# The nonlinear theory of the Buneman instability

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We construct exact nonlinear solutions which describe the evolution of the Buneman instability. We show that under well-defined initial conditions a self-similar solution is realized with an explosive increase with time of the amplitude of the potential; the formation of short-lived double layers which have been observed in numerical simulations is possibly connected with this solution.

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1. The standard scheme for studying the nonlinear dynamics of instabilities which is based upon the weak-turbulence approximation is inapplicable in the case of the so-called instabilities with stiff excitations even when the amplitude of the excited oscillations is finite barely above threshold. As an example of such type of instabilities one can adduce the modulational instability caused by the negative pressure of the plasmons (quanta of the Langmuir waves). As a result of the instability the plasmons are localized in cavitons—regions of a lowered plasma density. The localization is accompanied by the collapse of the cavitons to sizes where Landau damping of the plasmons trapped in the cavitons becomes important (Langmuir collapse[1]). Another, not less important example of an instability with hard excitation is the Buneman instability which is caused by the fact that the pressure in an electron current which moves relative to the ions at above-thermal velocity ($u_0 > (T/m)^{1/2}$) is negative.[2] Closely connected with the dynamics of the Buneman instability is the problem of the formation, in the plasma which is penetrated by the electron current, of double layers—regions where the charges are separated by distances of the order of the Debye radius and in the boundaries of which the electrical potential is changed by an appreciable amount comparable with or larger than the thermal energy of the particles.

For the occurrence of a double layer it is necessary that the electrons be accelerated in the direction in which the potential increases. This leads to a decrease in the electron density from the cathode end of layer to the anode end. This is possible only in the case where the electron current moves with superthermal speed (Bohm condition[3]), which is exactly the same as the condition for the occurrence of the Buneman instability. A numerical simulation of the instability[4] has shown that in its nonlinear stage the formation of short-lived nonstationary double layers occurs in which the drop in the potential rises with time explosively and reaches a magnitude which is at least two orders of magnitude larger than the thermal energy. Such behavior is well described by some particular self-similar solutions of the appropriate hydrodynamic equations,[4] but the problem of the conditions for reaching these solutions was not elucidated.

The aim of the present paper is to obtain exact nonlinear solutions which describe the evolution of the Buneman instability for arbitrary initial conditions. We show that under some well-defined initial conditions solutions with an explosive growth of the electrical potential in time are possible and agree with the numerical experiment and with the self-similar solutions found in Ref. 4.

2. For quasi-neutral perturbations the system of equations describing the Buneman instability has the form[4]

$$m_i\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}\right)=-e\frac{\partial\varphi}{\partial x}-\frac{T_i}{n}\frac{\partial n}{\partial x},\tag{1}$$

$$\frac{\partial n}{\partial t}+\frac{\partial}{\partial x}(nv)=0,\tag{2}$$

$$e\varphi=\frac{m_e I_e^2}{2}\left(\frac{1}{n^2}-\frac{1}{n_0^2}\right)+T_e\ln\frac{n}{n_0}.\tag{3}$$

Here $I_e$ is the electron current and $n_0$ the unperturbed electron density.

Since the characteristic time for the development of the instability is determined by the inertia of the ions, the process is slow in the scale of electron times and, as was already noted in Ref. 2, the flow of the electrons can be assumed to be stationary for quasi-neutral perturbations:

$$I_e=\text{const}.$$

Small deviations from stationarity are important only for non-quasi-neutral perturbations.

One can interpret Eqs. (1) to (3) as a system of equations of one-dimensional hydrodynamics with negative pressure (adiabatic index $\gamma=-2$), the solution of which can be obtained by means of a Legendre transformation (see Ref. 5). Introducing into the discussion the velocity potential $v=-\partial\psi/\partial x$ one can easily integrate the ion equation of motion. The integral of this equation is of the form

$$\frac{\partial\psi}{\partial t}+\frac{v^2}{2}+w=0,\tag{4}$$

where we have used the notation

$$w=\frac{e\varphi}{m_i}+\frac{T_i}{2m_i}\ln\frac{n^2}{n_0^2}=\frac{m_e I_e^2}{2m_i}\left(\frac{1}{n^2}-\frac{1}{n_0^2}\right)+\frac{T_e+T_i}{2m_i}\ln\frac{n^2}{n_0^2}.$$

