Collisions of electrons with hydrogen atoms in a strong magnetic field

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The cross sections for elastic scattering, excitation and ionization of electrons colliding with hydrogen atoms in a strong magnetic field ($\hbar\omega_c \gg E_B$, where ω_c is the cyclotron frequency and E_B the Bohr energy) are obtained. The results are asymptotically accurate with respect to the parameter $\ln(\hbar\omega_c/E_B) \gg 1$ in the essential energy ranges of the incident electrons. The rate of the energy loss is determined. The bound states of two electrons in the Coulomb field of a nucleus are found.

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The behavior of atomic systems in a strong magnetic field is of interest for solid-state physics and astrophysics. By strong we mean here a magnetic field such that the magnetic length λ is much less than the Bohr radius a, i.e., $\hbar\omega_c \gg E_B$, where ω_c is the cyclotron frequency and E_B is the Bohr energy. The spectrum and the wave functions of the hydrogen atom in a strong magnetic field were obtained with asymptotic accuracy by Hasegawa and Howard.[1] We consider here only states that pertain to the lower Landau band. Under these conditions, the transverse motion of the electron is determined entirely by the magnetic field and is characterized by the quantum number $-M_1$, which is the projection of the angular momentum on the direction of the magnetic field. Motion along the field takes place in an effective one-dimensional potential whose form depends on M_1. Thus, for each M_1 there is a series of levels corresponding to longitudinal motion. The lowest level of the atom corresponds to $M_1 = 0$ and is located below the bottom of the Landau band at an energy distance E_B/ν_0^2, where $\nu_0 = (1/2)\ln(a/\lambda)$. In the present paper we assume that $\nu_0 \ll 1$, i.e., the ionization energy of the atom greatly exceeds the Bohr energy. The electron localization region then takes the form of a needle with a characteristic transverse dimension λ and a longitudinal dimensional $\nu_0 a$. The excited levels with $M_1 = 0$ have the usual Bohr spectrum $E_\nu = -E_B/\nu^2$, $\nu \approx 1, 2, 3 \ldots$, and to each level there correspond two states, even and odd. The wave functions of these states have a length $\sim a$ in the field direction. The first excited level of the atom corresponds to the ground state in the well with $M_1 = 1$ and lies at an energy $2E_B/\nu_0$ higher than the ground state. It corresponds to a longitudinal wave function that is practically the same as the function of the ground state of the atom. The entire energy interval from the ground level with $M_1 = 0$ to levels with binding energy $\approx E_B$ is filled with levels corresponding to ground states in one-dimensional wells with different M_1.

The motion of the incident electron is essentially one-dimensional, so that the entire collision problem is thus one-dimensional. In the present paper we obtain the cross sections for elastic scattering, excitation and ionization of the atom, and also the rate of energy losses. We obtain asymptotically exact (in terms of the parameter $\nu_0 \ll 1$) expressions for the cross sections in the significant energy intervals of the incident electron, while for the elastic-scattering cross section in the case of slow collisions we obtain an expression that is valid also at $\nu_0 \sim 1$. It will be shown that there exist bound states of the electron on the atom, i.e., states of the negative hydrogen ion. The affinity energies are obtained as the poles of the scattering amplitude, assuming a strong differences between projections of the angular momenta of the electrons on the direction of the magnetic field. Under this assumption, the exchange effects are insignificant and by the same token the mutual orientation of the spins, while the correlation energy can be calculated exactly.

We note that for shallow donors in indium antimonide, the condition $\hbar\omega_c = E_B$ is reached already at relatively low value of the field $H \sim 700$ Oe. However, the satisfaction of the inequality $\nu_0 \ll 1$ calls for very strong magnetic fields (for example, in a field $H \sim 150$ kOe the donor ionization energy is $\sim 8E_B$). Under astrophysical conditions this inequality can be quite strong.

1. WAVE FUNCTIONS OF AN INCIDENT ELECTRON IN THE FIELD OF AN ATOM

The Schrödinger equation for two electrons located in a Coulomb field of a nucleus and in a magnetic field is

$$(H_1 + H_2 + V)\Psi = E\Psi,$$

where H_1 is the Hamiltonian of the atomic electron, H_2 is the Hamiltonian of the incident electron in a magnetic field and does not take into account the interaction with the atom, and V is the interaction of the incident electron with the atom (i.e., with the atomic electron and with the nucleus). As will be shown below, the principal role in the processes of interest to us are played by relatively large impact parameters $\rho \gg \lambda$. It is possible therefore to disregard exchange effects and assume that the coordinates $\mathbf{r}_1$ describe the atomic electron, while the coordinates $\mathbf{r}$ describe the incident electron. The wave function will be sought in the form

$$\Psi(\mathbf{r}_1, \mathbf{r}) = \sum_a \psi_a(\mathbf{r}_1)\mathcal{F}_a(\mathbf{r}). \tag{1}$$

Here a is the aggregate of the quantum numbers that characterize the atomic-electron state described by the function $\psi_a(\mathbf{r}_1)$. The function $\mathcal{F}_a(\mathbf{r})$, which pertains to the incident electron, is to be determined. Substituting the sum (1) in the Schrödinger equation, we

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obtain an infinite system of equations for $\mathcal{F}_a(\mathbf{r})$:

$$[H_z+V_{aa}(\mathbf{r})-(E-E_a)]\mathcal{F}_a(\mathbf{r})=-\sum_{a'}{}' V_{aa'}(\mathbf{r})\mathcal{F}_{a'}(\mathbf{r}), \tag{2}$$

where E_a is the energy of the atomic electron in the state a, and $V_{aa'}$ are the matrix elements of the interaction on the atomic-electron functions. The prime on the summation sign means that $a'\neq a$. The set of quantum numbers a includes the angular-momentum projection $-M_1$ on the direction of the magnetic field (which we take to be the z axis) and the principal quantum number ν, which depends on M_1 and is connected by definition with the level binding energy by the relation $E_\nu=-E_B/\nu^2$. All the considered states pertain to the lower Landau band, the bottom of this band is taken to be the null of the energy.

The matrix elements $V_{aa'}$ that are not diagonal in M_1 are small compared with the diagonal ones if $\rho\gg\lambda$. They can therefore be neglected if scattering without a change of M_1 is considered. At a fixed M_1, by virtue of the axial symmetry of the problem, the projection of the angular momentum of the external electron on the z axis ($-M$) is also fixed. We can therefore write

$$\mathcal{F}_a(\mathbf{r})=F_\nu(z)\Phi_M(\rho),$$

where $\Phi_M(\rho)$ is the wave function for the free electron in a magnetic field. Averaging Eq. (2) over the transverse motion of the incident electron, we obtain a system of one-dimensional equations:

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}-(E-E_\nu)+V_{\nu\nu}(z)\right]F_\nu(z)=-\sum_{\nu'}{}' V_{\nu\nu'}(z)F_{\nu'}(z). \tag{3}$$

The system (3) pertains to definite values of M and M_1. Here

$$V_{\nu\nu'}(z)=e^2\int\psi_\nu^*(\mathbf{r}_1)\Phi_M^*(\rho)\left(-\frac{1}{r}+\frac{1}{|\mathbf{r}-\mathbf{r}_1|}\right)\psi_{\nu'}(\mathbf{r}_1)\Phi_{M'}(\rho)\,d^2\rho d^3\mathbf{r}_1. \tag{4}$$

If the atom is in a state with $M_1=0$ or M_1 of the order of unity, and the incident electron is characterized by a large value of M, then the matrix element (4) can be written in the form

$$V_{\nu\nu'}(z)=e^2\int_{-\infty}^{\infty}\psi_\nu^*(z_1)\psi_{\nu'}(z_1)\left\{-\frac{1}{(\rho^2+z^2)^{1/2}}+\frac{1}{[\rho^2+(z-z_1)^2]^{1/2}}\right\}dz_1, \tag{5}$$

where $\psi_\nu(z_1)$ describes the longitudinal motion of the atomic electron. (The functions $\psi_\nu(z)$ are written out in Appendix I.) In the derivation of the formula we took into account the fact that $M\gg1$, i.e., the wave function $\Phi_M(\rho)$ of the transverse motion is quasiclassical and has a sharp maximum (of width $\sim\lambda$) at $\rho=(2M)^{1/2}\lambda$. The function $\Phi_M(\rho_1)$, on the contrary, is concentrated in the region $\rho_1\sim\lambda$. Under these conditions the matrix element of the potential depends on M only through ρ and on M_1 only via the value of ν.

