Transition radiation from rough interfaces

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The transition radiation from a rough interface $z=f(x,y)$ of two media is considered theoretically in the perturbation-theory approximation. The calculation is applicable to the interface of two media with only slightly differing refractive indices. The interface, however, can be arbitrary and some general regularities of the transition radiation from arbitrary rough interfaces can be studied. Expressions are obtained for the spectral energy densities of energy of the transition radiation in the case of two-dimensional statistically rough interfaces. It is shown that along with the longitudinal effect, in which the particle "feels" the roughness, a transverse effect arises if the longitudinal coherent length is of the order of or smaller than the height f_0 of the inhomogeneities. The transverse effect is due to phenomena which take place in a direction perpendicular to the motion. If a characteristic length l (correlation length) in the interface plane is introduced, the alteration of the formulas will be determined by the parameter l/ρ, where ρ stands for the transverse dimensions of the field. The spectral energy density of the transition radiation from a plane interface, and in particular its polarization, vary considerably, and the roughness of the interface can therefore be investigated. For either $l/\rho\to 0$ or $l/\rho\to\infty$, the particle is insensitive to the inhomogeneity of the surface in a plane perpendicular to the direction of motion.

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The well-known paper by Ginzburg and Frank,[1] devoted to the calculation of transition radiation, deals only with an ideally interface of two media. Actually, any interface differs to one degree or another from an ideally geometrical plane. The inhomogeneities on a plane interface can vary greatly, and can take the form of individual inhomogeneities insulated from one another, periodically disposed inhomogeneities, and statistical inhomogeneities. Theoretical investigations of these questions, in view of their complexity, are only in the initial stage, although even in the first experiments on transition radiation it was emphasized that the perfection of the finish of the interface greatly influences the polarization of the transition radiation.[5] Detailed experimental reports devoted to this question have been recently published (see, e.g., Ref. 6).

We disregard at present investigations of transition radiation from surfaces with different geometric shapes[7,8] or from boundaries having a transition layer.[9,10]

The present study is an attempt to take into account the influence of the statistical inhomogeneities of the interface on the transition radiation. To clarify the physical cause of this effect, we point out that in the case of a ideally flat interface the momentum transferred to the medium in the course of the radiation is always perpendicular to the interface. This follows from the homogeneity of the medium in directions parallel to the interface. If the interface is the xy plane, the momentum $\mathbf{q}_{\parallel}$ (in reciprocal centimeters) of a particle moving along the z axis with velocity v is transferred to the surface only along the motion. In the presence of inhomogeneities, the situation changes and the radiation can transfer to the interface both longitudinal and transverse momentum. As for the longitudinal momentum transfer, if it exceeds the "longitudinal momentum" of the surface inhomogeneities, then the influence of the inhomogeneities in the longitudinal direction on the transition radiation can be neglected. This situation is well known and was repeatedly discussed for various processes at higher energies.[11]

We shall be interested in effects produced in directions perpendicular to the motion. If a flat interface has in a direction transverse to the motion a characteristic inhomogeneity length l and a corresponding momentum uncertainty, the medium can receive in the transverse direction a momentum of the order of $1/l$. This in turn leads to a change in the Ginzburg-Frank equations. How substantial this change will be depends on the contribution made to the transition radiation by momentum transfers of the order of $1/l$ in the direction transverse to the motion.

The transverse distances that are effective in the radiation processes, are determined by the following expression (see, e.g., Ref. 11):

$$\rho=\lambdabar\beta\varepsilon^{1/2}(1-\beta^2\varepsilon)^{-1/2},\qquad(1)$$

where $\lambdabar=\lambda/2\pi$ is the wavelength of the radiated photon, ε is the dielectric constant of the medium, and $\beta=v/c$, where c is the speed of light in vacuum. The physical meaning of ρ is the following: if we expand the electric field of the uniformly moving particle in a Fourier integral with respect to time, it turns out that the spectral density of the particle field contains frequencies ω only for the collision parameters (the distance from the point at which the particle field is sought to the trajectory in a direction perpendicular to the particle motion) that are smaller than ρ. At larger collision parameters, the spectrum of the particle field contains practically no photons with frequencies exceeding ω.

Since the momentum effectively transferred in a transverse direction in the medium is of the order of $1/\rho$, we can expect the corrections necessitated by transverse effects to be determined by the parameter

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l/ρ, and these corrections should vanish as $l \to 0$ and $l \to \infty$, meaning in the absence of inhomogeneities.

The expression for the energy of the transition radiation from a rough interface will contain parameters that describe the surface. This enables us to study the properties of the surface by using a beam of charged particles.

INITIAL EXPRESSION FOR THE ENERGY OF TRANSITION RADIATION FROM AN INTERFACE OF ARBITRARY SHAPE

We derive the initial equation for the calculation of radiation produced when a charged particle crosses the interface $z=f(x,y)$ of two media (see Fig. 1). The function $f(x,y)$ describes small deviations of the interface, due to roughnesses, from the plane $z=0$; this plane would be the interface in the case of an ideal surface. The particle velocity $\mathbf{v}$ is directed along the z axis from the first medium with dielectric constant ε_1 to the second medium with dielectric constant ε_2.