The next step in getting the solution consists in changing to new independent variables, which we take to be $v$ and $w$. The function $\chi(v,w)$, defined by the relation

$$\chi=\psi-xv+t(v^2/2+w),$$

plays the role of generating function for that transforma-

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tion. Using (4) one shows easily that

$$d\chi=-(x-vt)\,dv+t\,dw,$$

and, thus, we have the equations

$$\frac{\partial\chi}{\partial w}=t,\quad v\frac{\partial\chi}{\partial w}-\frac{\partial\chi}{\partial v}=x, \qquad (5)$$

which we can consider as the formulae which change the old $(x, t)$ to the new $(v, w)$ independent variables of the problem.

The equation for the generating function $\chi(v,w)$ is obtained from the not yet used continuity equation (2), changing in it to the independent variables $v, w$. The result has the form

$$\frac{\partial\chi}{\partial w}+\frac{n}{dn/dw}\frac{\partial^2\chi}{\partial w^2}-\frac{\partial^2\chi}{\partial v^2}=0. \qquad (6)$$

The solution of the nonlinear set of Eqs. (1) to (3) has thus been reduced to the solution of the linear differential equation (6), for once we know the function $\chi(v,w)$ the transformation formulae (5) determine implicitly the solution of the problem $v=v(t,x)$, $w=w(t,x)$.

The solution of Eq. (6) is not particularly difficult. For the sake of simplicity we restrict ourselves to the case where the threshold for the Buneman instability is well exceeded, when

$$m_eI_e^2/2n^2\gg T_e+T_i.$$

Then

$$n=n_0(U/(w+U))^{1/2},\quad U=m_eI_e^2/2m_in_0^2.$$

In that case Eq. (6) can be written as follows:

$$2(w+U)\frac{\partial^2\chi}{\partial w^2}-\frac{\partial\chi}{\partial w}+\frac{\partial^2\chi}{\partial v^2}=0. \qquad (7)$$

Differentiating (7) with respect to $w$ and introducing the function $F=\partial\chi/\partial w$, we have for it the following equation in the variables $u=[2(w+U)]^{1/2}$ and $v'=iv$

$$\frac{\partial^2F}{\partial u^2}-\frac{\partial^2F}{\partial v'^2}=0.$$

The general solution of this equation has the form

$$\frac{\partial\chi}{\partial w}=f_1[v+i(2(w+U))^{1/2}]+f_2[v-i(2(w+U))^{1/2}]. \qquad (8)$$

Here $f_1$ and $f_2$ are arbitrary functions.

From (8) it follows that

$$\chi=\int_0^w dw'\{f_1[v+i(2(w'+U))^{1/2}]+f_2[v-i(2(w'+U))^{1/2}]\}+G(v).$$

One finds the form of the function $G(v)$ from Eq. (7). Evaluating $d^2G/dv^2$ from that equation and integrating over $v$ we get the following result for the function $\partial\chi/\partial v$ in which we are interested and which enters in the transformation equation (5):

$$\frac{\partial\chi}{\partial v}=\int_{i(2U)^{1/2}}^{v^+}dt\,f_1(t)+\int_{-i(2U)^{1/2}}^{v^-}dt\,f_2(t)-i(2(w+U))^{1/2}[f_1(v^+)-f_2(v^-)]; \qquad (9)$$

$$v^\pm=v\pm i(2(w+U))^{1/2}.$$

The functions $f_1, f_2$ are in each actual problem determined using the set of initial or boundary conditions.

In the present paper we consider two such problems. One of them, considered in the present section, is the time evolution of the oscillation mode which arises from an initial harmonic density and ion-velocity distribution. In such a solution the reflection of the electrons from the region of negative values of the potential, which plays the role of a potential energy hump, limits the growth in the amplitude of the potential $e\varphi_0$ to values comparable to the initial energy of the electron current $m_eu_0^2/2$.