Without the right-hand side, Eq. (3) for $\nu=\nu_0$ describes the scattering of an electron by a potential $V_{\nu_0\nu_0}(z)$. This potential is shallow (see below) in the sense that $\eta=(mV/\hbar^2)(\Delta z)^2\ll1$; here V is the characteristic value of the potential and Δz is the characteristic radius of its action. In this case, as is well known, the scattering amplitude is determined in first-order approximation by the value of the integral of the potential over all of z. In our case, however, it follows from (5) that

$$\int_{-\infty}^{\infty}V_{\nu\nu}(z)\,dz=0. \tag{6}$$

It is therefore necessary to take into account the next approximation in the parameter η. But terms of the same order arise when account is taken of the right-hand side of Eq. (3) (this corresponds to taking into account the polarization of the atom by the incident electron). For the functions $F_{\nu'}(z)$ that enter in it we can confine ourselves to the first nonvanishing approximation in the potential[1)] (it is assumed that $|E=E_{\nu_0}|\ll|E_{\nu_0}|$, i.e., $E\approx E_{\nu_0}$), i.e., we can attempt to determine them from the equation

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}-(E_{\nu_0}-E_{\nu'})\right]F_{\nu'}(z)=-V_{\nu'\nu_0}(z)F_{\nu_0}(z). \tag{7}$$

Equation (7) is easy to integrate,[2)] and when its solution is substituted in the right-hand side of (3) we obtain

$$-\left(\frac{d^2}{dz^2}+k^2\right)F(z)+\hat{U}(z)F(z)=0, \tag{8}$$

where the ν labels have been left out for simplicity, k is the wave vector of the incident electron ($\hbar^2k^3/2m=E-E_{\nu_0}$), $\hat{U}$ is a nonlocal potential[3)] in the form

$$\begin{gathered}\hat{U}=U_{\nu_0\nu_0}+\Delta\hat{U},\quad U_{\nu\nu'}=\frac{2m}{\hbar^2}V_{\nu\nu'},\\ \Delta\hat{U}(z)F(z)=-\sum_{\nu}{}' U_{\nu\nu'}(z)\int_{-\infty}^{\infty}dz'G_0^{(\nu,\nu_0)}(z,z')U_{\nu_0\nu_0}(z')F(z'),\end{gathered} \tag{9}$$

where $G_0^{(\nu,\nu_0)}$ is the free Green's function:

$$G_0^{(\nu,\nu_0)}(z,z')=\frac{1}{2\varkappa_{\nu\nu_0}}\exp(-\varkappa_{\nu\nu_0}|z-z'|),\quad \varkappa_{\nu\nu_0}^2=\frac{2m}{\hbar^2}(E_\nu-E_{\nu_0}).$$

We shall seek the solution of Eq. (8) in the form

$$F(z)=A_k(z)e^{ikz}, \tag{10}$$

where $A_k(z)$ is a function that varies slowly in the range of action of the potential (to the extent that the parameter η is small). By virtue of the condition (6) it is nevertheless necessary to take into account the deviation of $A_k(z)$ in this region from a constant. We therefore put

$$A_k(z)=A_k(0)[1-\varphi_k(z)],\quad |\varphi_k(z)|\ll1,\quad (|z|\lesssim\Delta z). \tag{11}$$

We write down next Eq. (8) in integral form (it is assumed that the electron is incident from the left):

$$F(z)=e^{ikz}-\frac{i}{2k}\int_{-\infty}^{\infty}dz'\exp(ik|z-z'|)\hat{U}(z')F(z') \tag{12}$$

and substitute $F(z')$ as given by Eqs. (10) and (11) under the integral sign. We then obtain for $A_k(0)$ the expression

$$A_k(0)=\left[1+\frac{i}{2k}\int_{-\infty}^{\infty}dz'e^{ik|z'|}\hat{U}(z')e^{ikz'}(1-\varphi_k(z'))\right]^{-1} \tag{13}$$

and for $\varphi_k(z)$ the equation (the increment $\Delta\hat{U}$ to the potential is neglected here)

$$\begin{aligned}\varphi_k(z)=&\frac{i}{2k}\int_{-\infty}^{\infty}dz'U_{\nu_0\nu_0}(z')(1-\varphi_k(z'))\\ &\times\{\exp[ik(z'-z+|z'-z|)]-\exp[ik(z'+|z'|)]\}.\end{aligned} \tag{14}$$

The second term in the square brackets in (13) is significant only at small k, when it becomes of the order of unity. Its exponential can therefore be replaced by unity and $A_k(0)$ can be represented in the form

$$A_k(0)=k/(k-ik_b),\quad k_b=-\tfrac{1}{2}\int_{-\infty}^{\infty}\hat{U}(z')(1-\varphi_k(z'))dz'. \tag{15}$$

The quantity k_b, as a pole of the scattering amplitude, determines the binding energy in the one-dimensional potential $\hat{U}(z)$. The fact that the potential is shallow means that $k_b\Delta z\ll 1$. The expression for $\varphi_k(z)$ is obtained from (14) if the small quantity $\varphi_k(z')$ in the integrand is neglected. To calculate k_b we need $\varphi_k(z')$ at $k\sim k_b\ll 1/\Delta z$. We then obtain from (15)

$$\varphi_k(z)=-\int_0^z dz'\int_0^{z'}dz''U_{\nu_0\nu_0}(z''),\quad k\to 0. \tag{16}$$

For k_b we have thus[4] $k_b=k_{b0}+\Delta k_b$,

$$k_{b0}=\tfrac{1}{2}\int_{-\infty}^{\infty}dz\Big[\int_z^{\infty}U_{\nu_0\nu_0}(z')dz'\Big]^2,$$
$$\Delta k_b=\tfrac{1}{2}\sum_{\nu}{}'\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}dz'U_{\nu_0\nu}(z)G_0^{(\nu,\nu_0)}(z,z')U_{\nu\nu_0}(z'). \tag{17}$$

We proceed to calculate k_b from Eqs. (17). To this end we consider first the diagonal matrix element $U_{\nu_0\nu_0}(z)$, which is determined by expression (5) (accurate to the factor $2m/\hbar^2$). In these calculations we use the functions (AI.1) from Appendix I. At large ρ (much larger than the localization region $\nu_0 a$ of the atomic electron in the ground state) this is the quadrupole potential

$$U_{\nu_0\nu_0}(z)=-\frac{(\nu_0 a)^2}{2a}\frac{\rho^2-2z^2}{(\rho^2+z^2)^{5/2}},\quad \rho\gg\nu_0 a. \tag{18}$$

