Some features of neutron scattering in superfluid helium

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The shape of the spectrum of inelastic neutron scattering by helium is investigated in the near-threshold regions where an important role is played by the i·teraction of the excitations. The attraction between the rotons at large momenta on the shape of the spectr. n is considered near its end point. The experimental data on neutron scattering with production of excitatior near the maximum of the dispersion curve at increased pressures are explained. The spectrum near the p: non-emission threshold is investigated in the case when the group velocity of the excitations exceeds the sound velocity in a certain region. It is shown that at arbitrarily low temperatures, the neutron-scatterin spectrum line corresponding to the production of one roton has a wing on the high-energy side.

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This article deals with the effects produced in inelastic scattering neutrons by superfluid-helium excitation-spectrum singularities due to the presence of various decay thresholds. We note first of all that observation of these singularities calls for a rather high measurement accuracy. One can hope, however, that the latest progress in experimental techniques will make such measurements possible.[1]

The first section of the paper considers the influence of attraction between rotons on the shape of the spectrum near its end point at large momenta. In the second section is investigated the roton spectrum near its maximum at increased pressure, when this maximum is close to the threshold value 2Δ. The third section is devoted to neutron scattering with production of a roton in the region where "Cerenkov" emission of a phonon by a roton is possible. It is shown in the last section that the neutron-scattering spectral line corresponding to production of one roton has a wing on the higher-energy side even at absolute zero temperature. This wing is connected with the simultaneous emission of a long-wave phonon.

We point out beforehand that the entire paper deals only with the case of very low temperatures, when the collision damping of the excitations can be neglected.

1. SHAPE OF THE He II SPECTRUM NEAR ITS END POINT

Pitaevskiĭ[2] has shown that Bose elementary excitations can have three possible decay thresholds. He has also advanced the hypothesis that in real liquid helium the spectrum terminates after its curve reaches an energy $\varepsilon_c=2\Delta$. At this point (at a momentum $p_c<2p_0$) the excitation can decay into two rotons. The available experimental data on neutron scattering do not contradict this assumption. The question is nonetheless not quite clear. The point is that, as will be shown below, a threshold of the type indicated can exist only if the sign of the interaction between the rotons at the threshold point corresponds to repulsion.[1)] Yet this sign is not known beforehand, and the only way of determining it reliably is to compare the theoretical spectrum curve in the threshold region with experiment. We shall therefore determine this curve by assuming, unlike in Ref. 2, that attraction exists between the rotons. It turns out that in this case the spectrum curve continues up to momentum values $p>2p_0$, and only then does the decay into two rotons with parallel momenta take place. Comparison with experiment will decide the choice between the two possibilities.

In the notation of Zawadowski, Ruvalds, and Solana,[3] the single-particle Green's function with allowance for roton-roton interaction is of the form

$$G^{-1}(\mathbf{p},\varepsilon)=\varepsilon-E_0(p)-2g_3^2F/(1-g_4F),\tag{1}$$

where $g_4<0$, g_3 are constants describing respectively the amplitudes for the scattering and decay of the excitations (a negative g_4 corresponds to attraction); $E_0(p)=\Delta+(p-p_0)^2/2\mu$ is the unperturbed spectrum; $F(p,\varepsilon)$ is the polarization operator,

$$F(\mathbf{p},\varepsilon)=\frac{i}{(2\pi)^4}\int d^3\mathbf{q}d\omega G^{(0)}(\mathbf{p}-\mathbf{q},\varepsilon-\omega)G^{(0)}(\mathbf{q},\omega).\tag{2}$$

Pitaevskiĭ and Fomin[3] have shown also that if $g_4<0$, then at $p<2p_0$ the equation $G^{-1}(\mathbf{p},\varepsilon)=0$ has two solutions.[2)] One of them corresponds to energies $\varepsilon\to2\Delta$ and describes the upper branch of the spectrum, i.e., the one above the fundamental branch. It will be explained in detail in the last section of the article that this branch is damped and is of no interest to us (see also Ref. 5). For the fundamental branch, however, the energy is rigorously less than 2Δ at $p<2p_0$.

This means that the lower branch of the spectrum cannot terminate at a point $p_c<2p_0$, $\varepsilon_c=2\Delta$, as assumed in Ref. 2. The solution obtained in Ref. 3 is bounded by the condition $p<2p_0$. We shall calculate the Green's function (1) in the region $|p-2p_0|\ll p_0$ (we shall see that this is the end point of the spectrum).