In accordance with the above remarks we look for the solution of Eqs. (1) to (3), linearized in the amplitude of the perturbations, in the following form:

$$v=\nu_0e^{\delta t}\cos\varkappa x,$$
$$\frac{\delta n}{n_0}=-\frac{w}{2U}=\frac{\nu_0}{(2U)^{1/2}}e^{\delta t}\sin\varkappa x, \qquad (10)$$

$\delta=\varkappa(2U)^{1/2}$ is the growth rate of the instability.

It follows from the solution that in linear stage of the instability the following relation is satisfied:

$$t=\frac{\partial\chi_L}{\partial w}=\frac{1}{2\varkappa(2U)^{1/2}}\left\{\ln\frac{v-iw/(2U)^{1/2}}{\nu_0}+\ln\frac{v+iw/(2U)^{1/2}}{\nu_0}\right\}.$$

By matching with the given linearized solution we determine the functions $f_1, f_2$ in the nonlinear solution (8), (9). As a result we have

$$\frac{\partial\chi}{\partial w}=\frac{1}{2\varkappa(2U)^{1/2}}\left\{\ln\frac{v-i\xi}{\nu_0}+\ln\frac{v+i\xi}{\nu_0}\right\},$$
$$\frac{\partial\chi}{\partial v}=\frac{1}{\varkappa}\arcsin\frac{\xi}{(v^2+\xi^2)^{1/2}}-\frac{v}{\varkappa(2U)^{1/2}} \qquad (11)$$
$$+\frac{v}{2\varkappa(2U)^{1/2}}\left\{\ln\frac{v+i\xi}{\nu_0}+\ln\frac{v-i\xi}{\nu_0}\right\},$$

where we have used the notation

$$\xi(w)=(2(w+U))^{1/2}-(2U)^{1/2}.$$

Substituting (11) into the transformation equations (5) and inverting them, we get the following solution of the required problem:

$$v(t,x)=\nu_0e^{\delta t}\cos\left[\varkappa x-\frac{v(t,x)}{(2U)^{1/2}}\right], \qquad (12)$$

$$e\varphi/m_i={}^1/_2\left\{\nu_0e^{\delta t}\sin\left[\varkappa x-\frac{v(t,x)}{(2U)^{1/2}}\right]-(2U)^{1/2}\right\}^2-U, \qquad (13)$$

$$n=n_0\left(1+\frac{e\varphi}{m_iU}\right)^{-1/2}. \qquad (14)$$

The exponential growth with time of the perturbations is thus conserved also in the nonlinear stage of the instability, but for large amplitudes the symmetry between the positive and negative phases of the potential is lost (see Fig. 1). The limitation on the amplitude in the given regime is connected with the appearance of electrons which are reflected from the potential energy hump—an effect not taken into account in the original hydrodynamic equations (1) to (3). The condition for reflection of the electron current is

$$e\varphi_{min}\to-m_iU=-m_eu_0^2/2,\quad n_{max}\to\infty$$

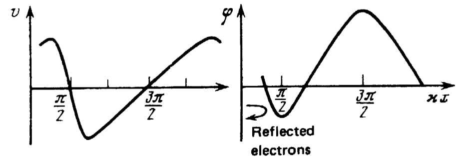


FIG. 1.

($u_0$ is the unperturbed current speed), and the corresponding maximum amplitude of the potential reaches the value $e\varphi_{max} = \frac{3}{2} m_e u_0^3$.

The given solution describes quite satisfactorily the initial evolution of the Buneman instability in the numerical experiment, the preliminary results of which were given in Ref. 4. In that experiment we studied the behavior of a plasma in a plane one-dimensional layer of size 100 Debye lengths under conditions of continuous injection into the layer of an electron current with speed $u_0 = 1.8 (T_e/m_e)^{1/2}$. The Buneman instability developed in the plasma and led to a growth of micropulsations of the electrical field with time. At a sufficiently large amplitude of the field the harmonic with a wavelength comparable to the length of the computing interval, which can be assumed with a high degree of accuracy to be quasineutral, splits off in the spectrum of the oscillations. The amplitude of the potential in this stage grows with time exponentially until there is a reflection of the electron current from the minimum of the potential. The maximum value of the amplitude of the potential in this stage increases to a magnitude $e\varphi_{max} = (5 \text{ to } 6) \times T_e$, which agrees with the analtyical result $e\varphi_{max} = \frac{3}{2} m_e u_0^2$.