The width of the region of its action is $\Delta z\sim\rho$, and the characteristic value is $V\sim e^2(\nu_0 a)^2/\rho^3$, so that $\eta\sim\nu_0^2 a/\rho\ll 1$. At small ρ ($\rho\ll\nu_0 a$) the potential is represented as a sum of potentials of a charged filament and a point charge of opposite sign:

$$U_{\nu_0\nu_0}(z)=\frac{2}{a}\Big[-\frac{1}{(\rho^2+z^2)^{1/2}}+\frac{2}{\nu_0 a}\exp\Big(-\frac{2|z|}{\nu_0 a}\Big)\ln\frac{\nu_0 a}{\rho}+\frac{1}{\nu_0 a}t\Big(\frac{2z}{\nu_0 a}\Big)\Big], \tag{19}$$

where

$$t(x)=e^{|x|}E_1(|x|)+e^{-|x|}E^*(|x|)+e^{-|x|}(2\ln 2-C),$$
$$t(0)=2\ln 2-C,\quad t(|x|\to\infty)\to 2/|x|.$$

Here E_1 and E^* are integral exponential functions, and C is the Euler constant. The last two terms in (19) describe a potential produced at the point z by a filament whose linear charge density is $-e[\psi_{\nu_0}(z)]^2$. The filament potential has a characteristic effective range $\Delta z\sim\nu_0 a$ and a characteristic value $V\sim(e^2/\nu_0 a)\ln(\nu_0 a/\rho)$, so that $\eta\sim\nu_0\ln(\nu_0 a/\rho)$. This value of small if

$$\ln\frac{\nu_0 a}{\rho}\ll\frac{1}{\nu_0}=2\ln\frac{\nu_0 a}{\lambda}. \tag{20}$$

The point-charge potential prevails over the filament potential in the region $|z|<\nu_0 a/\ln(\nu_0 a/\rho)$, which in turn is much smaller than the radius r_0 of the bound state in a one-dimensional Coulomb potential cut off at a depth $e^2/\rho(r_0\sim a/\ln(a/\rho))$. In this region the amplitude of the wave function $A_k(z)$ has therefore no time to change.

We turn now to the off-diagonal elements of the potential $U_{\nu_0\nu}$ that determines the value of Δk_b. The product $\psi_{\nu_0}(z)\psi_\nu(z)$ is small for all the excited states $\nu\gg 1$ and for the continuous spectrum states at $E_\nu\ll|E_{\nu_0}|$, i.e., at $|\nu|\gg\nu_0$, since the wave functions of these states are small in the localization region of the ground-state function ($|z|\sim\nu_0 a$). Therefore the main contribution to the result of the summation of the intermediate states in the second formula of (17) is made by the states of the continuous spectrum with $E_\nu\sim|E_{\nu_0}|$, for which the wave functions in the region $|z|\sim\nu_0 a$ are determined by expression (AI.5) of Appendix I. Then the matrix elements of the potential $U_{\nu_0\nu}(z)$ can be easily calculated in the limiting cases of small and large[5] ρ:

$$U_{\nu_0\nu}(z)=\frac{8\,\nu_0(\nu_0 a)^{1/2}(q_\nu\nu_0 a)}{\sqrt{\pi}\,[1+(q_\nu\nu_0 a)^2]^2}\frac{z}{(\rho^2+z^2)^{3/2}}\quad \rho\gg\nu_0 a;$$
$$U_{\nu_0\nu}(z)=\frac{4}{a}\ln\Big(\frac{\nu_0 a}{\rho}\Big)\psi_{\nu_0}(z)\psi_\nu(z),\quad \rho\ll\nu_0 a\quad \Big(\ln\frac{\nu_0 a}{\rho}\gg 1\Big). \tag{21}$$

Using now Eqs. (17)–(19) and (21) we can calculate k_b:

$$k_b=\frac{3\pi}{32}\frac{\nu_0}{a}\Big(\frac{\nu_0 a}{\rho}\Big)^3,\quad \frac{\Delta k_b}{k_b}=5/6,\quad \rho\gg\nu_0 a; \tag{22}$$

$$k_b=\frac{3}{2}\frac{\nu_0}{a}\Big(\ln\frac{\nu_0 a}{\rho}\Big)^2,\quad \frac{\Delta k_b}{k_b}=1/3,\quad \rho\ll\nu_0 a\quad \Big(\ln\frac{\nu_0 a}{\rho}\gg 1\Big). \tag{23}$$

We note that the binding energy of an electron in the hydrogen atom was calculated with a computer.[2] A multiparameter variational problem was solved for the three-dimensional Schrödinger equation and it was found that the bound state of the electron in the atom (this state exists at $H=0$ and has a binding energy $\approx 0.055E_B$) vanishes in the strong-field limit. Kadomtsev and Kudryavtsev[3] considered heavy atoms and used the self-consistent-field method. It can be concluded from their results that such a bound state is preserved in a strong field. Actually, they obtained the electron-affinity energy for the hydrogen atom at $M_1=0$ and $M=1$. They noted at the same time that they did not take into account the exchange correction. It can be shown, however, that under their assumption $\nu_0\ll 1$ this approximation is parametrically exact (at $\nu_0\ll 1$ the exchange interaction must be taken into account only when the values of M and M_1 coincide and, moreover, M and M_1 are good quantum numbers). Equation (23) shows that at large values of M, at which this equation is valid, there exists a bound state. If it is applied to the values $M\sim 1$, then it yields an affinity energy $(9/64)E_B/\nu_0^2$ that is close to the value $(1/8)E_B/\nu_0^2$ obtained in Ref. 3.

2. ELASTIC SCATTERING

From Eq. (12) follow asymptotic expressions for the function $F(z)$:

$$F(z)=e^{ikz}+f^{\pm}e^{\pm ikz},\quad z\to\pm\infty,$$

where the elastic scattering amplitudes $f^{\pm}$ are determined by the formula

$$f^{\pm}=-\frac{i}{2(k-ik_b)}\int_{-\infty}^{\infty}dze^{\mp ikz}\hat{U}(z)e^{ikz}(1-\varphi_k(z)). \tag{24}$$

The cross sections for the forward (σ^+) and backward

(σ^-) elastic scattering

$$\sigma^{\pm}=2\pi\int_0^\infty |f^{\pm}|^2\rho\,d\rho$$

take different forms, depending on the value of k. We consider first forward scattering. We can separate here three regions of variation of k:

1) $k\ll v_0/a$; 2) $v_0/a\ll k\ll 1/v_0a$; 3) $k\gg 1/v_0a$.

