Tunneling transformation of whistler waves in an inhomogeneous plasma

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We consider the propagation of whistler waves in a stratified plasma in a constant magnetic field which is directed at right angles to the density gradient. We make a detailed study of the corresponding WKB solutions of the Maxwell equations, on the basis of which we elucidate the conditions for the transformation of different wave branches into one another. We find expressions for the transformation coefficients. In the framework of the WKB method we obtain a complete solution of the problem of the passage of a whistler wave through a smooth transition layer taking into account transformation effects. Using these results we discuss the conditions for waveguide propagation of whistler waves and we indicate the ensuing consequences for nonlinear self-focusing.

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1. INTRODUCTION

The present paper is devoted to the study of some effects which arise when whistler waves propagate in an inhomogeneous plasma; these effects (called tunneling transformation in what follows) are in our opinion of basic interest, and have also have a value for applications to a number of problems connected with the propagation of whistlers in the magnetosphere, of plasma waveguides, in self-focusing, and so on.

In a magnetoactive plasma the component of the wave vector along the direction of an inhomogeneity is, in general, not a unique function of the parameters of the medium so that one can speak of several branches of oscillations. The properties of these branches, for example, the conditions for total internal reflection, the direction of the group velocity, and so on, are different. If the points where one branch changes to another lie on the real axis, a wave transformation occurs in those points and is characterized by a change in some of its properties. In a number of cases, however, these points lie in the complex plane. In the framework of geometric optics the transformation of one branch into another is then impossible. Nonetheless, from the point of view of wave theory these branches are not isolated: if at $x=-\infty$ there exists one branch, as $x\to\infty$ we get a superposition of several branches. When using the WKB method one can observe transitions between branches, using an approach called in quantum mechanics the "method of complex classical trajectories" (Ref. 1, § 52). Among a number of effects considered by this method we have, for instance, above-barrier reflection and various kinds of tunneling transitions related to the processes studied in the present paper. This is the reason for the terminology suggested by us.

An exact statement of the problem and a derivation of the basic equations in a form convenient for us, and also their solution by the WKB method, are contained in Sec. 2. Here we also study the different branches of the oscillations. In Sec. 3 we consider the case when the point where two branches merge lies on the real axis. The process taking place in the vicinity of this point is the total internal reflection of a wave, accompanied by its transition to another branch. In Sec. 4 we consider the case where the transition points lie in the complex plane, i.e., when a tunneling transformation is possible. We obtain general expressions for the transformation coefficients. We consider in Sec. 5 a typical example of reflection and refraction of a wave incident upon a smooth transition layer with decreasing density, and evaluate on the basis of the general theory all the transformation coefficients. Using these results we discuss in Sec. 6 the conditions for waveguide propagation of whistler waves and the consequences following from this for nonlinear self-focusing. The effects of the tunneling transformation lead here to results different from those obtained earlier on the basis of geometric optics or of a parabolic equation (in the theory of self-focusing—the non-linear Schrödinger equation). In particular, it turns out that in channels with an increased plasma density, oriented along an external magnetic field, waveguide propagation is, strictly speaking, impossible for any frequencies, owing to the tunneling transformation. (From the point of view of the "classical" theory[2] such channels are ideal waveguides at $\omega < \omega_H/2$.) If, however, the width of the channel is sufficiently large compared to the longitudinal wavelength and frequency is not too close to $\omega_H/2$, then the energy leakage is small so that one can speak of "quasi-waveguides." The quantitative results obtained by us enable us to estimate simply their quality which vanishes as $\omega\to\omega_H/2$.

2. BASIC EQUATIONS AND THEIR SOLUTION BY THE WKB METHOD

We assume that the plasma density and hence the electron Langmuir frequency ω_p is a function of a single coordinate, say x, while the electron gyro-frequency $\omega_H=\text{const}$. We choose the z-axis in the direction of the external magnetic field. The electric field of a monochromatic wave of frequency ω [time dependence $\exp(-i\omega t)$] satisfies the equation

$$\operatorname{grad}\operatorname{div}\mathbf{E}-\nabla^2\mathbf{E}=\omega^2c^{-2}(\hat{\varepsilon}\mathbf{E}), \tag{2.1}$$

where $\hat{\varepsilon}$ is the permittivity tensor

$$\hat{\varepsilon}=\begin{pmatrix}\varepsilon & -ig & 0\\ ig & \varepsilon & 0\\ 0 & 0 & \eta\end{pmatrix},$$