To clarify the physical aspect of the question and to obtain general results without describing specifically the properties of the surface, we use perturbation theory. We construct this theory in analogy with light-scattering theory (see, e.g., Ref. 12), replacing in the latter the scattered wave by the moving-particle field which we expand in accordance with the universal procedure in a Fourier integral with respect to time. The radiation problem reduces then to the problem of scattering of an assembly of monochromatic waves that make up the field of the moving particle (for more details, see Ref. 11, § 30). For the scattering effect to be small, it must be assumed that the dielectric constants of the two media differ insignificantly. A more rigorous test of the validity of the calculations will be given below.

Thus, the perturbation-theory calculation developed below for the transition radiation, while not specifying the surface, is applicable to a rather limited group of interfaces between two media whose refractive indices differ little. Such are, for example, the interfaces between solid particles and the corresponding immersion liquids, etc. Although the transition-radiation yield is proportional to the square of the difference between the refractive indices of the two media, and is consequently strongly suppressed in media with close refractive indices, the calculation method employed makes it possible to cope with the physical picture of the phenomenon and to obtain general formulas that are valid for all interfaces. It is obvious that many qualitative conclusions become valid also for interfaces between two media with greatly differing optical properties. Moreover, for an approximate quantitative estimate of the radiation in the case of greatly differing properties of the media one can use an interpolation formula based on replacing in the final expressions the perturbation-theory factor corresponding to radiation from a plane boundary by the exact expression for transition radiation from a plane interface.

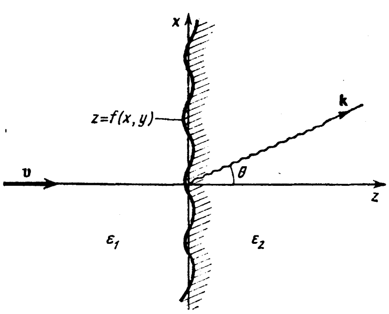

FIG. 1.

The radiation energy at large distances R_0 in the frequency interval $d\omega$ and in the solid-angle interval $d\Omega$ for an interface of arbitrary shape is determined by the usual expressions of classical electrodynamics with account taken of the dielectric constant of the medium:

$$dI(\omega, \mathbf{k}) = c\sqrt{\varepsilon_0}|\mathbf{E}_\omega|^2 R_0^2 d\Omega d\omega, \tag{2}$$

where $\varepsilon_0=(\varepsilon_1+\varepsilon_2)/2$ is the arithmetic mean the dielectric constants of the two media. By $\mathbf{E}_\omega$ we denote the intensity of the radiation field of frequency ω at large distances from the interface; this intensity is determined from Maxwell's macroscopic equations (see, e.g., Ref. 11, Eq. 30.12):

$$\mathbf{E}_\omega = -\frac{e^{ikR_0}}{4\pi R_0}\left[\mathbf{k}\times\left[\mathbf{k}\times \int_{-\infty}^{\infty} \mathbf{E}_\omega'(\mathbf{R}) e^{-i\mathbf{k}\mathbf{R}}\varepsilon'(\mathbf{R})\,d\mathbf{R}\right]\right], \tag{3}$$

where $\mathbf{E}'(\mathbf{R})$ is the Fourier component of the field of the uniformly moving particle at the point $\mathbf{R}(x, y, z)$ in a medium with average dielectric constant ε_0 (see e.g., Ref. 11, Eq. 30.14):

$$\mathbf{E}_\omega'(\mathbf{R}) = \frac{ie}{2\pi^2 v}\int_{-\infty}^{\infty} \frac{\omega\mathbf{v}/c^2 - \mathbf{k}'/\varepsilon_0}{k'^2-\omega^2\varepsilon_0/c^2} \exp(ik_x'x+ik_y'y+ik_z'z)\,dk_x'dk_y', \tag{4}$$

$\varepsilon'(\mathbf{R})$ is the deviation of the dielectric constant from ε_0, i.e., in our case

$$\begin{aligned} \varepsilon'(\mathbf{R}) &= \varepsilon_1-\varepsilon_0, \quad -\infty<z<f(x, y) \\ \varepsilon'(\mathbf{R}) &= \varepsilon_2-\varepsilon_0, \quad f(x, y)<z<\infty \end{aligned}, \tag{5}$$

and e is the charge of the electron. In (3) the wave vector of the emitted photon is designated $\mathbf{k}=\omega\mathbf{n}\sqrt{\varepsilon_0}/c$ ($\mathbf{n}$ is a unit vector in the direction of $\mathbf{k}$), and the wave vector of the incident pseudophoton is designated $\mathbf{k}'(k_x', k_y', k_z'=\omega/v)$.