Upon integration with respect to ω, Eq. (2) takes the form

$$F(\mathbf{p},\varepsilon)=-\frac{1}{(2\pi)^3}\int\frac{d^3\mathbf{q}}{E_0(q)+E_0(|\mathbf{p}-\mathbf{q}|)-\varepsilon-i0}.\tag{3}$$

We shall use the following device for the calculations[6,7]: accurate to terms of fourth order in $p-p_0$ we can write

$$(p-p_0)^2/2\mu\approx(p^2-p_0^2)^2/8p_0^2\mu.$$

The integration in (3) is next extended to infinity, the result being

$$F(\mathbf{p},\varepsilon)=-\frac{p_0^2\mu}{\pi p}\ln\frac{(p^2/4p_0\mu^{1/2})^{1/2}+[S+(2\Delta-\varepsilon+S^2)^{1/2}+p^2/4p_0\mu^{1/2}]^{1/2}}{[S+(2\Delta-\varepsilon+S^2)^{1/2}]^{1/2}} \qquad (4)$$

$$S=(p^2-4p_0^2)/8p_0\mu^{1/2}\approx(p-2p_0)/2\mu^{1/2}. \qquad (5)$$

The threshold p_c, ε_c is characterized by the presence of branch points of the Green's functions at $p=p_c$ and $\varepsilon=\varepsilon_c$. It is seen from (4) and (5) that in the region $p>2p_0$ only a square-root branch point $(2\Delta-\varepsilon+S^2)^{1/2}=0$ is possible. This corresponds to a threshold of type b in Pitaevskii's paper.[2] It follows from this condition that

$$\varepsilon_c=2\Delta+S_c^2=2\Delta+(p_c-2p_0)^2/4\mu. \qquad (6)$$

The value of p_c is determined from the condition $G^{-1}(p_c,\varepsilon_c)=0$; this yields

$$p_c=2p_0+8p_0\exp\left(-\frac{E_0(2p_0)-2\Delta}{E_0(2p_0)-2\Delta-2g_3^2/g^4}\frac{4\pi\hbar^3}{p_0\mu|g_4|}\right). \qquad (7)$$

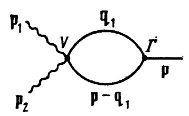
For the obtained formulas to be valid we must have $g_4 \ll 4\pi\hbar^3/p_0\mu$.

We see that at small negative g_4 we actually have $p_c>2p_0$ and $p_c-2p_0\ll p_0$. The spectrum near the threshold is determined by the poles of the Green's function (1), and this yields

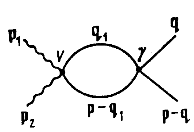
$$\varepsilon=\varepsilon_c+\frac{p_c-2p_0}{2\mu}(p-p_c). \qquad (8)$$

The spectrum near the threshold is thus close to linear. At the threshold point, a decay takes place into two rotons with parallel momenta $p_1=p_2=p_c/2$. As seen from (7), p_c-2p_0 (meaning also $\varepsilon_c-2\Delta$) increases exponentially with increasing $|g_4|$. Assuming by way of estimate $g_4=-1.2\times10^{-38}$ erg·cm^3 (see Ref. 1) and $2g_3^2/g_4\sim1$ K, we obtain $p_c-2p_0\approx1.9\times10^{-2}p_0$ and $\varepsilon_c-2\Delta\approx23$ mK.

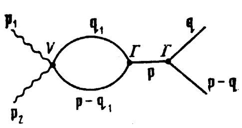
We consider now inelastic neutron scattering with production of excitations near a threshold of this type. Let p and ε be the momentum and energy lost by the neutron. Production of one excitation is possible at $\varepsilon\leqslant\varepsilon_c$ and $p\leqslant p_c$ [see diagram (9)]. When $\varepsilon\geqslant2\Delta$, two excitations with energy $\geqslant\Delta$ each can be produced [see (10) and (10′)].



(9)



(10)



(10′)

In diagrams (9)–(10′), the wavy lines correspond to the neutron, Γ and γ are the exact vertex functions, and V is the amplitude for neutron scattering by a free atom.

The probability of neutron scattering with production of one excitation with energy ε is

$$dw=2\pi|N_1M|^2\delta(E_1-E_2-\varepsilon)(2\pi\hbar)^{-3}d^3\mathbf{p}_2. \qquad (11)$$

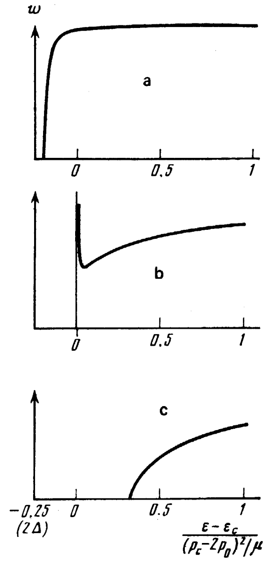


FIG. 1. Dependence of the probability of inelastic neutron scattering on the energy transfer ε at fixed momentum-transfer values near the threshold momentum p_c: 1) $p-p_c=-0.5(p_c-2p_0)$; b) $p=p_c$; c) $p-p_c=0.5(p_c-2p_0)$.