In regions 1) and 2) we can put $k=0$ in the expression for φ_k and in the term that contains the polarization increment $\Delta\hat{U}$ in Eq. (24). In the term containing $U_{v_0v_0}$, in the case of forward scattering, the exponentials cancel out and k does not enter likewise in this term. Therefore in regions 1) and 2) the expression for the amplitude f^+ takes the simple form

$$f^+=ik_b/(k-ik_b).$$

Figures 1a and 1b show the schematic form of $|f^+|^2$ as a function of ρ for the regions 1) and 2) of the variation of k. It is seen from this figure that in region 1) the principal contribution cross section is made by $\rho \gg v_0a$, and using Eq. (22) for k_b, we obtain

$$\sigma^+=4\pi(v_0a)^2\left(\frac{v_0}{ka}\right)^{1/3}\alpha,\quad \alpha=\frac{\pi}{96}\frac{(6\pi)^{1/3}}{\sqrt{3}}\approx 0.134,\quad k\ll\frac{v_0}{a}. \tag{25}$$

In region 2) the cross section is determined by the values $\rho\sim v_0a$ at which the exact values of $k_b(\rho)$ cannot be calculated in explicit form. We can therefore write σ^+ in this region of k only accurate to a coefficient of the order of unity:

$$\sigma^+=\frac{2\pi}{k^2}\int_0^\infty k_b^2(\rho)\rho d\rho\sim\frac{v_0^4}{k^2},\quad \frac{v_0}{a}\ll k\ll\frac{1}{v_0a}. \tag{26}$$

In region 3), the contribution to the amplitude due to allowance for the polarization is negligibly small and the calculation leads to the result (which we write down again accurate to within a coefficient)

$$\sigma^+=\frac{\pi}{32k^6}\int_0^\infty\left(\int_{-\infty}^{\infty}[U_{v_0v_0}(z)]^2dz\right)^2\rho d\rho\sim\frac{1}{a^4k^6},\quad k\gg\frac{1}{v_0a}. \tag{27}$$

Only at very large k ($k\gtrsim 1/(\lambda a)^{1/2}$), when the cross section is determined by the values $\rho\sim\lambda$, does the dependence of σ^+ on k take the usual asymptotic form $\sigma^+\propto k^{-2}$ ($\sigma^+\sim\lambda^2/(ka)^2$). The reason is the allowance for the approximate character of Eq. (6) [cf. Eqs. (4) and (5)]. The integral in it is not strictly equal to zero and is only exponentially small at $M\gg 1$. At very large values of k, the main contribution to the amplitude is made by the term $\sim(\hbar^2/mk)I$, where I is the integral (6). This contribution is of the order of $(ka)^{-1}$ at $\rho\sim\lambda$.

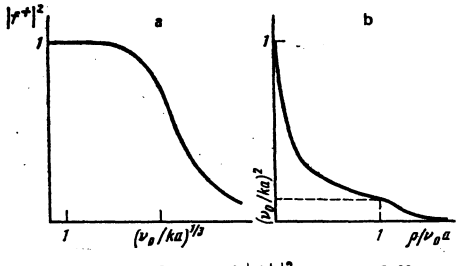

FIG. 1. Dependence of $|f^+|^2$ on ρ in different regions of variation of k: a) $k\ll v_0/a$, b) $v_0/a\ll k\ll 1/v_0a$.

We turn now to backward scattering. The amplitude f^- can be written in the form $f^-=f^++\Delta f$. The main contribution to Δf is determined by the integral (this is valid at $k\ll 1/\lambda$, see below):

$$\Delta f=-\frac{i}{2(k-ik_b)}\int_{-\infty}^{\infty}dz'U_{v_0v_0}(z')e^{2ikz'}. \tag{28}$$

With the aid of expression (5) and (AI.1), the integral is easily calculated, and we obtain

$$\Delta f=\frac{2i}{(k-ik_b)a}\frac{(kv_0a)^2}{1+(kv_0a)^2}K_0(2k\rho), \tag{29}$$

where K_0 is a Macdonald function; Δf can exceed f^+ at large ρ, but nevertheless in the region 1), as can be shown, the contribution of Δf to the cross section is negligibly small. Therefore $\sigma^-=\sigma^+$ in region 1). In regions 2) and 3) the quantity k_b in the denominator can be neglected in the essential region of values of ρ. The contribution of Δf to the cross section takes in this case the form (the cross-section term containing the product $f^+\Delta f$ is inessential)

$$\Delta\sigma=2\pi\int_0^\infty|\Delta f|^2\rho d\rho=4\pi(v_0a)^2\frac{v_0^2}{2[1+(kv_0a)^2]^2},\quad k\gg\frac{v_0}{a}. \tag{30}$$

As seen from a comparison of (30) and (26), $\Delta\sigma$ remains smaller than σ^+ at $k\ll 1/a$, so that the scattering is isotropic. At $k\gg 1/a$ the scattering is anisotropic, and backward scattering predominates ($\sigma^-=\Delta\sigma\gg\sigma^+$). Only at $k\gg 1/\lambda$ does σ^+ become larger than σ^-. In this case $\sigma^-=\Delta\sigma=\pi\lambda^2/2(ka)^2(k\lambda)^4$ and is determined by the scattering of electrons[6] with $M=0$. The general form of the cross section for elastic scattering, as a function of k, is shown in Fig. 2.

We note that the cross section diverges as $k\to 0$. The reason is that in the case of uniform motion an electron with zero energy is totally reflected from an arbitrarily weak potential of the atom. Electrons with arbitrarily large impact parameters thus undergo total reflection. It can be shown that the divergence takes

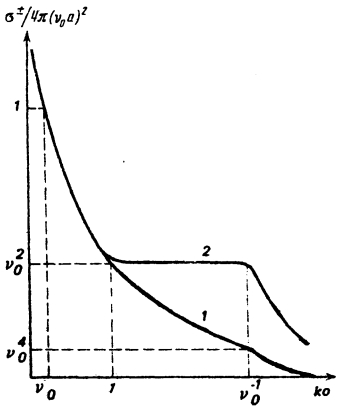

FIG. 2. General form of the cross section for elastic scattering as a function of the energy E_0 of the incident electron. The abscissas are the values of $ka=(E_0/E_B)^{1/2}$ (E_B is the Bohr energy). σ^+ (1) and σ^- (2) are respectively the forward and backward scattering cross sections; σ^+ is given by Eqs. (25), (26), (27); $\sigma^-\approx\sigma^+$ at $ka\ll 1$, and at $ka\gg 1$ we have $\sigma^-=\Delta\sigma$ [Eq. (30)]; $ka=1/v_0$ corresponds to an energy E_0 equal to the atoionization energy.

place in an arbitrarily weak magnetic field. This effect is similar to the appearance of a bound state in an arbitrarily weak magnetic field in any attracting three-dimensional potential.[4] Divergence exists at an arbitrary law of variation of the potential with distance, with the exception of a zero-radius potential and of long-range attracting potentials that decrease more slowly than $1/r^2$.

The plateau in the plot of $\sigma^-(k)$ is due to the mutual cancellation of two factors: the decrease of the characteristic values of the impact parameters with increasing k ($\rho \sim 1/k$), and the increase of Δf at these values of ρ. The last factor is connected with the specific property (6) of the potential, owing to which the interaction of the electron with the atom becomes stronger when the wave function begins to vary within the range of the action of the potential.

For the values $\rho \gg \nu_0 a$, which determine the cross section at $k \ll \nu_0/a$, the polarization of the atom shell is small, even if $\nu_0 \sim 1$. We can therefore obtain an exact expression for σ in slow collisions, without assuming ν_0 to be small. In analogy with the procedure used above, it is easy to obtain

$$\sigma^- = \frac{\pi^2 (4\pi)^{2/3}}{24\sqrt{3}} \left(\frac{Q^2/a + p}{ka} \right)^{2/3}, \quad k \ll v_0/a.$$

Here Q is the quadrupole moment of the atom, and p is the polarizability of the atom in an electric field directed along the magnetic field. At $\nu_0 \ll 1$ these quantities are given by $Q \to (\nu_0 a)^2/2$ and $p \to 5\nu_0(\nu_0 a)^3/4$.