Integrating (3) with respect to z and using (5), we obtain

$$\mathbf{E}_\omega = \frac{e(\varepsilon_2-\varepsilon_1)}{8\pi^3 v\varepsilon_0}\frac{e^{ikR_0}}{R_0}\int_{-\infty}^{\infty}\frac{[\mathbf{k}\times[\mathbf{k}\times \omega\mathbf{v}/c^2-\mathbf{k}'/\varepsilon_0]]}{q_\parallel(k'^2-\omega^2\varepsilon_0/c^2)}\exp[iq_\parallel f(x,y)]$$

$$\times\exp\{i(k_x'-k_x)x+i(k_y'-k_y)y\}\,dk_x'\,dk_y'\,dx\,dy, \tag{6}$$

$$q_\parallel = \omega/v - k_z. \tag{7}$$

Substituting (6) in (2) we obtain an expression for the spectral energy density of the transition radiation: $I = dI(\omega,\mathbf{k})/d\Omega d\omega$.

In the case of a plane interface $z=f(x,y)=0$ we obtain from (6) the conservation law $k_x'=k_x$ and $k_y'=k_y$. This means that when a photon is emitted in the θ direction (θ is measured from the z axis, with $0 \leq \theta \leq \pi$) momentum is transferred to the interface only in the longitudinal direction, while the transverse momentum carried away by the radiated photon is compensated for by the momentum of the pseudophoton. In this case it is possible to carry out the integration and obtain the transition-radiation equations for one interface

$$I_{pl} = \frac{e^2|\varepsilon_2-\varepsilon_1|^2}{4\pi^2 c\varepsilon_0^{3/2}}\beta^2\sin^2\theta\left|\frac{1-\beta^2\varepsilon_0-\beta\sqrt{\varepsilon_0}\cos\theta}{(1-\beta^2\varepsilon_0\cos^2\theta)(1-\beta\sqrt{\varepsilon_0}\cos\theta)}\right|^2 \quad (8)$$

and for two interfaces.[2] From a comparison of Eq. (8) with the exact transition-radiation equations it follows that in addition to the condition

$$\left|\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}\right| \ll 1 \quad (9)$$

it is necessary to satisfy one more condition

$$\left|\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}\right| \ll \frac{\lambdabar}{l_{coh}}\cos\theta. \quad (10)$$

The coherent length

$$l_{coh} = \frac{\lambdabar\beta\sqrt{\varepsilon_0}}{1-\beta\sqrt{\varepsilon_0}\cos\theta} \quad (11)$$

is defined here as the reciprocal of the momentum (7) longitudinally transferred to the interface when a photon is emitted in the θ direction. In the classical analysis this corresponds to the length of the trajectory of the radiating particle that plays a role in the formation of the transition radiation (see, e.g., Refs. 11 and 13). For nonrelativistic particles $l_{coh} \sim \lambdabar\beta\sqrt{\varepsilon_0}$ and condition (10) at $\cos\theta > \beta\sqrt{\varepsilon_0}$ is practically always weaker than the condition (9); for relativistic particles the condition (10) can be much stronger than the condition (9).

TRANSITION RADIATION FROM STATISTICALLY ROUGH INTERFACES

We shall investigate the transition radiation from rough surfaces for statistically uneven interfaces $z=f(x,y)$, which are described by the distribution functions (12) of the deviations of the surface points from the plane $z=0$.[14-17]

In many cases the distribution of the deviations is approximated by a normal distribution (the Gauss law). For the height-distribution density $f(x,y)$ one uses the two-dimensional normal distribution

$$W(f(\mathbf{r}),f(\mathbf{r}')) = \frac{1}{2\pi(1-F^2)^{1/2}}\exp\left\{-\frac{f^2(\mathbf{r})-2Ff(\mathbf{r})f(\mathbf{r}')+f'^2(\mathbf{r}')}{2f_0^2(1-F^2)}\right\}, \quad (12)$$

$W(f(\mathbf{r}),f(\mathbf{r}'))$ is the probability that at two points defined by radius vectors $\mathbf{r}(x,y)$ and $\mathbf{r}(x',y')$ the heights of the surface turn out to be equal to f and f'. In (12) the interface is defined by two parameters, the mean squared deviation of the heights $f_0^2=\overline{f^2}$ (the bar denotes averaging over the surface) of the roughness from the plane $z=0$, and the correlation coefficient F. The correlation coefficient is defined as the mean value of the product of the heights in two spatially separated points $\mathbf{r}(x,y)$ and $\mathbf{r}'(x',y')$:

$$\overline{f(\mathbf{r})f'(\mathbf{r}')} = f_0^2 F\left(\frac{|x-x'|}{l_x},\frac{|y-y'|}{l_y}\right). \quad (13)$$

Since f_0^2 is the mean squared height, the correlation coefficient F at $\mathbf{r}=\mathbf{r}'$ is equal to unity. If the distance between the points $\mathbf{r}$ and $\mathbf{r}'$ exceeds the characteristic lengths l (called the correlation radii) for which the height correlation vanishes, the function F tends to zero. In most papers on light scattering by statistical inhomogeneities of a surface, the following expression is used for the correlation coefficient:

$$F = \exp\left\{-\frac{|\mathbf{r}-\mathbf{r}'|^2}{l^2}\right\}, \quad (14)$$

for which the correlation radius l is the distance over which the correlation decreases by a factor e. The fact that the correlation function depends on the coordinate difference expresses the statistical homogeneity of the interface. On the other hand, in the case of statistically isotropic surfaces we have $f_x - l_y = l$. We shall consider just such interfaces.