For diagram (9), the matrix element M is finite, therefore the behavior of dw near the decay threshold is determined by the renormalization constant

$$N_1=[-\operatorname{Res}_{\varepsilon}G(\mathbf{p},\varepsilon)]^{1/2}\propto(2\Delta-\varepsilon+S^2)^{1/4}.$$

Inasmuch as in the production of one excitation the quantities ε and p are connected by Eq. (8), it follows that if the momentum transfer p is given we have

$$dw\propto p-p_c.$$

The line intensity corresponding to production of one excitation decreases thus in proportion to $p-p_c$ when the threshold momentum is approached.

The probability of production of two excitations is

$$dw=2\pi|N_2^2M|^2\delta(E_1-E_2-\varepsilon(q)-\varepsilon(|\mathbf{p}-\mathbf{q}|))(2\pi\hbar)^{-6}d^3\mathbf{q}d^3\mathbf{p}_2, \qquad (12)$$

N_2 corresponds to an excitation with energy close to 2Δ and has no singularities. The diagram (10) likewise yields no singularity. The behavior of dw is determined by diagram (10′). After integrating the δ-function with respect to $\mathbf{q}$ we obtain near the threshold

$$w\propto|G(\mathbf{p},\varepsilon)|^2I(\mathbf{p},\varepsilon), \qquad (13)$$

$$I(\mathbf{p},\varepsilon)=\begin{cases}4\pi^2p_0^2\mu/p, & 2\Delta-\varepsilon+S^2\geqslant0\\ \dfrac{4\pi p_0^2\mu}{p}\left(\dfrac{\pi}{2}-\arcsin\dfrac{S}{(\varepsilon-2\Delta)^{1/2}}\right), & 2\Delta-\varepsilon+S^2<0.\end{cases} \qquad (14)$$

Plots of the scattering probabilities vs. ε at different values of the momentum p near p_c are shown in Fig. 1. The single line vanishes at a momentum transfer $p=p_c$, while the continuous spectrum has a singularity at the point $\varepsilon=\varepsilon_c$:

$$w\propto(\varepsilon-\varepsilon_c)^{-1/2}.$$

2. ELEMENTARY-EXCITATION SPECTRUM NEAR THE MAXIMUM OF INCREASED PRESSURE

In this section we consider the behavior of the spectrum near the first maximum at pressures such that the height of this maximum approaches the doubled roton minimum 2Δ. This situation was experimentally investigated in Ref. 9. According to the data obtained there, the roton minimum falls off almost linearly with increasing pressure, while the energy Δ_1 increases slowly at the maximum. As a result, this maximum energy exceeds 2Δ at a certain pressure $P-P_0\approx 18.6$ atm, so that the excitations near the maximum can break up into two rotons. The experimental $\Delta_1(P)$ plot given in Ref. 9 has no singularity whatever near $P=P_0$. In our opinion, however, this is due to the insufficient experimental accuracy. More accurate measurements should reveal a peculiarly singular behavior of $\Delta_1(P)$ in this region.

We shall carry out the analysis for two different signs of the roton-roton interaction: $g_4>0$ (repulsion) and $g_4<0$ (attraction), since the question of the sign of the interaction in this momentum region has not been solved as yet. The accuracy of the experiments on inelastic neutron scattering with "maxon" production[9] does not permit as yet a choice between these two possibilities. We hope that this choice will become possible after a detailed comparison of the theory presented here with experiment. There is every reason for assuming that the interaction between the rotons is weak in the considered region, so that the spectrum is substantially distorted by the interaction only near the decay threshold, i.e., near the energy $\varepsilon=2\Delta(P)$.

The dependence of the excitation energy ε on the pressure P and on the momentum p is determined by the poles of the Green's function $G(\mathbf{p},\varepsilon,P)$. The latter, taking into account the roton-roton interaction, is of the form

$$G^{-1}(p,\varepsilon,P)=\varepsilon-E_0(p,P)-2g_3^2F(\mathbf{p},\varepsilon,P)/[1-g_4F(\mathbf{p},\varepsilon,P)]. \quad (15)$$

The bare energy $E_0(p,P)$ can be determined by extrapolating the obtained experimental function $\varepsilon(p,P)$ from the region $P<P_0$ into the region $P>P_0$:

$$E_0(p,P)=2\Delta_0+\lambda(P-P_0)-(p-p_1)^2/2\mu_1.$$

According to the data of Ref. 9, the constant λ is small. The remaining notation is the same as in Sec. 1.

At energies ε close to $2\Delta(P)$ and momenta p near p_1 we obtain after integrating in (3)

$$F(\mathbf{p},\varepsilon,P)=\alpha\ln[(2\Delta(P)-\varepsilon)/a],\quad \alpha=p_0^2\mu/2\pi\hbar^3p_1,\quad a=(p_0^2/\mu)(p_1/p_0)^2(1-p_1^2/4p_0^2). \quad (16)$$

When ε approaches $2\Delta(P)$, the term in (15), which describes the roton-roton interaction, causes the spectrum to deviate substantially from the "unperturbed" spectrum $E_0(p,P)$. The point P_c, ε_c at which the excitation energy reaches the value $2\Delta(P_c)$ is a branch point of the Green's function, or the threshold of decay into two rotons.