the components of which at $\omega_H(m_e/m_i)^{1/2}\ll\omega<\omega_H \ll\omega_p$ are equal to

$$\varepsilon=\alpha/(1-u^2),\quad g=\alpha/u(1-u^2),\quad \eta=-\alpha/u^2, \tag{2.2}$$

where

$$\alpha=\omega_p^2/\omega_H^2\gg1,\quad u=\omega/\omega_H<1. \tag{2.3}$$

Solving Eq. (2.1) in Cartesian coordinates and dropping the factor $\exp(-i\omega t)$, we look for the components E_x and E_y in the form

$$E_x=f(x)\exp(i\omega pz/c),\quad E_y=iF(x)\exp(i\omega pz/c). \tag{2.4}$$

We then put from (2.1) the following basic set of equations:

$$\begin{gathered}\frac{d^2\Phi}{dx^2}-\frac{d}{dx}(\ln\alpha)\frac{d\Phi}{dx}=-\frac{\omega^2}{c^2}\frac{\eta}{\varepsilon}[(\varepsilon-p^2)\Phi+p^2gF],\\ \frac{d^2F}{dx^2}=-\frac{\omega^2}{c^2}\frac{1}{\varepsilon}\{g\Phi+[\varepsilon(\varepsilon-p^2)-g^2]F\},\end{gathered} \tag{2.5}$$

where

$$\Phi(x)=\varepsilon f(x)+gF(x). \tag{2.6}$$

From the equations $\operatorname{div}(\hat{\varepsilon}\mathbf{E})=0$ and $\mathbf{H}=-i(c/\omega)\operatorname{curl}\mathbf{E}$ it follows that

$$\begin{gathered}E_z=i\frac{c}{\omega}\frac{1}{p\eta}\frac{d\Phi}{dx}\exp\left(i\frac{\omega}{c}pz\right),\quad H_z=\frac{c}{\omega}\frac{dF}{dx}\exp\left(i\frac{\omega}{c}pz\right),\\ H_x=-ipF(x)\exp\left(i\frac{\omega}{c}pz\right),\quad H_y=\frac{1}{p}\Phi(x)\exp\left(i\frac{\omega}{c}pz\right).\end{gathered} \tag{2.7}$$

We shall assume the medium to be slowly varying, assuming that $\alpha=\alpha(\mu x)$, where μ is a small parameter. After changing to a new argument $w=\mu x$ we look for a solution of the set (2.5) by the WKB method in the form of the vector

$$\psi(w)=\begin{Bmatrix}\Phi(w)\\ F(w)\end{Bmatrix}=\varphi(w)\exp\left[\frac{i}{\mu}\frac{\omega}{c}\int q(w)dw\right], \tag{2.8}$$

where $\varphi(w)$ is a two-component vector which we can expand in powers of μ:

$$\varphi(w)=\varphi^{(0)}(w)+\mu\varphi^{(1)}(w)+\mu^2\varphi^{(2)}(w)+\dots \tag{2.9}$$

From the set (2.5) we get for the components of $\varphi^{(0)}(w)$ a set of algebraic equations which are the same as the equations for the vector $\varphi(w)$ in a homogeneous medium:

$$\begin{pmatrix}\eta(\varepsilon-p^2) & p^2g\eta\\ g & \varepsilon(\varepsilon-p^2)-g^2\end{pmatrix}\varphi^{(0)}(w)=\varepsilon q^2\varphi^{(0)}(w). \tag{2.10}$$

Equating the determinant of this set to zero we get the following expression for q^2:

$$q_{1,2}^2=-\frac{\alpha}{u^2}+\left(\frac{1}{2u^2}-1\right)p^2\mp\frac{p}{2u^2}(p^2-4\alpha)^{1/2} \tag{2.11}$$

(the minus sign corresponds to q_1, the plus sign to q_2). The corresponding refractive indices are

$$n_{1,2}=(p^2+q_{1,2}^2)^{1/2}=\frac{1}{2u}[p\mp(p^2-4\alpha)^{1/2}], \tag{2.12}$$

so that

$$n_1,n_2=\alpha/u^2=|\eta|. \tag{2.13}$$

Using (2.11) and (2.12) we find the solution of the set (2.10) and it is convenient for us to write it in the form

$$\varphi^{(0)}(q_k)=C(w)q_k^{-1/2}(p^2-4\alpha)^{-1/4}\chi(q_k),\quad k=1,2, \tag{2.14}$$

where $C(w)$ is an arbitrary scalar factor and $\chi(q_k)$ the two-dimensional vector

$$\chi(q_k)=\begin{Bmatrix}pn_k\\ 1\end{Bmatrix}[p-(-1)^k(p^2-4\alpha)^{1/2}]^{1/2}, \tag{2.15}$$

and the w-dependence of q_k is given by Eq. (2.11), where $\alpha=\alpha(w)$.