In analogy with the theory of light scattering by rough surfaces (see, e.g., Ref. 14), we substitute (2) in (6) and average the latter over an assembly of rough surfaces with the aid of the distribution function (12). Introducing the notation

$$k_\rho^2 = k_x^2+k_y^2, \quad k_\rho'^2 = k_x'^2+k_y'^2,$$
$$q_\perp^2 = (k_x'-k_x)^2+(k_y'-k_y)^2, \quad \eta = |\mathbf{r}-\mathbf{r}'|/l, \quad (15)$$

we obtain the following expressions for the spectral energy densities of the transition radiation with parallel polarization (the electric vector lies in the radiation plane that contains the wave vector k of the radiated quantum and the normal to the plane $z=0$) and perpendicular polarization (the electric vector is perpendicular to the plane of radiation):

$$I^{\parallel} = \alpha\int_{-\infty}^{\infty}\left(\frac{(k_x'k_x+k_y'k_y)k_z/k_\rho-\omega k_\rho(1-\beta^2\varepsilon_0)/v}{k_\rho'^2+\omega^2(1-\beta^2\varepsilon_0)/v^2}\right)^2 G(k_x',k_y')\,dk_x'\,dk_y',$$
$$I^{\perp} = \alpha\frac{k^2}{k_\rho^2}\int_{-\infty}^{\infty}\left(\frac{k_y'k_x-k_x'k_y}{k_\rho'^2+\omega^2(1-\beta^2\varepsilon_0)/v^2}\right)^2 G(k_x',k_y')\,dk_x'\,dk_y',$$
$$G(k_x',k_y') = l^2\int_0^{\infty} J_0(q_\perp l\eta)\exp\left\{-\frac{f_0^2}{l_{coh}^2}[1-F(\eta)]\right\}\eta\,d\eta,$$
$$\alpha = \frac{e^2|\varepsilon_2-\varepsilon_1|^2}{8\pi^3 c\beta^2\varepsilon_0^{1/2}}\frac{l_{coh}^2}{\lambdabar^2}, \quad (16)$$

where $F(\eta)$ is defined by (14).

We consider now the case of weak roughness:

$$f_0^2 \ll l_{coh}^2. \quad (17)$$

The integral $G(k_x',k_y')$, after expanding the exponential in the integrand and recognizing that[18]

$$\int_0^{\infty} J_0(q_\perp l\eta)\eta\,d\eta = \frac{\delta(q_\perp)}{q_\perp l^2}, \quad (18)$$

can be represented in the form

$$G(k_x',k_y') \approx \left(1-\frac{f_0^2}{l_{coh}^2}\right)\frac{\delta(q_\perp)}{q_\perp}+\frac{f_0^2}{l_{coh}^2}\frac{l^2}{2}\exp\left\{-\left(\frac{q_\perp l}{2}\right)^2\right\}. \quad (19)$$

The transition to an ideally flat interface corresponds to the limit $f_0 \to 0$. In this case expression (19) leads to the conservation law $q_\perp=0$. This means that the momentum transfer is perpendicular to the interface. In the case of strong roughness, when an inequality inverse to (17) is satisfied:

$$f_0^2 \gg l_{coh}^2 \quad (20)$$

a contribution to the interval (16) is made only by those regions of η for which $F(\eta)$ differs little from unity. This corresponds to complete correlation of the height at the two points of a rough surface. Near $\eta \ll 1$, expanding $F(\eta)$ in a series and retaining the first nonvanishing term, we obtain for the integral with respect to η:

$$G(k_x', k_y') \approx \frac{L^2}{2} \exp\left\{-\left(\frac{q_\perp L}{2}\right)^2\right\}, \tag{21}$$

where

$$L = ll_{coh}/f_0 \tag{22}$$

is a certain effective dimension which is smaller than the correlation radius l by a factor l_{coh}/f_0. Expressions (21) and (19), if the term with the factor $(1-f_0^2/l_{coh}^2)$ is discarded in the latter, have the same exponential dependence on $q_\perp$, and this enables us to deduce hereafter all the results for the case of strong roughness (21) from the results for weak roughness (19), by making the formal substitution $l \to L$ and by setting the factor f_0^2/l_{coh} equal to unity.