a. Consider the case $g_4>0$ (repulsion). The single-excitation spectrum reaches here the decay threshold

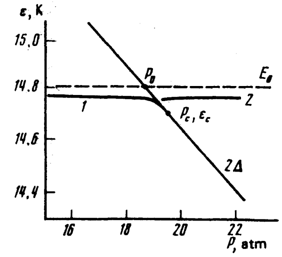


FIG. 2. Pressure dependence of the energy of the excitations at a fixed momentum $p-p_1$. Curve 1 describes the position of the maximum of the spectrum at pressures $P<P_c$. Curve 2 corresponds to the peaks in the continuous scattering spectrum at pressures $P>P_c$. The parameter values are $g_4=0.2\times10^{-38}$ erg.cm^3 and $2g_3^2/g_4=0.1$ K. The 2Δ curve and the dashed line are taken from Ref. 9.

$\varepsilon_c=2\Delta(P_c)$ at a pressure $P_c>P_0$ and at an energy $\varepsilon_c<2\Delta(P_c)$:

$$P_c=P_0+2g_3^2/\eta g_4,\quad \varepsilon_c=2\Delta_0-2g_3^2/g_4;\quad \eta=\frac{d}{dP}2\Delta(P)|_{P=P_0}. \quad (17)$$

If $\varepsilon<2\Delta(P)$, then $G(\mathbf{p},\varepsilon,P)$ has poles determined by the implicit equation

$$\varepsilon(p,P)=2\Delta(P)-a\exp\left\{-\frac{2g_3^2/g_4-[\varepsilon-\varepsilon_c+(p-p_1)^2/2\mu_1]}{\alpha g_4[\varepsilon-\varepsilon_c+(p-p_1)^2/2\mu_1]}\right\}. \quad (18)$$

When the pressure approaches P_c, the maximum on the dispersion curve $\Delta_1(P)$ approaches exponentially the value $\varepsilon_c=2\Delta(P_c)$; with decreasing pressure, however, $\Delta_1(P)$ approaches monotonically $E_0(p_1,P)$ (see Fig. 2, curve 1).

The maxon energy as a function of the momentum has the following behavior near the threshold. When $P>P_c$, the spectrum consists of two unconnected pieces: there are no poles of the Green's function in the momentum interval $|\mathbf{p}-\mathbf{p}_1|<[2\mu_1\eta(P-P_c)]^{1/2}$ (this circumstance was first noted by Nepomnyashchiĭ[10]). The maxon energy $\varepsilon(p,P)$ approaches the value $2\Delta(P)$ exponentially.

The analytic continuation of the Green's function into the energy region $\varepsilon>2\Delta(P)$ has neither real nor complex poles on the physical sheet. In this sense the branch point of the function $G(\mathbf{p},\varepsilon,P)$ is the end point of the spectrum. Nonetheless, the neutron scattering intensity has in this region maxima of finite width, which can be regarded as a continuation of the spectrum in the indicated region.

Let us see how the neutron scattering probability should behave as a function of energy in the case of a momentum loss $p=p_1$ at various pressures P. At an energy transfer $\varepsilon<2\Delta(P)$, one excitation with energy $\Delta_1(P)$ is produced [the probability of this process is described by formula (11)]. When the critical pressure is approached, the line intensity decreases in proportion to $(2\Delta-\varepsilon)\{\ln[(2\Delta-\varepsilon)/a]\}^2$. With decreasing temperature, the line width in this region becomes arbitrarily small. In the continuous spectrum $\varepsilon>2\Delta(P)$, however, rather pronounced peaks appear starting with a certain pressure P_1 ($P_0<P_1<P_c$) (see Fig. 3). These peaks are due

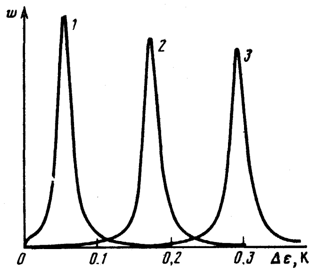


FIG. 3. Plot of continuous neutron-scattering spectrum at a fixed momentum transfer $p = p_1$ at various pressures: 1) $P = P_c$, 2) $P - P_c = 1$ atm, 3) $P = P_c = 2$ atm. The parameter values are $g_4 = 0.2\times10^{-38}$ erg-cm^3, $2g_3^2/g_4 = 0.1$ K.