This stage, the main consequence of which is the appearance of a large number of reflected electrons, plays the role so to speak of a preparation for the main (explosive) phase of the Buneman instability. The spatial distribution of the potential in this phase is typical for a double layer, i.e., the region of negative values which leads to a reflection of the electron current vanishes. The amplitude of the potential increases with time explosively, $\varphi_0 \propto (t_0 - t)^{-2}$, to values $e\varphi_{max} \sim (100 \text{ to } 300) \times T_e$, followed by a collapse of the instability. As a result of this process, a short-lived double layer is formed, at the limits of which the electrons are accelerated up to energies about two orders of magnitude larger than the injection energy.

3. In this section we consider the solution of the original set of Eqs. (1) to (3) in the form of dynamical potential bursts which have the character of an explosion in which the amplitude $e\varphi_0$ increases to values considerably exceeding the initial energy of the electron current. Such a solution is possible when there is no reflection of electrons from the potential energy hump and, hence, no negative phase in the spatial distribution of the potential. The initial conditions leading to explosive bursts of the potential correspond to the presence in the electron density distribution of a gap

$$n_e(0, x) = n_0 \left(1 - \varepsilon/2 \operatorname{ch} \frac{x}{L}\right), \quad \varepsilon \ll 1. \tag{15}$$

In the region of the gap, the electrons are accelerated; this is due to a local increase in the potential

$$e\varphi = \frac{m_e u_0^2}{2} \left( \frac{n_0^2}{n_e^2} - 1 \right) \approx \frac{m_e u_0^2}{2} \varepsilon / \operatorname{ch} \frac{x}{L}.$$

In the experiment, such regions of local electron acceleration are produced due to pinching in the electron current; to obtain this one uses a strong longitudinal magnetic field. Double layers just in the pinching regions occur in the first place (see Ref. 6).

We choose for the sake of simplicity the initial condition for the velocity in the form

$$v(0, x) = 0. \tag{15'}$$

Below we show that the main singularities of the solution that develops from the initial conditions (15), (15′), namely an explosion of the amplitude of the solution with time, a spatial distribution of the density and the velocity, and others is the same as in the numerical simulation.[4] We turn to obtaining the nonlinear solution.

It follows from the transformation equations (5) that the initial conditions (15), (15′) in the case considered lead to the following equations for $f_1, f_2$:

$$f_1[i(2(w+U))^{1/2}] + f_2[-i(2(w+U))^{1/2}] = 0, \tag{16}$$

$$\int\limits_{i(2U)^{1/2}}^{i(2(w+U))^{1/2}} dt f_1(t) + \int\limits_{-i(2U)^{1/2}}^{-i(2(w+U))^{1/2}} dt f_2(t) - i(2(w+U))^{1/2}$$
$$\times [f_1(i(2(w+U))^{1/2}) - f_2(-i(2(w+U))^{1/2})] = -L \operatorname{Arch} \frac{\varepsilon (w+U)^{1/2}/2}{(w+U)^{1/2} - U^{1/2}}. \tag{17}$$

Using the condition $f_2(it) = -f_1(-it)$ in (17) and changing to a new independent variable $\xi = [2(w+U)]^{1/2} - (2U)^{1/2}$, we get for the function $Z(\xi)$ which is defined by the relation

$$Z(\xi) = \int\limits_0^{\xi} dt f_1(it),$$

the following differential equation:

$$(\xi + (2U)^{1/2})^2 \frac{dZ}{d\xi} - Z = -i \frac{L}{2} \operatorname{Arch} \frac{\varepsilon (\xi + (2U)^{1/2})/2}{\xi}. \tag{18}$$