After substituting (19) in (16), we change over to the new variables:

$$k_x' = k_\rho' \cos\varphi', \quad k_y' = k_\rho' \sin\varphi'. \tag{23}$$

Integrating with respect to φ' with the aid of the equation[19]

$$\int_0^{2\pi} e^{P\cos x} \cos(-mx)\,dx = 2\pi I_m(P), \tag{24}$$

where $I_m(P)$ is a modified Bessel function, we obtain for the spectral energy densities of the transition radiation

$$\begin{aligned} I^{\parallel} = &\left(1-\frac{f_0^2}{l_{coh}^2}\right) I_{pl} + \alpha\pi \frac{f_0^2}{l_{coh}^2} \frac{l^2}{\rho^2} \int_0^\infty q'\,dq' \frac{E(q')}{(q'^2+1)^2} \left\{ [q^2(1-\beta^2\varepsilon_0) \right. \\ &\left. + q'^2 k_z^2 \rho^2] I_0\left(\frac{l^2 q}{2\rho^2} q'\right) - 2q' k_z \rho \left(q(1-\beta^2\varepsilon_0)^{1/2} + \frac{k_z \rho^3}{l^2 q} \right) I_1\left(\frac{l^2 q}{2\rho^2} q'\right) \right\}, \\ I^{\perp} = &\frac{2\pi\alpha}{q} \frac{f_0^2}{l_{coh}^2} \frac{\rho^2}{\lambda^2} \int_0^\infty q'^2\,dq' \frac{E(q')}{(q'^2+1)^2} I_1\left(\frac{l^2 q}{2\rho^2} q'\right). \end{aligned} \tag{25}$$

We have introduced in (25) the dimensionless quantities

$$q' = \rho k_\rho', \quad q = \rho k_\rho, \quad E(q') = \exp[-l^2(q'^2+q^2)/4\rho^2]. \tag{26}$$

Starting from expressions (25), we estimate the influence of the roughnesses of the interface on the transition radiation. To this end, we break up the integration region into two parts. In the first region

$$q' \ll 2\rho^2/l^2 q \tag{27}$$

the Bessel functions can be approximated by the expression[20]

$$I_m(P) \approx \frac{1}{\Gamma(m+1)} \left(\frac{P}{2}\right)^m. \tag{28}$$

After substituting (28) in (25), we note that the integrand is not exponentially small only under the condition

$$l^2(q'^2+q^2)/4\rho^2 \lesssim 1, \tag{29}$$

while the q' determined from (29) should be contained in the q' interval of the inequality (27). In the second integration region

$$q' \gg 2\rho^2/l^2 q \tag{30}$$

the Bessel functions can be replaced by the asymptotic expression[20]

$$I_m(P) = \frac{e^P}{(2\pi P)^{1/2}} [1 + O(P^{-1})] \tag{31}$$

and after substituting (31) in (25) we find that the result is not exponentially small if the inequality

$$l^2(q'-q)^2/4\rho^2 \lesssim 1 \tag{32}$$

is satisfied, where the q' should be contained in the region of the inequality (30).

It follows from the inequalitities (27), (29), and (30), (32) that under the condition

$$\frac{l^2 q^2}{4\rho^2} = \frac{l^2 \sin^2\theta}{4\lambda^2} \ll 1 \tag{33}$$

the essential region of integration is the first region, since the contribution from the second region is exponentially small. The condition (33) is always satisfied if the radiated wavelength exceeds the correlation radius. On the other hand, if $\lambda \ll l$ this condition will be satisfied for all $\theta \ll 2\lambda/l$. We obtain

$$\begin{aligned} I^{\parallel} = &\left(1-\frac{f_0^2}{l_{coh}^2}\right) I_{pl} + \alpha\pi \frac{f_0^2}{l_{coh}^2} \frac{l^2}{\rho^2} \int_0^Q q'\,dq' \frac{E(q')}{(q'^2+1)^2} \\ &\times \left\{ q^2(1-\beta^2\varepsilon_0) + \frac{1}{2} q'^2 \left[k_z^2\rho^2 - \frac{l^2 k_z}{\rho} q^2 (1-\beta^2\varepsilon_0)^{1/2} \right] \right\}, \\ I^{\perp} = &\frac{\alpha\pi}{2} \frac{f_0^2}{l_{coh}^2} \frac{l^2}{\lambda^2} \int_0^Q q'^3\,dq' \frac{E(q')}{(q'^2+1)^2}. \end{aligned} \tag{25'}$$

Here $Q = 2\rho^2/l^2 q$, and $E(q')$ is defined in (26). Integration of Eqs. (25′) leads to expressions that contain integral exponential functions[20] of two arguments: $(l^2/\Delta\rho^2)$ and $(l^2/4\rho^2 + \lambda^2/l^2 \sin^2\theta)$. Since the condition (33) is satisfied, the second argument is always large. This simplifies the integral exponentials, and to obtain lucid expressions we consider the limiting cases of the integral exponential function of the first argument. When the following inequality is satisfied

$$\frac{l^2}{4\rho^2} = \frac{l^2(1-\beta^2\varepsilon_0)}{4\lambda^2\beta^2\varepsilon_0} \ll 1, \tag{34}$$

i.e., when the transverse dimension of the field of the particle are large compared with the correlation radius, we obtain for the spectral densities of the transition-radiation energy

$$\begin{aligned} I^{\parallel} &= \left(1-\frac{f_0^2}{l_{coh}^2}\right) I_{pl} + I^{\perp}\cos^2\theta, \\ I^{\perp} &= I_{pl} \frac{f_0^2}{l_{coh}^2} \frac{l^2}{4\rho^2} \frac{(1-\beta^2\varepsilon_0\cos^2\theta)^2 \ln(2\rho/l)}{\sin^2\theta(1-\beta^2\varepsilon_0)(1-\beta^2\varepsilon_0-\beta\sqrt{\varepsilon_0}\cos\theta)^2}. \end{aligned} \tag{35}$$