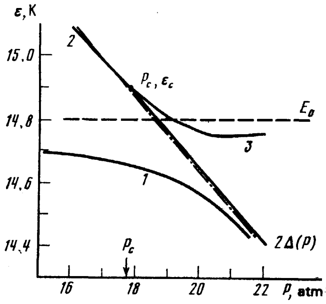


FIG. 4. Pressure dependence of the energy of the excitations at a fixed momentum transfer $p = p_1$. The curves 1 ($\Delta_1(P)$) and 2 describe the positions of the maxima in the upper and lower branches of the spectrum. Curve 3 corresponds to the peaks in the continuous neutron-scattering spectrum. The parameter values are $g_4 = 0.2\times10^{-38}$ erg-cm^3 and $2g_3^2/g_4 = -0.1$ K. The $2\Delta(P)$ curve and the dashed lines are taken from Ref. 9. The dash-dot line is a plot of $2\Delta(P) - E_b$.

to the proximity of the poles of the analytic continuation of the function $G(\mathbf{p}, \varepsilon, P)$, which lie on the unphysical sheet of its Riemann surface near the real semiaxis $\varepsilon > 2\Delta(P)$. We emphasize however, that these maxima have a finite width even as the temperature $T \to 0$. The scattering intensity at $\varepsilon > 2\Delta(P)$ is described by Eq. (12). The behavior of dw is determined by the matrix element of the diagram (10′), so that

$$w \propto |\Gamma G(\mathbf{p}_1, \varepsilon, P)|^2 = \frac{\Gamma_0^2}{2\alpha\pi g_3^2}\frac{\xi}{(\varepsilon-\zeta)^2+\xi^2}, \quad (19)$$

$$\xi = 2\alpha g_3^2 \,\mathrm{Im}\left[\ln\frac{2\Delta-\varepsilon}{a} \Big/ \left(1 - \alpha g_4 \ln\frac{2\Delta-\varepsilon}{a}\right)\right],$$

$$\zeta = E_0(p_1, P) + 2\alpha g_3^2 \,\mathrm{Re}\left[\ln\frac{2\Delta-\varepsilon}{a} \Big/ \left(1 - \alpha g_4 \ln\frac{2\Delta-\varepsilon}{a}\right)\right]. \quad (20)$$

Far from $2\Delta(P)$ the dependence of ξ and ζ on ε is very weak, so that the "line" in the continuous spectrum has an almost Lorentz shape. The line half-width

$$\xi \approx 2\alpha\pi g_3^2 \Big/ \left(1 - \alpha g_4 \ln\frac{\eta(P-P_0)}{a}\right)$$

increases somewhat with increasing P, the height of the peak decreases in proportion to $1/\xi$, and the position of the maximum approaches $E_0(p_1, P)$ logarithmically with increasing P (see curve 2 of Fig. 2).

Thus, at $g_4 > 0$ the set of lines $\Delta_1(P)$ at $P < P_c$ and of the peaks in the continuous spectrum at $P > P_c$ describes well the plot obtained in Ref. 9 and shown dashed in Fig. 2, provided that the parameters g_4 and g_3 are properly chosen (thus, if we choose $2g_3^2/g_4 = 0.1$ K, then at $g_4 \leq 0.2\times10^{-38}$ erg·cm^3 the theoretical curves are indistinguishable from the experimental one within the limits of the accuracy of Ref. 9). Nonetheless, characteristic singularities should be observable at a higher experimental accuracy. We note, for example, that the $\Delta_1(P)$ curve should be tangent to the $2\Delta(P)$ curve at the point $P = P_c$, and should approach the tangency point from above, so that a characteristic downward bending of the $\Delta_1(P)$ curve takes place.

We consider now the case $g_4 < 0$ (attraction). After finding the poles of the function $G(\mathbf{p}, \varepsilon, P)$ in the region $\varepsilon < 2\Delta(P)$ we obtain two branches of the spectrum. Both branches are described by Eqs. (17) and (18), in which now $g_4 < 0$. At a fixed momentum $p = p_1$ the energy dependence on the pressure for both branches is shown in Fig. 4 (curves 1 and 2). The lower $\Delta_1(P)$ curve (1) describes the fundamental single-particle branch of the spectrum. It exists at all pressures (unlike the case $g_4 > 0$). When the pressure exceeds P_0, the energy $\Delta_1(P) \to 2\Delta(P) - E_b$, where E_b plays the role of the binding energy:

$$E_b = a\exp(-1/\alpha|g_4|).$$

The line intensity tends then to a constant small limit. On the other hand, when the pressure is lower than P_0, the $\Delta_1(P)$ curve (1) approaches $E_0(p_1, P)$ monotonically. The upper branch (2), which can be called a bound state of two rotons, lies entirely in the energy region $2\Delta(P) - E_b < \varepsilon(p_1, P) \leq 2\Delta(P)$. It exists only at $P < P_c$, when $\varepsilon(p_1, P) \to 2\Delta(P)$, the line intensity tends to zero like $(2\Delta - \varepsilon)[\ln(2\Delta - \varepsilon)/a)]^2$.

The dependence of the energy of the excitations on the momentum for the upper branch is the following. When $P > P_c$, the upper branch consists of two unconnected pieces located in the momentum region $|\mathbf{p} - \mathbf{p}_1| > [2\mu_1\eta(P - P_c)]^{1/2}$. It approaches the energy $2\Delta(P)$ exponentially. We note that the upper branch is always damped, since decays with phonon emission are possible.