The solution of this equation leads to the following formulae for the derivatives $\partial\chi/\partial w$ and $\partial\chi/\partial v$:

$$\frac{\partial\chi}{\partial w} = -\frac{iL}{2(2U)^{1/2}} \{\operatorname{Arch} z_- - \operatorname{Arch} z_+\}, \tag{19}$$

$$\frac{\partial\chi}{\partial v} = -\frac{L}{2} \left\{ \left(1 + \frac{iv}{(2U)^{1/2}}\right) \operatorname{Arch} z_- + \left(1 - \frac{iv}{(2U)^{1/2}}\right) \operatorname{Arch} z_+ \right.$$
$$\left. - \frac{\varepsilon}{2} \arcsin z_-^{-1} - \frac{\varepsilon}{2} \arcsin z_+^{-1} \right\} - \frac{\pi L \varepsilon}{4}, \tag{20}$$

where

$$z_\pm = (U/2)^{1/2} \varepsilon [(2(w+U))^{1/2} \pm iv - (2U)^{1/2}]^{-1}.$$

The terms in the formula for $\partial\chi/\partial v$ which are proportional to $\varepsilon$ are important only in the vicinity of the singularity, when the amplitude of the solution is rather large. Hence we can simplify them, assuming that $|z_\pm^{-1}| \gg 1$. In that case

$$\arcsin z_-^{-1} = i^{-1} \ln (2iz_-^{-1}), \quad \arcsin z_+^{-1} = i^{-1} \ln (iz_+/2),$$

where the branches of the arcsin function are chosen using the condition that the solution must be single-valued in $x$. This solution is determined by the transformation formulae which connect the variables $x$ and $t$ with the hydrodynamic characteristics of the plasma $n$ and $v$:

$$t = -i \frac{L}{2(2U)^{1/2}} \{\operatorname{Arch} z_- - \operatorname{Arch} z_+\}, \tag{21}$$

$$x - \frac{iL\varepsilon}{4} \ln \frac{z_+}{z_-} = \frac{L}{2} \{\operatorname{Arch} z_- + \operatorname{Arch} z_+\}. \tag{22}$$

Inverting the transformation formulae we get the final solution of the problem considered:

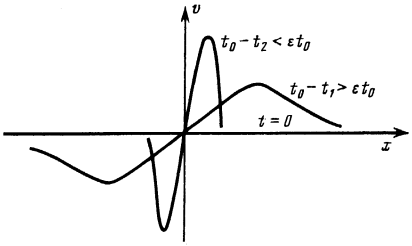


FIG. 2.

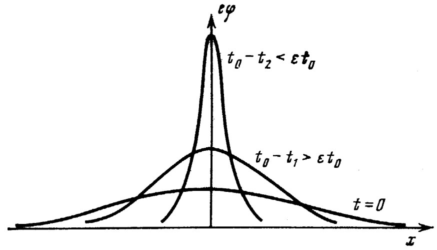


FIG. 3.

$$(2(w+U))^{1/2}-(2U)^{1/2}=(2U)^{1/2}\left(\frac{n_0}{n}-1\right)$$

$$=\varepsilon\left(\frac{U}{2}\right)^{1/2}\mathrm{ch}\frac{x^*}{L}\cos\frac{(2U)^{1/2}}{L}t\left[\mathrm{ch}^2\frac{x^*}{L}\cos^2\frac{(2U)^{1/2}}{L}t+\mathrm{sh}^2\frac{x^*}{L}\sin^2\frac{(2U)^{1/2}}{L}t\right]^{-1}, \quad (23)$$

$$v=\varepsilon\left(\frac{U}{2}\right)^{1/2}\mathrm{sh}\frac{x^*}{L}\sin\frac{(2U)^{1/2}}{L}t\left[\mathrm{ch}^2\frac{x^*}{L}\cos^2\frac{(2U)^{1/2}}{L}t+\mathrm{sh}^2\frac{x^*}{L}\sin^2\frac{(2U)^{1/2}}{L}t\right]^{-1}, \quad (24)$$

$$x^*=x-\frac{L\varepsilon}{2}\mathrm{arctg}\frac{v}{(2(w+U))^{1/2}-(2U)^{1/2}}.$$