In the case of a strong roughness (20) we obtain

$$\begin{aligned} I^{\parallel} &= I^{\perp}\cos^2\theta, \\ I^{\perp} &= I_{pl} \frac{l^2}{f_0^2} \frac{(1+\beta\sqrt{\varepsilon_0}\cos\theta)^2 \ln(2\rho f_0/ll_{coh})}{4\sin^2\theta(1-\beta^2\varepsilon_0-\beta\sqrt{\varepsilon_0}\cos\theta)^2} \end{aligned} \tag{35'}$$

under the conditions

$$\frac{L^2}{4\lambda^2}\sin^2\theta = \frac{l^2 l_{coh}^2}{4\lambda^2 f_0^2}\sin^2\theta \ll 1, \tag{33'}$$

$$\frac{L^2}{4\rho^2} = \frac{l^2}{4\rho^2}\frac{l_{coh}}{f_0^2} \ll 1, \tag{34'}$$

where L is defined by (22).

We analyze now the expressions (35). For nonrelativistic particles ($\beta\sqrt{\varepsilon_0} \ll 1, \rho \sim \lambda\beta\sqrt{\varepsilon_0}, l_{coh} \sim \lambda\beta\sqrt{\varepsilon_0}$), the transition-radiation formulas change in order of magnitude for the emission angles

$$\sin\theta \lesssim \frac{f_0 l \ln^{1/2}(2\lambda\beta\sqrt{\varepsilon_0}/l)}{2\lambda^2\beta^2\varepsilon_0}.$$

When the foregoing inequality is satisfied, as well as (17) and (34), which limit β from below, not only is complete depolarization of a radiation observed, but the intensity of the transition radiation can also greatly exceed the radiation on a flat boundary. When the inverse inequality is satisfied, the radiation intensity is $I = I^{\parallel} + I^{\perp} \approx (1 - f_0^2/l_{coh}^2) I_{pl}$.

In the case of strong roughnesses and when (20) and (34′) are satisfied for nonrelativistic particles at emission angles

$$\sin\theta \lesssim \frac{l}{2f_0}\ln^{1/2}\left(\frac{2f_0}{l}\right)$$

the transition radiation is likewise completely depolarized and exceeds the radiation from a flat boundary. When the opposite inequality is satisfied, the intensity of the radiation is suppressed compared with the intensity on the plane interface. The same condition with a coefficient on the order of unity holds also for relativistic particles in the last case. As for the emission of relativistic particles from interfaces with weak inhomogeneities, the emission angles at which the enhancement effect is observed are decreased in comparison with the presented expression for rough surfaces by a factor $(1 - \beta\sqrt{\varepsilon_0}\cos\theta)^2$.

Thus, an investigation of the angular dependence of the intensity of transition radiation of nonrelativistic particles, for both strong and weak roughnesses, can yield valuable information on the values of f_0 and l that characterize the surface.

At large values of the argument of the integral exponential function. i.e., when the inequality inverse to (34) is satisfied (the transverse dimensions of the field is small compared with the correlation radius),

$$\frac{l^2}{4\rho^2} = \frac{l^2(1-\beta^2\varepsilon_0)}{4\lambda^2\beta^2\varepsilon_0} \gg 1, \tag{36}$$

we have

$$I^{\parallel} = I_{pl} + I^{\perp}\cos^2\theta,$$
$$I^{\perp} = I_{pl}\cdot 2\frac{f_0^2}{l^2}\frac{(1-\beta\sqrt{\varepsilon_0}\cos\theta)^2}{\sin^2\theta(1-\beta^2\varepsilon_0-\beta\sqrt{\varepsilon_0}\cos\theta)^2}. \tag{37}$$

These expressions are valid for weak as well as for strong roughness, but in the case of strong roughness the validity conditions are changed: (33) is replaced by (33′), and (36) takes the form

$$\frac{L^2}{4\rho^2} = \frac{l^2 l_{coh}^2}{4\rho^2 f_0^2} \gg 1. \tag{36′}$$

For the emission angles [subject to satisfaction of conditions (20) and (36), (36′) which limit the values of l and f_0]

$$\sin\theta \lesssim \sqrt{2}\frac{f_0}{l}(1 - \beta\sqrt{\varepsilon_0}\cos\theta)$$

the radiation is completely depolarized and exceeds the transition radiation from a plane interface; at large angles, the radiation tends to that from a flat interface for both strong and weak roughness.