Any analytic continuation of the Green's function into the energy region $\varepsilon > 2\Delta(P)$, just as in the case $g_4 > 0$, has no poles on the physical sheet of its Riemann surface. The neutron scattering probability has in this region maxima of finite width. The sign of g_4 has practically no effect on the positions of the peaks in the continuous spectrum at high pressures. In contrast to the repulsion case, however, the width of the peaks decreases slightly with increasing pressure, and their height increases. In general, however, the width of the peaks is several times larger in the case of attraction than in the case of repulsion. The positions of the maxima in the continuous spectrum are shown as functions of the pressure in Fig. 4 (curve 3).

The experimental data[9] show a decrease of the scattering intensity with increasing pressure (at $P > P_0$). It is possible that this indicates that the rotons are repelled.

Although, as seen from Figs. 2 and 4, the behavior of the spectrum curves are different in repulsion and attraction, it is not always easy to distinguish between these two possibilities primarily because of the rather low scattering intensities in the region $\varepsilon \sim 2\Delta$. There is no doubt, however, that in principle it is possible to observe the described singularities in experiment.

We emphasize that, just as in the preceding section, observation of the predicted phenomena calls for research at sufficiently low temperature. For the downward bend of the $\Delta_1(P)$ curve to be noticeable, it is necessary that the temperature width of the lines be less than the characteristic quantity $2g_3^2/g^4$, which we have assumed in the calculations to be 0.1 K. The temperature width of the lines decreases exponentially with decreasing temperature (see, e.g., Refs. 11 and 12). At $T=1.2$ K, the half-width is ~0.1 K, and at $T=0.9$ K it already one-tenth as small, so that the predicted phenomena should become noticeable.

3. ROTON DAMPING DUE TO "CERENKOV" EMISSION OF PHONONS

According to the experimental data,[13] the group velocity of the rotons, starting with a certain momentum $p_c > p_0$, reaches the speed of sound c at low temperatures. The properties of the spectrum near the threshold beyond which production of a phonon by a quasiparticle becomes possible were investigated in a number of studies.[2,14,15] It was shown, in particular, that near this threshold the phonon damping is $\Gamma \propto (p-p_c)^3$. It was assumed in these papers that the group velocity of the rotons only reaches c, but does not exceed it. According to the data of Ref. 13, however, there is evidence that when the temperature is raised and the energy of the roton minimum decreases, the roton group velocity exceeds the speed of sound c in a certain momentum interval. Nonetheless, this question is not completely clear. In this section we consider the properties of the spectrum in the region of supersonic rotons (assuming that such a momentum region exists).

We choose the bare spectrum $\varepsilon_0(p)$ in the form of a cubic parabola, whose tangent has a slope equal to c at the point p_c, ε_c, and which lies above this tangent in the momentum interval $p_c < p < p_c'$. Below the threshold p_c, the shape of the spectrum is given by a Landau parabola. Thus,

$$\varepsilon_0(p)=\Delta+\beta(p-p_0)^2, \quad p<p_c,$$

$$\varepsilon_0(p)=\varepsilon_c+c(p-p_c)+\beta(p-p_c)^2-\frac{\beta}{p_c'-p_c}(p-p_c)^3, \quad p>p_c. \tag{21}$$

At momenta $p_c < p < p_c'$ the rotons can emit phonons. We calculate the roton damping by perturbation theory. From the momentum and energy conservation laws it follows that the roton $\mathbf{p}$ can decay into a roton $\mathbf{p}'$ and a phonon $\mathbf{q}$ such that

$$(\varepsilon_0(p)-\varepsilon_0(p'))/(p-p')\geqslant c. \tag{22}$$

The probability of such a process is

$$dw=\frac{2\pi}{\hbar}|V_{fi}|^2\delta(\varepsilon(p')+cq-\varepsilon)\frac{V^2d^3\mathbf{p}'d^3\mathbf{q}}{(2\pi\hbar)^6}, \tag{23}$$

where V is the volume of the helium; the matrix element of the operator of the effective interaction between the roton and the long-wave phonon is chosen in the form[16]

$$V_{fi}=-i\frac{(2\pi\hbar)^3}{V^{1/2}}\delta(\mathbf{p}-\mathbf{p}'-\mathbf{q})\left(\frac{qc}{2\rho}\right)^{1/2}\left(p\cos\theta+\frac{\rho}{c}\frac{\partial\varepsilon}{\partial\rho}\right) \tag{24}$$

(ρ is the helium density and θ is the angle between the vectors $\mathbf{p}$ and $\mathbf{q}$).