The main features of the solution given here is the absence of a negative phase of the potential (and hence of reflected electrons) and the explosion of the amplitude of the solution with time. The time of the explosion is $t_0=\pi L/2(2U)^{1/2}$, and close to the singularity ($x^* \ll L$, $t \approx t_0$) the solution can be simplified and written in the form

$$\frac{n_0}{n}-1=\frac{\varepsilon}{2}\frac{\pi(t_0-t)}{2t_0}\left[\frac{x^{*2}}{L^2}+\frac{\pi^2(t_0-t)^2}{4t_0^2}\right]^{-1}, \quad (25)$$

$$v=\varepsilon\left(\frac{U}{2}\right)^{1/2}\frac{x^*}{L}\left[\frac{x^{*2}}{L^2}+\frac{\pi^2(t_0-t)^2}{4t_0^2}\right]^{-1}, \quad (26)$$

$$x^*=x-\frac{L\varepsilon}{2}\mathrm{arctg}\frac{x^*/L}{\pi(t_0-t)/2t_0}. \quad (27)$$

The nature of the solution depends essentially on how closely we approach the singularity, i.e., in fact on the magnetide of the deviation $t_0 - t$. When $(t_0 - t)/t_0 \gg \varepsilon$ the last term on the right-hand side of Eq. (27) is negligibly small, i.e., $x^* \approx x$ and Eqs. (25), (26) describe the explosive growth of the amplitude of the density variation and the velocity with time

$$\frac{|\delta n|}{n_0}\sim\frac{t_0}{t_0-t} \qquad \frac{v}{(U)^{1/2}}\sim\frac{t_0}{t_0-t}$$

while the characteristic spatial dimension decreases linearly with time

$$x\sim L\frac{t_0-t}{t_0}.$$

In the opposite limiting case $(t_0 - t)/t_0 \ll \varepsilon$ the term $x^*$ in Eq. (27) is negligibly small, i.e.,

$$\frac{x^*}{L}=\frac{\pi}{2}\frac{t_0-t}{t_0}\mathrm{tg}\frac{2x}{L\varepsilon}$$

and we get from Eqs. (25) and (26) a self-similar solution with separable variables

$$\frac{n_0}{n}\approx\frac{\varepsilon}{\pi}\frac{t_0}{t_0-t}\cos^2\frac{2x}{L\varepsilon}, \quad (28)$$

$$v\approx\frac{\varepsilon}{\pi}\left(\frac{U}{2}\right)^{1/2}\frac{t_0}{t_0-t}\sin\frac{4x}{L\varepsilon}, \quad (29)$$

$$e\varphi=2m_iU\frac{n_0^2}{n^2}. \quad (30)$$

The dynamics of the spatial velocity and potential distribution is illustrated in Figs. 2 and 3 for the solution obtained here and the law for the explosive growth of the potential amplitude $\varphi \sim (t_0 - t)^{-2}$ and the spatial distribution of the potential agree well both with the results of the numerical simulation of Ref. 4 and with the self-similar solution found there.

We have shown in the present paper that such a solution can be realized, traced the dynamics of reaching the self-similar regime, and we found the connection of the main characteristics of the solution with the initial conditions.

We must emphasize that the solution (25) to (27) is not applicable in the immediate vicinity of the singularity $t \to t_0$, since the quasineutrality condition $\partial^2\varphi/\partial x^2 \ll 4\pi en$, used in the initial Eqs. (1) to (3), is then violated. One shows easily, using the solution (25) to (27), that the condition for the violation of the quasineutrality has the same form

$$\frac{t_0-t}{t_0}\leqslant\left(\varepsilon\frac{u_0^2}{\omega_{0e}^2L^2}\right)^{1/3}$$

as in the case when this violation occurs in the initial phase of the self-similar solution when $t_0 - t \gg \varepsilon t_0$; $\omega_{0e} = (4\pi e^2 n_0/m_e)^{1/2}$ is the plasma frequency.

Finally, the solution obtained for the potential is reversible ($\varphi \to 0$ as $x \to \pm\infty$). Irreversibility, the jump in the potential which is characteristic of a double layer, arises in those cases when the energy for the particle acceleration in the double layer can be borrowed either from an external current generator[4] or from an external electrical field (fixed potential difference at the boundaries of the plasma layer).


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