{\partial^2 n_1'}{\partial t^2}-\alpha\Delta n_e'=0,$$

$$\frac{\partial^2}{\partial t^2}(n_1'-n_e')-\omega_p^2(n_1'-n_e')+\beta\Delta n_e'=0, \tag{15}$$

where

$$\alpha=\frac{1}{m}\frac{\rho_1^{(n)2}+\rho_{11}^{(s)}\rho^{(n)}}{\rho_1\rho^{(n)}}\frac{\partial P_e}{\partial n_e},\quad \beta=\frac{n_e}{\rho_e}\frac{\partial P_e}{\partial n_e},\quad \omega_p^2=\frac{4\pi c^2 n_e}{m}\frac{\rho_1}{\rho_e},$$

and n_1' and n_e' are the solute-ion and electron densities. The compatibility condition for these equations yields a dispersion equation that has two roots. When $\alpha/\beta \ll 1$, the first root,

$$u_2^2=\omega^2/k^2=\beta+\omega_p^2/k^2, \tag{16}$$

represents oscillations of the electrons alone, while the second root,

$$u_3^2=\frac{\alpha}{1+\beta k^2/\omega_p^2}, \tag{17}$$

represents oscillations of the solute concentration along with the electrons. Here $\mathbf{k}$ is the wave vector of the propagating waves. It is evident from (16) that the electron oscillations can propagate only when $\omega > \omega_p$, whereas according to (17) the concentration waves can propagate only when $\omega \leqslant \omega_p(\rho_e/\rho_1)^{1/2}$. Thus, neither of these waves can propagate at frequencies ω such that $\omega_p(\rho_e/\rho_1)^{1/2} \leqslant \omega \leqslant \omega_p$.

We shall show that other disturbances—the so-called fourth sound—can propagate in precisely this frequency interval. The equations resulting from linearizing Eqs. (1) with $\mathbf{v}_n = 0$ have wave solutions, and the dispersion equation for these longitudinal waves takes the form

$$u_4^4-u_4^2\left[\rho_{22}^{(s)}\frac{\partial\mu_2}{\partial\rho_2}+\rho_{11}^{(s)}\frac{\partial\mu_1}{\partial\rho_1}+\frac{c^2}{k^2\lambda^2}\right]$$

$$+\rho_{22}^{(s)}\frac{\partial\mu_2}{\partial\rho_2}\left(1-\frac{\rho_{12}^{(s)2}}{\rho_{11}^{(s)}\rho_{22}^{(s)}}\right)\left(\rho_{11}^{(s)}\frac{\partial\mu_1}{\partial\rho_1}+\frac{c^2}{k^2\lambda^2}\right)=0, \tag{18}$$

where

$$\lambda^{-2}=\frac{4\pi e^2}{m^2c^2}\rho_{11}^{(s)}=\frac{4\pi e^2}{m^*}N_s.$$

When $\rho_{12}^{(s)}$ is very small as compared with both $\rho_{11}^{(s)}$ and $\rho_{22}^{(s)}$ the waves of the superfluid solvent and those of the solute become independent. The propagation velocity is

$$u_4^2(2)=\rho_{22}^{(s)}\,\partial\mu_2/\partial\rho_2 \tag{19}$$

for disturbances of the neutral component, and

$$u_4^2(1)=\rho_{11}^{(s)}\frac{\partial\mu_1}{\partial\rho_1}+\frac{c^2}{k^2\lambda^2} \tag{20}$$

for disturbances of the charged component. As is evident from (20), the waves of the charged component can propagate only when $\omega > \omega_p(\rho_e/\rho_1)^{1/2}$. We also note that the waves of the charged component are always accompanied by the propagation of longitudinal components of the electric field.

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