3. INELASTIC SCATTERING ($\Delta M = 0$)

We consider first the case $E_0 = E_I \gg E_B$ (E_0 is the energy of the incident electron, $E_I = -E_{\nu_0}$ is the ionization threshold). In this region the Born approximation can be used to find the cross section for inelastic scattering, i.e., we can leave out of Eq. (3) all the diagonal potentials and retain in the right-hand side only the term with $\nu = \nu_0$. Indeed, the values $\rho \gtrsim \nu_0 a$ are the ones that determine the cross section (see below). In this case $E_I \gg h^2 k_b^2/2m$, and since $E_0 \gg E_I$ we can neglect the elastic scattering in the initial state and put $F_{\nu_0} = \exp(ikz)$. Scattering in the final state can also be neglected, inasmuch as for the most probable transitions the energy of the incident electron at $E_0 - E_I \gg E_B$ is much larger than the Bohr energy after the collision, while the energy of the electron bound to the excited atom is of the order of the Bohr energy at $\rho \sim \nu_0 a$. Thus, the amplitude $f_\nu^\pm$ of the transition of the atomic electron from the ground state into the state ν [with scattering of the incident electron forward (+) and backward (−)] is given by

$$\begin{aligned} f_\nu^\pm &= -\frac{i}{2k_\nu} \int_{-\infty}^{\infty} dz U_{\nu\nu_0}(z) \exp[i(k \mp k_\nu) z] \\ &= +\frac{2i}{k_\nu a} K_0[(k \mp k_\nu)\rho] \varphi_{\nu\nu_0}(k \mp k_\nu), \\ \varphi_{\nu\nu_0}(k') &= \int_{-\infty}^{\infty} dz \psi_{\nu_0}(z) e^{ik'r} \psi_\nu(z). \end{aligned} \tag{31}$$

Here $k_\nu = (2m/\hbar^2)^{1/2}(E_0 - E_I = E_\nu)^{1/2}$ is the wave vector of the incident electron after the collision.

We turn first to ionization. Assuming $E_\nu \gg E_B$, we use the functions $\varphi_\nu(z)$ (AI. 5) of Appendix I) and obtain for $\varphi_{\nu\nu_0}(k')$

$$|\varphi_{\nu\nu_0}(k')|^2 = \frac{16\nu_0 a}{\pi} \frac{(q_\nu \nu_0 a)^2 (k'\nu_0 a)^2}{\{1+[(q_\nu - k')\nu_0 a]^2\}^2 \{1+[(q_\nu + k')\nu_0 a]^2\}^2} \tag{32}$$

for the odd states ν, and for the even ones this expression must be multiplied by $(k'\nu_0 a)^2/[1 + (q_\nu \nu_0 a)^2]$. For the cross section of a transition to a state of definite parity in a unit interval of the values of q_ν we then have[7)]

$$\sigma_\nu^\pm = 2\pi \frac{k_\nu}{k} \int_0^\infty |f_\nu^\pm|^2 \rho d\rho = \frac{4\pi}{k_\nu k (k \mp k_\nu)^2 a^2} |\varphi_{\nu\nu_0}(k \mp k_\nu)|^2. \tag{33}$$

Equation (32) has a simple form in two limiting cases: $E_0 - E_I \ll E_I$ and $E_0 \gg E_I$. In the first region, the scattering is isotropic and the transitions of the atomic electron into states of different parity (at a given energy E_ν) are equally probable. We then obtain for the total cross section of the transition in a unit interval of values of q_ν

$$\sigma_\nu = 4\pi(\nu_0 a)^2 \frac{8}{\pi} \frac{\nu_0^4 a (q_\nu a)^2}{k_\nu a}, \quad E_0 - E_I \ll E_I. \tag{34}$$

Integrating this expression with respect to q_ν, we obtain the total ionization cross section ($Q = [k^2 - (1/\nu_0 a)^2]^{1/2}$):

$$\sigma_I = \int_0^Q dq_\nu \sigma_\nu = 4\pi(\nu_0 a)^2 \frac{4\nu_0^2}{3\pi} \frac{E_0 - E_I}{E_I}, \quad E_0 - E_I \ll E_I. \tag{35}$$

In the region $E_0 \gg E_I$, the largest probability is possessed by transitions with $E_\nu \sim E_I$. Scattering in these transitions (as in all transition with $E_\nu \ll E_0$) is anisotropic: forward scattering predominates and the atomic electron goes over mainly into odd states. The value of σ_ν in this case is given by

$$\sigma_\nu = 4\pi(\nu_0 a)^2 \frac{16\nu_0^3 a}{\pi (k\nu_0 a)^2} \frac{(q_\nu \nu_0 a)^2}{[1+(q_\nu \nu_0 a)^2]^4}, \quad E_0 \gg E_I, E_\nu; \tag{36}$$

and for σ_I we then have

$$\sigma_I = 4\pi(\nu_0 a)^2 \frac{\nu_0^2}{2} \frac{E_I}{E_0}, \quad E_0 \gg E_I. \tag{37}$$

The general form of the ionization cross section is shown in Fig. 3.

In the case of slow collisions, the interaction in the

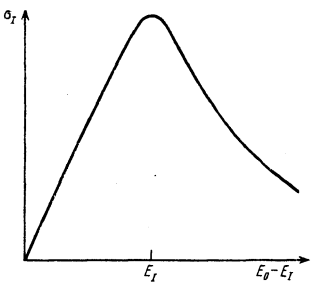

FIG. 3. General form of ionization cross section as a function of the excess of the incident-electron energy E_0 over the ionization threshold E_I. σ_I is determined by Eq. (35) at $E_0 - E_I \ll E_I$ and by Eq. (37) at $E_0 \gg E_I$.

final state between the incident electron and the atom is substantial. The total inelastic-scattering cross section is determined by the excitation of the atom, and can be estimated at

$$\sigma_{ex}\sim 4\pi(\nu_0 a)^2\nu_0^4[(E_0-E_I+E_B)/E_B]^{1/2}.$$

The variation $\sigma_\nu \propto k_\nu$ is the general law for the threshold behavior of a reaction cross section in the case of a short-range interaction (this relation may be easily established in analogy with the proof for the three-dimensional case (Ref. 5, p. 698 of Russ. original).

4. INELASTIC SCATTERING ($\Delta M \neq 0$)

The probabilities of excitation with $\Delta M \neq 0$ are relatively small (proportional to λ^2), but these transitions are of interest inasmuch as at $E_0 < E_I - E_B$ other inelastic processes are impossible. When these transitions are considered, it is necessary to use Eq. (3) in the lowest order in λ. We consider first the transition with $\Delta M = 1$. The system (3) is then reduced to two equations: the equation for the function F_{ν_1} that describes the transition to the ground state with $M_1 = 1$:

$$-\left(\frac{d^2}{dz^2}+k_1^2\right)F_{\nu_1}+\hat{U}_{\nu_1\nu_1}F_{\nu_1}=-U_{\nu_1\nu_0}F_{\nu_0},\quad k_1^2=\frac{2m}{\hbar^2}(E_0+E_{\nu_0}-E_{\nu_1}) \tag{38}$$

and the Eq. (8) for F_{ν_0}. The diagonal potential $\hat{U}_{\nu_1\nu_1}$ hardly differs from the potential $\hat{U}$ [Eq. (9)], while the off-diagonal potential takes at $\rho \gg \lambda$ the form

$$U_{\nu_1\nu_0}=\frac{\sqrt{2}\lambda\rho}{a}\int_{-\infty}^{\infty}\frac{\psi_{\nu_0}^2(z_1)\,dz_1}{[\rho^2+(z-z_1)^2]^{1/2}}. \tag{39}$$