The damping connected with the phonon emission is $\Gamma = \hbar w/2$. After integrating dw over all possible $\mathbf{p}'$ and $\mathbf{q}$ we get

$$\Gamma(\varepsilon,p)=2\pi^2\left(p+\frac{\rho}{c}\frac{\partial\varepsilon}{\partial\rho}\right)^2\frac{q_{max}^3(\varepsilon,p)-q_{min}^3(\varepsilon,p)}{3\rho(2\pi\hbar)^3}, \tag{25}$$

where $q_{max}(\varepsilon,p)$ and $q_{min}(\varepsilon,p)$ are determined from the condition (22); this yields

$$q_{max}=\Delta p+[(\Delta\varepsilon-c\Delta p)/\beta]^{1/2},$$

$$q_{min}=\begin{cases}0, & \Delta p\leqslant{}^2/_3(p_c'-p_c)\\ \Delta p-{}^1/_3(p_c'-p_c)(1+\sqrt{3}\sin\varphi-\cos\varphi), & \Delta p>{}^2/_3(p_c'-p_c)\end{cases}, \tag{26}$$

$$\varphi=\frac{1}{3}\arccos\left(1-\frac{27}{2}\frac{\Delta\varepsilon-c\Delta p}{\beta(p_c'-p_c)^2}\right), \tag{27}$$

where have introduced the notation $\Delta p = p - p_c$ and $\Delta\varepsilon = \varepsilon - \varepsilon_c$.

Figure 5 shows the dependence of the damping on the momentum p, calculated along the bare $\varepsilon_0(p)$ curve. First, near the lower threshold, Γ increases with increasing $\Delta p (\Gamma \propto \Delta p^3$ at $\Delta p \ll p_c' - p_{-c}$, in agreement with the results of Refs. 2, 14, and 15), and reaches a maximum at $\Delta p \approx 0.85(p_c' - p_c)$. On the other hand when the momentum approaches the upper threshold p_c', the damping tends rapidly to zero: $\Gamma \propto (p_c' - p)^{1/2}$.

We examine now in greater detail the spectrum near the upper threshold p_c', ε_c'. We shall show that in a certain small vicinity of this threshold the spectrum is distorted by a phonon-roton interaction in such a way that at the point p_c', ε_c' the group velocity of the rotons again becomes equal to the sound velocity c. The Green's function near the upper threshold is of the form

$$G^{-1}(\mathbf{p},\varepsilon)=v_0(p-p_c')-(\varepsilon-\varepsilon_c')-2ai[\varepsilon-\varepsilon_c'-c(p-p_c')]^{1/2},$$
$$v_0=c-\beta(p_c'-p_c), \quad a=\pi^2(p_c'-p_c)^2\times\left(p+\frac{\rho}{c}\frac{\partial\varepsilon}{\partial\rho}\right)^2\Big/\rho(2\pi\hbar)^3\beta^{1/2}. \tag{28}$$

This corresponds to a threshold type b in Pitaevskiĭ's paper.[2]

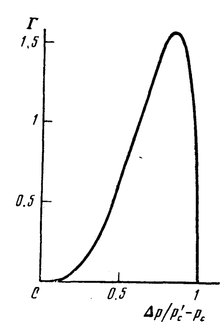


FIG. 5. Momentum dependence of the roton damping, calculated along the bare curve $\varepsilon_0(p)$.

In the momentum region $p_c' - a^2/(c-v_0) < p < p_c'$ the Green's function (28) has no poles. Nonetheless, the probability of inelastic neutron scattering at such momentum transfers has maxima at energy values

$$\varepsilon(p) = \varepsilon_c' + c(p - p_c') + (c - v_0)^2 (p - p_c')^2 / 2a^2. \quad (29)$$

Above the threshold p_c', ε_c' the equation $G^{-1}(\mathbf{p}, \varepsilon) = 0$ has real solution, i.e., there exists an undamped spectrum $\varepsilon(p)$ that takes near the threshold $(p - p_c' \ll a^2(c - v_0))$ the form

$$\varepsilon(p) = \varepsilon_c' + c(p - p_c') - (c - v_0)^2 (p - p_c')^2 / 4a^2. \quad (30)$$

As seen from (29) and (30), near the upper threshold the spectrum of the excitation is tangent to the straight line $\varepsilon = \varepsilon_c' + c(p - p_c')$. We note that near the upper threshold the region in which the spectrum differs substantially from the bare spectrum $\varepsilon_0(p)$ is very small in size.

A most recent paper[1] reports an unsuccessful attempt to observe roton damping in the investigated region. Since the damping Γ connected with the phonon emission does not exceed 10 mK according to Ref. 1, the momentum region in which the roton group velocity can exceed that of sound is very small. An estimate with the aid of (25) yields

$$p_c' - p_c \leqslant 0.04 p_0.$$

It may be of interest in this connection to repeat experiments of the indicated type at higher pressures. With increasing pressure, the effective mass of the roton decreases, and the group velocity increases, and this favors observation of this damping.

4. SCATTERING OF NEUTRONS BY ROTONS WITH EMISSION OF A LONG-WAVE PHONON

We consider, at $T=0$, the scattering of a neutron in liquid helium with a momentum loss $\mathbf{p}$, whereby the energy loss exceeds somewhat the roton energy $\varepsilon(p)$ at the given momentum. A roton with momentum $\mathbf{p} - \mathbf{q}$ and a long-wave phonon $\mathbf{q}$ are then produced. The diagram of such a process is obtained from the diagrams of the main process—scattering with production of one roton—by adding an external phonon line with momentum $\mathbf{q}$, which branches away from the external roton line.