Thus, when the conditions (33) and (33′) are satisfied for weak and strong roughness, respectively, the equations presented give a clear idea of the influence of the roughness on the transition radiation.

We proceed now to an investigation of the opposite case. When the inequality inverse to (33) is satisfied,

$$\frac{l^2 q^2}{4\rho^2} = \frac{l^2\sin^2\theta}{4\lambda^2} \gg 1, \tag{38}$$

which can take place only if $l \gg \lambda$, the contribution from the first integration region is exponentially small, and consequently the essential integration region is the second one. We obtain

$$I^{\parallel} = \left(1 - \frac{f_0^2}{l_{coh}^2}\right) I_{pl} + \alpha\left(\frac{\pi}{q}\right)^{1/2}\frac{f_0^2}{l_{coh}^2}\frac{l}{\rho}\int_Q^\infty \sqrt{q'}\,dq'\frac{E_1(q')}{(q'^2+1)^2}$$
$$\times\left\{[q(1-\beta^2\varepsilon_0)^{1/2} - q'k_z\rho]^2 - 2q'\frac{k_z^2\rho^4}{l^2 q}\right\}, \tag{25″}$$
$$I^{\perp} = \alpha\frac{2\pi^{1/2}}{q^{1/2}}\frac{f_0^2}{l_{coh}^2}\frac{\rho^3}{\lambda^2 l}\int_Q^\infty q'^{5/2}\,dq'\frac{E_1(q')}{(q'^2+1)^2};$$
$$E_1(q') = \exp[-l^2(q'-q)^2/4\rho^2].$$

When the transverse dimensions of the field are large compared with the correlation radius [the condition (34)], the exponential in the integrand can be replaced by unity because of condition (32), and we can put in the denominator $q'^2 + 1 \approx q'^2$. The integrals are easy to calculate, and we obtain for the spectral energy densities of the transition radiation

$$I^{\parallel} = \left(1 - \frac{f_0^2}{l_{coh}^2}\right) I_{pl} + I^{\perp}\cos^2\theta,$$
$$I^{\perp} = I_{pl}\frac{f_0^2}{l_{coh}^2}\frac{l^2}{4\rho^2}\frac{(1-\beta^2\varepsilon_0\cos^2\theta)^2}{\sqrt{2\pi}\sin^2\theta(1-\beta^2\varepsilon_0)(1-\beta^2\varepsilon_0-\beta\sqrt{\varepsilon_0}\cos\theta)^2}. \tag{39}$$

For the case of a strong roughness we have

$$I^{\parallel} = I^{\perp}\cos^2\theta,$$
$$I^{\perp} = I_{pl}\frac{l^2}{2f_0^2}\frac{(1+\beta\sqrt{\varepsilon_0}\cos\theta)^2}{\sqrt{2\pi}\sin^2\theta(1-\beta^2\varepsilon_0-\beta\sqrt{\varepsilon_0}\cos\theta)^2} \tag{39′}$$

under conditions (34′) and

$$\frac{L^2}{4\lambda^2}\sin^2\theta = \frac{l^2 l_{coh}^2}{4\lambda^2 f_0^2}\sin^2\theta \gg 1. \tag{38′}$$

Equations (39) and (39′) differ by insignificant factors from expressions (35) and (35′) which were investigated in detail. This indicates that if the inverse conditions (33), (33′) and (38), (38′) are satisfied, the physical results nevertheless differ little, thereby greatly simplifying the entire analysis. In the inverse limiting case [i.e., under the condition (36)], it follows from the exponential in the integrand in (25″) that in the essential region of the integration of (32) the predominant contribution to the integral is determined by the vicinity of the point q. Calculation by the Laplace method[21] leads to the following result:

$$I^{\parallel}=I_{\rm pl}-I^{\perp}\cos^2\theta,$$
$$I^{\perp}=I_{\rm pl}\cdot 2\frac{f_0^2}{l^2}\frac{(1-\beta\sqrt{\varepsilon_0}\cos\theta)^2}{\sin^2\theta(1-\beta^2\varepsilon_0-\beta\sqrt{\varepsilon_0}\cos\theta)^2}. \quad (40)$$

These equations describe also the case of a strong roughness, but in place of (36) and (38), the conditions satisfied are (36′) and (38′). Expressions (40) differ from the previously investigated Eqs. (37) only in the sign in front of $I^{\perp}\cos^2\theta$ in the first equation.

The equations obtained yield simple solutions of the problem of the influence of roughnesses on transition radiation if condition (38) is satisfied for weak roughness and condition (38′) is satisfied for strong roughness.