We have used here the fact that $\nu_1 \approx \nu_0$. Equation (38) is easily solved since the solutions of the corresponding homogeneous equation were already obtained in Sec. 1. Obtaining in the usual manner a solution satisfying the boundary conditions

$$F_{\nu_1}(z)\to g^{\pm}e^{\pm ik_1 z},\quad z\to\pm\infty,$$

we obtain for the excitation amplitudes $g^{\pm}$

$$g^{\pm}=-\frac{i}{2k_1}\int_{-\infty}^{\infty}dz F_{\nu_1}^{\mp}(z)\,U_{\nu_1\nu_0}(z)F_{\nu_0}(z), \tag{40}$$

where $F_{\nu_i}^{\mp}$ describes inelastic scattering of an electron with wave vector k_1 incident from the right (−) and from the left (+). Substituting in (40) Eqs. (10), (15), (39), and (AI.1), we get for $g^{\pm}$:

$$g^{\pm}=-i\sqrt{2}\frac{k}{(k-ik_b)(k_1-ik_b)}\frac{\lambda}{a}\frac{(k\mp k_1)K_1[(k\mp k_1)\rho]}{1+[(k\mp k_1)\nu_0 a/2]^2}, \tag{41}$$

where K_1 is a Macdonald function. The excitation cross section σ_1 is expressed in terms of the amplitudes in the following manner:

$$\sigma_1=\sigma_1^{+}+\sigma_1^{-},\quad \sigma_1^{\pm}=2\pi\frac{k_1}{k}\int_0^{\infty}|g^{\pm}|^2\rho\,d\rho. \tag{42}$$

We consider first the behavior of the cross section near the excitation threshold at $k_0 \ll \nu_0/a$ (i.e., $E_1 \ll \nu_0^3 E_{01}$; E_1 is the excess of the initial energy of the electron over the threshold, $E_{01}=E_{\nu_1}-E_{\nu_0}\approx 2E_B/\nu_0$ is the excitation energy). In this case the main contribution to the cross section is made by values $\rho \gg \nu_0 a$. Then $k=k_{01}=(2mE_{01}/\hbar^2)^{1/2}\gg k_b$ and Eq. (42) yields, when (22) is used for k_b,

$$\sigma_1^{\pm}=4\pi\lambda^2\frac{\nu_0^{1/2}}{\sqrt{2}k_1 a}\int_0^{\infty}\frac{x^7K_1^2(x)\,dx}{x^6+\beta}, \tag{43}$$

where $\beta=8(3\pi/32)^2\nu_0^5/(k_1a)^2$. At the very threshold when $\beta \gg 1$ we obtain

$$\sigma_1^{\pm}=4\pi\lambda^2\frac{k_1 a}{\nu_0^{5/2}}\gamma,\quad k_1 a\ll\nu_0^{5/2}, \tag{44}$$

where γ is the numerical factor, $\gamma=(32)^2 8\sqrt{2}/105\pi^2 \approx 11.2$. The maximum of the integrand corresponds here to $\rho\sim 1/k_{01}\sim\nu_0^{1/2}a$. At $\beta \ll 1$ Eq. (43) yields (the number under the logarithm sign was obtained by numerical integration)

$$\sigma_1^{\pm}=4\pi\lambda^2\frac{\nu_0^{1/2}}{3\sqrt{2}k_1 a}\ln\frac{0.4k_1 a}{\nu_0^{1/2}},\quad \nu_0^{5/2}\ll k_1 a\ll\nu_0. \tag{45}$$

The main contribution is made here by the values $\rho \sim \nu_0 a(\nu_0/k_1 a)^{1/3}$. We now proceed to the energy region for which $k_1 \gg \nu_0/a$. For this region the values $\rho \ll \nu_0 a$ are significant, namely $\ln(\nu_0 a/\rho)\sim(k_1 a/\nu_0)^{1/2}$. At sufficiently large k_1 ($k_1 \gtrsim 1/\nu_0 a$) these values of ρ fall in an interval in which the inequality (20) is violated and our analysis is strictly speaking incorrect. We turn therefore first to the region $\nu_0/a \ll k_1 \ll 1/\nu_0 a$. In this region the characteristic values $\rho \ll 1/(k \mp k_1)$ and we can use the expansion of the Macdonald function $K_1(x) \to 1/x$. The upper limit in the integral must be replaced here by $\sim\nu_0 a$. Calculation of the integral yields

$$\sigma_1^{\pm}=4\pi\lambda^2\frac{\pi}{2\sqrt{3}}\frac{1}{(k_1 a)^{1/2}(ka\sqrt{\nu_0})}\delta\left(\frac{k_1}{k}\right),\quad \nu_0\ll k_1 a\ll\frac{1}{\nu_0}, \tag{46}$$

where $\delta(x)=(1-x^{3/2})/(1-x^2)$ as a function that changes within a considered interval from 1 at $k_1 \ll k_{01}$ to 4/3 at $k_1 \gg k_{01}$. Let us examine, finally, the region $k_1 \gg 1/\nu_0 a$. In this region the Born approximation is valid, a fact corresponding to the same formula (41) at $k_b = 0$. The integral (42) then diverges at the lower limit. The reason for the divergence is that at $\rho \sim \lambda$ it is incorrect to treat the transverse motion quasiclassically, i.e., to replace (4) by (5) in the potential. The cross section can be obtained with logarithmic accuracy, by cutting off the integral for small ρ at the value $\rho \sim \lambda$. We then obtain[8)]

$$\sigma_1^{+}=4\pi\lambda^2\frac{1}{(k_1 a)^2}\ln\frac{ka^2}{\lambda};\quad \sigma_1^{-}\ll\sigma_1^{+};\quad k_1 a\gg\frac{1}{\nu_0}. \tag{47}$$

Equations (44)–(47) determine the cross section for the excitation $\Delta M = 1$ in the entire energy region. The general form of the cross section is shown in Fig. 4.

We emphasize once more than the main contribution to the cross section for the considered transition, in the region of its maximum, is made by $\rho \gg \nu_0 a$ (we recall that $\nu_0 a$ is the largest geometrical dimension of the scatterer and constitutes the length of the wave function of the ground state of the atomic electron along the magnetic field). The reason is that at large ρ the potential $\hat{U}$ contains bound states with low binding energy ($k_b \propto \rho^{-3}$ at $\rho \gg \rho_0 a$) and the energy of the slow electrons is almost at resonance with them.

We consider now briefly transitions with change of the angular momentum projection $\Delta M_1 > 1$. To estimate the amplitude of such an excitation we can use Eq. (41) with the natural substitutions $U_{\nu_1\nu_0}(z) \to U_{\nu_{M_1}\nu_0}(z)$, $k_1 \to k_{M1}$.