The probability of the main process is

$$dw_0 = w_0 \delta(E_1 - E_2 - \varepsilon(p)) V (2\pi\hbar)^{-3} d^3\mathbf{p}_2, \quad (31)$$

where $E_{1,2}$ and $\mathbf{p}_{1,2}$ are the energy and momentum of the neutron before and after the scattering, and $p = |\mathbf{p}_1 - \mathbf{p}_2|$. The probability of a process with additional emission of a soft phonon, calculated by perturbation theory, is

$$dw_1 = w_0 |G(\mathbf{p}, \varepsilon) V_{fi}|^2 \delta(\varepsilon - \varepsilon(q) - \varepsilon(|\mathbf{p} - \mathbf{q}|)) V^2 (2\pi\hbar)^{-6} d^3\mathbf{p}_2 d^3\mathbf{q}, \quad (32)$$

where $\varepsilon = E_1 - E_2$, and the matrix element V_{fi} of the phonon-roton interaction operator is given by Eq. (24). Integrating with respect to q in (32), we obtain

$$dw_1 = w_0 \frac{\pi c}{\rho (2\pi\hbar)^3} I(p) (\varepsilon - \varepsilon(p)) \frac{V d^3\mathbf{p}_2}{(2\pi\hbar)^3} \quad (33)$$

$$I(p) = \frac{2(3c^2 + v^2)}{3(c^2 - v^2)^3} \left(p\frac{c}{v} + \frac{\rho}{c}\frac{\partial\varepsilon}{\partial\rho} \right)^2 - \frac{4pc}{v(c^2 - v^2)^2} \left(p\frac{c}{v} + \frac{\rho}{c}\frac{\partial\varepsilon}{\partial\rho} \right) + \frac{2p^2}{v^2(c^2 - v^2)}, \quad (34)$$

where $\mathbf{v} = \partial\varepsilon/\partial\mathbf{p}$.

It is seen from (33) that the probability of production of an additional phonon is larger the larger the energy difference $\varepsilon - \varepsilon(p)$. Thus, even at $T=0$, that line in the neutron-scattering spectrum which corresponds to production of one roton has a linear wing on the high-energy side. The slope of this wing depends on the momentum transfer p. This dependence is determined by the function $I(p)$. Near the roton minimum, when the group velocity is low, $I(p)$ increases linearly with increasing v. When $|v| \to c$, however, the function $I(p)$ tends to infinity in proportion to $(c - |v|)^{-3}$. Perturbation theory cannot be used in this region, since it is possible to calculate dw_1 as a quantity of higher order than dw_0.

At finite temperature, we express the scattering spectrum in the form

$$dw = w_0 \left[\frac{\gamma}{(\varepsilon - \varepsilon(p))^2 + \gamma^2} + K(\varepsilon - \varepsilon(p)) \right] \frac{V d^3\mathbf{p}_2}{(2\pi\hbar)^3}, \quad (35)$$

where $K = \pi c I(p)/\rho(2\pi\hbar)^3$. The presence of a linear wing leads to a shift of the maximum of the line from $\varepsilon(p)$ towards higher energies by an amount $\varepsilon' \approx K\gamma^3/2$. A numerical calculation shows, however, that this effect is very small at low temperatures. Thus, at $\gamma \sim 1$ K the shift of ε' near the roton minimum does not exceed 1.6 mK; it becomes of the order of $\gamma/2$ only at $c - |v| \approx 0.1c$.

In conclusion, I am deeply grateful to L. P. Pitaevskiĭ for constant help and advice.

1) States of a system of two rotons with nonzero total momentum are classified by the values of the helicity m, i.e., the projection of the angular momentum on the momentum direction (for details see Refs. 3 and 4). In the questions considered here an important role is played by the sign of the interaction in states with $m = 0$, as will in fact be implied hereafter.

2) Such a picture of the spectrum can be regarded as the result of hybridization between bound states of two rotons produced at $g_4 < 0$, and the one-proton branch of the spectrum. We note that Tüttö and Zawadowski[8] believe that one can speak of such a hybridization regardless of the sign of g_4, under the condition that the quantity $g_{4\text{eff}} = g_4 + 2g_3^2/[\varepsilon - E_0(p)]$ is negative. Tüttö and Zawadowski start here from the premise that the two-particle Green's function has a pole $1 - g_{4\text{eff}}F = 0$. Such a terminology, however, seems inconsistent to us. The point is that at $g_3 \neq 0$ the poles of the single-particle and two-particle Green's functions coincide, so that a distinction between the single-particle and two-particle excitations can be made only in the limit as $g_3 \to 0$. But then the condition for the existence of a two-particle branch is precisely a negative g_4. On the other hand the condition $g_{4\text{eff}} < 0$ is automatically satisfied for any part of the spectrum close to 2Δ.

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