For an exact calculation of expressions (25), which is valid also for the intermediate cases [i.e., without the limitations (33), (33′) and (38), (38′)], we substitute the Bessel function in the form of the series[20]

$$I_m(P)=\sum_{\nu=0}^{\infty}\frac{(P/2)^{m+2\nu}}{\nu!\Gamma(m+\nu+1)}. \quad (41)$$

By interchanging the integration and summation, using the equation[19]

$$\int_0^{\infty}\frac{x^s e^{-\gamma x}}{(x+t)^u}dx=\Gamma(s+1)\gamma^{(u-s)/2-1}t^{(s-u)/2}e^{\gamma t/2}W_{k,m}(\gamma,t); \quad (42)$$

$$2k=-s-1,\ \ 2m=-s-u+1,\ \ |\arg t|<\pi,\ \ \operatorname{Re}\gamma>0,\ \ \operatorname{Re}s>-1,$$

where $W_{k,m}(\gamma t)$ is a Whittaker function, we obtain for the case of a weak roughness the following expressions:

$$I^{\parallel}=\left(1-\frac{f_0^2}{l^2_{\rm coh}}\right)I_{\rm pl}+\mu\sum_{\nu=0}^{\infty}\frac{1}{\nu!}\left(\frac{l^2}{4\rho^2}\right)^{\nu/2}\left(\frac{l\sin\theta}{2\lambda}\right)^{2\nu}$$
$$\times\left\{\frac{l}{\rho}(1-\beta^2\varepsilon_0)\sin^2\theta W_{-(\nu+2)/2,(\nu-1)/2}\left(\frac{l^2}{4\rho^2}\right)\right.$$
$$\left.+\cos\theta\left[(2\nu+1)\cos\theta-\left(\frac{l^2}{4\rho^2}\right)\beta^2\varepsilon_0\sin^2\theta\right]W_{-(\nu+3)/2,\nu/2}\left(\frac{l^2}{4\rho^2}\right)\right\},$$
$$I^{\perp}=\mu\sum_{\nu=0}^{\infty}\frac{1}{\nu!}\left(\frac{l^2}{4\rho^2}\right)^{\nu/2}\left(\frac{l\sin\theta}{2\lambda}\right)^{2\nu}W_{-(\nu+3)/2,\nu/2}\left(\frac{l^2}{4\rho^2}\right), \quad (43)$$
$$\mu=\frac{\alpha\pi}{2}\frac{f_0^2}{l^2_{\rm coh}}\frac{l\rho}{\lambda^2}\exp\left\{\frac{l^2}{4\rho^2}\left(\frac{1}{2}-q^2\right)\right\}.$$

In the case of strong roughness, there is no term with a factor $(1-f_0^2/l^2_{\rm coh})$ in the parallel component of the spectral energy density of the transition radiation, the factor $f_0^2/l^2_{\rm coh}$ in μ is replaced by unity, and l is replaced by L.

Let us consider the limiting cases of the exact expressions. At large values of the argument of the Whittaker functions, i.e., when the inequality (36) or (36′) is satisfied for weak and strong roughness, respectively, we have the asymptotic expansion[19]

$$W_{k,m}(P)=e^{-P/2}P^k[1+O(P^{-1})]. \quad (44)$$

After substituting (44) in (43), the requirement that the sums must converge leads to a limitation on the radiation angle θ, similar to the condition (33) or (33′), and we obtain for the spectral energy densities of the transition radiation expressions that coincide with Eqs. (37) for both strong roughness.

In the second limiting case of small values of the argument of the Whittaker function, i.e., when the inequality (34) or (34′) is satisfied, the restriction to the first terms of the series in (43) also leads to a condition that coincides with (33) or (33′), and for the spectral densities we obtain expressions (35) and (35′) for weak and strong interface roughness, respectively.

Summarizing the foregoing, we can state that the physical picture of the radiation from a rough interface is determined by both longitudinal and transverse effects.

From the analysis of the expressions derived for the spectral energy densities of the transition radiation it follows that two cases can be singled out. In the first, the transverse dimensions of the particle field are large compared with the correlation radius [expressions (35), (39) for weak roughness and (35′), (39′) for strong roughness]. In expressions (35) and (39) for the parallel component of the spectral energy density of the transition radiation, the first term leads to Eq. (8) for $I_{\rm pl}$ with an additional factor $(1-f_0^2/l^2_{\rm coh})$ that influences the radiation because of effects connected with the longitudinal dimensions of the inhomogeneities. The second term leads to an additional contribution to radiation, compared with a plane interface, on account of transverse effects. In the expressions for strong roughnesses there should be no limiting transition in the expression for a plane interface. The difference from the case of weak roughness lies in the fact that the spectral density of the transition-radiation energy in the case of perpendicular polarization exceeds the spectral energy density in the case of parallel polarization by a factor $1/\cos^2\theta$.

In the second case the transverse dimensions of the particles are small compared with the correlation radius [expressions (37) and (40)]. In this case the equations for the weak and strong roughnesses do not differ from each other and go over in the limit into the equations for a plane interface. The expressions for $I^{\perp}$ are larger the larger $f_0^2/l^2_{\rm coh}$. The additional contribution $I^{\parallel}$ to the radiation is due mainly to transverse effects. These increments to (37) and (40) are equal, but in (37) they increase the spectral energy density of the transition radiation from the plane interface, and in (40) they decrease it. As $l\to\infty$ they vanish and lead to the formula for the plane interface, namely $I^{\parallel}=I_{\rm pl}, I^{\perp}=0$.

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