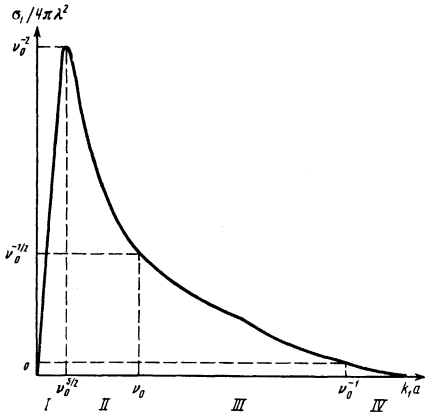

FIG. 4. Dependence of the excitation cross section σ_1 on the energy E_0 of the incident electron for the transition $M_1 = 0 \to M_1 = 1$. The abscissas are the values of $k_1a = [(E_0 - E_{01})/E_B]^{1/2}$ where E_{01} is the excitation energy and E_B is the Bohr energy. The behavior of the cross section is different in different ranges of variation of this parameter: I) $k_1a \ll \nu_0^{5/2}$, $\sigma_1 \propto (k_1a)$, Eq. (44); II) $\nu_0^{5/2} \ll k_1a \ll \nu_0$, $\sigma_1 \propto (k_1a)^{-1}\ln(k_1a)$, Eq. (45); III) $\nu_0 \ll k_1a \ll \nu_0^{-1}$, Eq. (46); IV) $k_1a \gg \nu_0^{-1}$, $\sigma_1 \propto (k_1a)^{-2}$, Eq. (47). The ordinates are the characteristic values of the cross sections in these regions in the unit of the square of the magnetic length λ.

The nondiagonal potential $U_{\nu M_1 \nu_0}$ contains the factor λ^{M_1} and at small ρ is proportional to $(\rho)^{-M_1}$. Therefore the integral (42) diverges at small ρ in power-law fashion and this divergence must be cut off at $\rho \sim \lambda$. As a result, σ_{M_1} is proportional to λ^2. At $\rho \sim \lambda$ we have $k_b \sim 1/\nu_0 a$. Therefore at $\Delta M_1 > 1$ we have $\sigma_{M_1} \sim \lambda^2 \nu_0^2 \xi(M_1)$ (this value corresponds to the maximum of σ_{M_1}, which takes place at $k \sim 1/\nu_0 a$), where $\xi(M_1)$ is a factor determined by the overlap of the functions of the transverse motion of the atomic electron with angular momentum projections 0 and M_1. This factor decreases rapidly (exponentially) with increasing M_1. Thus, the cross sections for excitation of the levels with $M_1 \neq 1$ are small compared with σ_1. Moreover, even the total cross section for the excitation into all levels with $M_1 \neq 1$ is small compared with σ_1, for although the effective density of the states increases with increasing M_1 (in proportion to M_1), the cross section increases exponentially at the same time.

5. ENERGY LOSSES

The results reported above allow us to obtain the value of the effective deceleration B:

$$B = \sum_{\nu} (E_\nu - E_{\nu_0})\sigma_\nu.$$

At incident-electron energies lower than the excitation energy of an atom with $\Delta M = 0$, the main process that determines the value of B is excitation with $\Delta M = 1$. In this case

$$B = (E_{\nu_1} - E_{\nu_0})\sigma_1, \quad E_0 < E_I - E_B$$

and the dependence of B on E_0 is determined entirely by the corresponding σ_1 dependence (see Fig. 4). At $E_0 > E_I - E_B$ the principal transitions are those with $\Delta M = 0$. In a narrow range of variation of E_0 ($|E_0 - E_I| \lesssim E_B$) the losses are determined by the excitation processes, and in the entire $E_0 - E_I \gg E_B$ region they are determined by ionization processes. If E_0 exceeds E_I only insignificantly ($E_0 - E_I \ll E_I$), the energy loss for all the transitions is practically equal to E_I, so that

$$B = E_I\sigma_I, \quad E_I \gg E_0 - E_I \gg E_B,$$

where σ_I is the total ionization cross section (see Fig. 3). At high energies $E_0 \gg E_I$ the principal losses are determined by the processes of ionization into states with energy $\sim E_I$, for which the cross sections are given by Eq. (36). The effective deceleration is in this case

$$B = \int_0^\infty \sigma_\nu (E_I + \hbar^2 q_\nu^2/2m)\, dq_\nu = 4\pi E_B/k^2. \tag{48}$$

We note that B (48) does not depend on the magnetic field. The reason is that with increasing field intensity the total cross section of the inelastic processes decreases in proportion to ν_0^2 [Eq. (37)], and the characteristic loss of energy by a fast electron ($\sim E_I$) increases by the same number of times.

Equation (48) is valid in a more general case than considered in the present paper. It sufficies to satisfy the inequality $a \gg \lambda$ [and not $\ln(a/\lambda) \gg 1$]. The can be demonstrated by deriving (48) by the usual method of calculating the energy losses of fast electrons with aid of the sum rule (Ref. 5, p. 715 of Russ. original). It can be shown similarly that the total cross section σ_r of the inelastic processes for fast electrons without assuming smallness of ν_0 is of the form $\sigma_r = 4\pi(ka)^{-2} \times \langle \nu_0 | z^2 | \nu_0 \rangle$, where the matrix element is taken between the functions of the ground state of the atomic electron. At $\nu_0 \ll 1$ the latter is equal to $(\nu_0 a)^2/2$ and we arrive at Eq. (37).

We note in conclusion that the below-threshold resonance effects which were not considered above do not affect the values of the cross sections. Resonances of this type lead to a strong increase of the scattering amplitude only for narrow intervals of ρ and k.

We are grateful to B.A. Aronzon and E.Z. Meĭlikhov for initiating this work.

APPENDIX I

At $\nu_0 \ll 1$ the normalized wave functions $\psi_\nu(z)$ are simple in form.

For the ground stage (ν_0)

$$\psi_{\nu_0}(z) \to (\nu_0 a)^{-1/2} \exp(-|z|/\nu_0 a); \tag{AI.1}$$

for excited states of the discrete spectrum ($\nu \to 1, 2, 3, \ldots$) we have for odd states

$$\psi_\nu(z) \to (2/\nu^3 a)^{1/2} (z/\nu a) \exp(-|z|/\nu a) L_{\nu-1}^{(1)}(2|z|/\nu a), \tag{AI.2}$$

where L are Laguerre polynomials; for even states at $|z| \gg \nu_0 a$ the function $\psi_\nu(z)$ is described by Eq. (AI.2) in which we make the substitution $z \to |z|$, and for $|z| \lesssim \nu_0 a$ we get

$$\psi_\nu(z) \to (2\nu_0^2/\nu^3 a)^{1/2} (|z|/\nu_0 a - 1); \tag{AI.3}$$

for the states of the continuous spectrum (normalization to $\delta(q_\nu - q_\nu')$, $q_\nu = 1/|\nu| a$)

$$\psi_\nu(z)=(4\pi)^{-1/2}\exp(-\pi/2q_\nu a)\,[W_{i/q_\nu a,\,1/2}(-2iq_\nu z)\exp(i\theta)+\text{c.c.}],$$

where W is a Whittaker function. The functions ψ_ν are chosen in the form of even and odd standing waves. For the odd wave $\theta=\arg\Gamma(i/q_\nu a)$, and for the even one

$$\theta=\arg\Gamma(i/q_\nu a)-\arcsin\{\pi/[\pi^2+\zeta^2(q_\nu)]^{1/2}\};$$
$$\zeta(q_\nu)=[1-\exp(-2\pi/q_\nu a)]/2\nu_0.$$

For $|z|\lesssim\nu_0 a$ and $q_\nu\nu_0 a\ll 1$ in the case of an even wave

$$\psi_\nu(z)\to\{q_\nu a[1-\exp(-2\pi/q_\nu a)]/2(\pi^2+\zeta^2)\}^{1/2}(1-|z|/\nu_0 a),$$

and in the case of an odd wave

$$\psi_\nu(z)\to(z/a)\{2q_\nu a/[1-\exp(-2\pi/q_\nu a)]\}^{1/2}. \qquad \text{(AI.4)}$$

At $q_\nu a\gg 1$ the effects of the Coulomb interaction can be disregarded, and the functions $\psi_\nu(z)$ take the form of the functions corresponding to potential $-e^2\delta(z)/\nu_0$; for the even wave

$$\psi_\nu(z)\to\pi^{-1/2}\cos(q_\nu|z|+\operatorname{arctg}(1/q_\nu\nu_0 a)), \qquad \text{(AI.5)}$$

and for the odd wave

$$\psi_\nu(z)\to\pi^{-1/2}\sin(q_\nu z).$$

In all formulas (AI. 1–5) no account was taken of the logarithmic divergence of $d\psi/dz$ as $z\to 0$ for the even states (this influences little the value of ψ).

APPENDIX II

SPECTRUM OF ELECTRONS AFTER IONIZATION

In the case $E_0\gg E_I$, the transitions for which the incident electron loses a considerable fraction of its energy make no substantial contribution to the total ionization cross section. They can be of interest in themselves, however. The quantity σ_ν has its principal maximum at $E_\nu\ll E_0$. With further increase of E_ν, the cross section σ_ν decreases, but as seen from (33) it begins to increase again at $E_\nu\sim E_0$. At $E_0-E_\nu\ll E_0$, when the incident electron loses practically all its energy, we obtain for σ_ν:

$$\sigma_\nu=4\pi(\nu_0 a)^2\frac{\nu_0^2}{\pi k_\nu(k\nu_0 a)^7},\quad E_0\gg E_I,\quad E_0-E_\nu\ll E_0. \qquad \text{(AII.1)}$$

The scattering becomes isotropic for all these values of E_0, and the atomic electron has as before the largest probability of going over to odd states.

We continue the analysis of the case $E_0-E_I\gg E_B$. Equation (AII. 1), just as all the results above, is valid so long as the energy of the incident electron after the collision is much larger than the Bohr energy, i.e., so long as $k_\nu a\gg 1$. If $k_\nu a\lesssim 1$, then in the final state it is necessary to take into account the interaction of the incident electron with the nucleus.[9] The scattering amplitude $f_\nu^\pm$ differs in this case from $f_\nu^\pm$ determined by expression (31) only by the factor $F_\nu(0)$, where $F_\nu(0)$ is the value at $z=0$ of the wave function of an electron scattered by a Coulomb potential $\propto 1/(\rho^2+z^2)^{1/2}$ (see Sec. 1). The latter can be easily obtained from expressions (5.34–35) of Ref. 1.[10] As a result, the divergence of the cross section $\sigma_\nu\propto k_\nu^{-1}$ (34), (AII.1) is cut off as $k_\nu\to 0$ at the value $k_\nu\sim 1/a$ by the factor

$$(k_\nu a/2\pi)\,[1-\exp(-2\pi/k_\nu a)]\,J(k_\nu, k).$$

Here J is a quantity of the order of unity, which is equal to 1 at $k_\nu a\gg 1$ [in this case the entire factor is equal to 1 and the result (33) is valid], and depends weakly (logarithmically) on the value of k at $k_\nu a\lesssim 1$.

If after the collision the low energy is possessed by the atomic electron ($q_\nu a\lesssim 1$), then after calculating the matrix element $U_{\nu\nu_0}$ it is necessary to use the functions (AI. 4), which take into account the long-range character of the nuclear field. As a result, the cross sections σ_ν obtained above must be simply multiplied by the expression

$$(2\pi/q_\nu a)\,[1-\exp(-2\pi/q_\nu a)]^{-1}.$$

We note, however, that processes with $q_\nu a\lesssim 1$ do not influence the total cross sections of the ionization in the considered case $E_0-E_I\gg E_B$.

Transitions with excitation of the atom at $E_0-E_I\gg E_B$ are much less probable than transitions with ionization. In analogy with the procedure used above for ionization we can obtain for the total excitation cross section σ_{ex}: $\sigma_{ex}/\sigma_I\sim((E_0-E_I)/(E_B)^{-3/2}$ for $E_B<E_0-E_I<E_I$ and $\sim\nu_0^3$ for $E_0>E_I$).

[1] When finding the binding energy of an electron in an atom (see below) this is permissible if the condition (20) is satisfied.

[2] It is assumed that the atom is in the ground state ν_0 prior to the collision, so that at all $\nu'\neq\nu_0$ the value of $F_{\nu'}(z)$ should contain only diverging waves as $z\to\pm\infty$. It is this requirement which serves as the boundary condition for the solution Eq. (7).

[3] Actually at $\ln(\nu_0 a/\rho\ll 1/\nu_0 a/\nu_0$ and $k\ll 1/\nu_0 a$ the function $F(z')$ in (9) can be taken outside the integral sign at the point z, so that the potential $\hat{U}$ is in fact local in this case.

[4] It is seen from Eqs. (17) and k_b does not depend on the sign of the potential (it enters in it quadratically). This means that particles with unlike charge have in a strong magnetic field bound states in the atom, and furthermore with different binding energies at equal masses (at $M\gg 1$).

[5] Only the matrix element that connects the ground state with an odd state of the continuous spectrum was written out for $\rho\gg\nu_0 a$. The characteristic value of the analogous element for an even state is smaller by a factor $\rho/\nu_0 a$ (at $q_\nu\sim 1/\nu_0 a$).

[6] Equation (29) cannot be used here to find $\Delta\sigma$, inasmuch as at $k\gtrsim 1/\lambda$ it becomes important to take into account the finite character of the localization of the wave function of the transverse motion of the incident electron [this width is $\sim\lambda$, see Eq. (5)]. For $k\gg 1/\nu_0 a$, however, scattering is primarily by the field of the nucleus (the interaction of the electrons is weak), for which the quantity Δf [see Eq.(28)] can be easily calculated also with account taken of the indicated circumstance. The calculation leads to the following expression for Δf:

$$\Delta f=(i/ka)M!\;\Psi(M+1,\,1;\,2k^2\lambda^2),$$

where Ψ is a confluent hypergeometric function of the second kind. For $1/\nu_0 a\ll k\ll 1/\lambda$ this expression goes over to Eq. (29), and for $k\gg 1/\lambda$ (more accurately, $k\gg M/\lambda$) it yields

$$\Delta f=iM!/(ka)\,(2k^2\lambda^2)^{M+1}.$$

It is this which leads to the result given in the text.

[7] It is seen from Eqs. (31) and (33) that the values that determine the cross sections $\sigma_\nu^\pm$ are $\rho\sim 1/(k\mp k_\nu)\gtrsim\nu_0 a$.

[8] For $k\ll 1/\lambda$ ($\ln(1/k\lambda)\gg 1$) the logarithm in (47) should be replaced by $1/2\ \nu_0$.

[9] The interaction with the atomic electron can in this case be

disregarded, since $q_\nu a \gg 1$. In the initial state, on the other hand, the interaction with atom can be neglected as before.

[10] For $\lambda \ll \rho \ll a$ we have

$$|F_\nu(0)|^2 = \frac{\pi k_\nu a}{2} \frac{1-\exp(-2\pi/k_\nu a)}{\pi^2+\zeta^2(k_\nu, \rho)},$$

$$\zeta(k_\nu, \rho) = (1-\exp(-2\pi/k_\nu a))[\ln(k_\nu\rho)+2C+\operatorname{Re}\psi(i/k_\nu a)].$$

Here C is the Euler constant, ψ is the logarithmic derivative of the Γ function; $\operatorname{Re}\psi(ix) \to -C$ at $x \ll 1$ and $\to \ln x \gg 1$.

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