The scalar factors in (2.14) and (2.15) are chosen such that the x-component of the Poynting vector $\Pi=(c/8\pi)\operatorname{Re}[\mathbf{E}\times\mathbf{H}^*]$ has the simplest expression. Indeed, substituting (2.7) into Π_x we get

$$\Pi_x=\frac{c^2}{8\pi\omega}\operatorname{Im}\left[F\frac{dF^*}{dx}-\frac{1}{p^2\eta}\Phi^*\frac{d\Phi}{dx}\right]. \tag{2.16}$$

Changing to the variable w and evaluating (2.16) in the zeroth approximation in μ by means of (2.8), (2.9), (2.14), and (2.15) we find, using Eqs. (2.12) and (2.13), that

$$\Pi_x(q_k)=\frac{c}{4\pi}\frac{q_k}{|q_k|}(-1)^{k-1}|C(w)|^2. \tag{2.17}$$

It follows from the equation $\operatorname{div}\Pi=0$ that in our case $\Pi_x=\text{const}$. From (2.17) we then get the identity $C(w)=\text{const}$ [it is clear from (2.8) that without loss of generality we can assume the phase of $C(w)$ to be constant]. We note that this identity is also obtained from the condition that the set of equations for the components of $\varphi^{(1)}(w)$ from (2.9), which we can obtain from the set (2.5) in the next order in μ, has a solution.

The general solution of the set (2.5) in the lowest WKB approximation can thus be written as a linear combination of solutions of the form

$$\psi(x)=\begin{Bmatrix}\Phi(x)\\ F(x)\end{Bmatrix}=Cq_k^{-1/2}(p^2-4\alpha)^{-1/4}\chi(q_k)\exp\left[\pm i\frac{\omega}{c}\int q_k dx\right], \tag{2.18}$$

where $C=\text{const}$, while $\chi(q_k)$ is given by Eq. (2.15).

We note also that it follows from (2.17) that the Poynting vector and the wave vector have in the same direction when $q=q_1$ and are oppositely directed when $q=q_2$ relative to the external magnetic field.

It follows from Eq. (2.11) that the graphs of the transverse wave number q as functions of α have at constant p and u the form shown in Fig. 1,

$$\alpha_0=p^2u(1-u),\quad q_0^2=\frac{p^2}{u^2}\left(\frac{1}{4}-u^2\right), \tag{2.19}$$

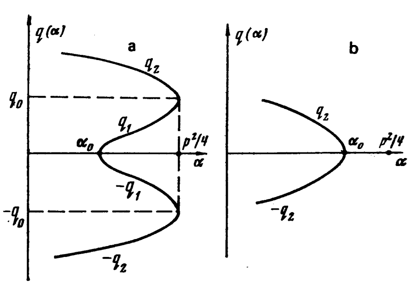

FIG. 1. Transverse wave number q as function of the dimensionless density α when $u<1/2$ (a) and when $u>1/2$ (b).

with $\alpha_0 \gg 1$ [see (2.3)].

To find the conditions for the applicability of the WKB solution (2.18) we substitute it into the set (2.5). Dropping terms containing α'' and α'^2, we require that after cancelling the exponential factor there remain only those terms which leads to expression (2.11) for q^2. Taking into account the inequalities [see Fig. 1 and Eq. (2.11)] $|q_1| \leq |q_2|$ ($u<\frac{1}{2}$) and $|q_2| \leq |q_1|$ ($u>\frac{1}{2}$), we find that the above-mentioned requirement is satisfied, if

$$\left|\frac{d(\ln\alpha)}{dx}\right| \ll 2\frac{\omega}{c}|q_k|\frac{|p^2-4\alpha|}{|p+(p^2-4\alpha)^{1/2}|p}, \tag{2.20}$$

where $k=1$ when $u<\frac{1}{2}$ and $k=2$ when $u>\frac{1}{2}$.

This is the required condition for the applicability of the WKB solution (2.18). It is clear that it is not satisfied near the special points where $\alpha=\alpha_0$ and $\alpha=p^2/4$ (Fig. 1).

In the vicinity of the point where $\alpha=\alpha_0$ we have according to (2.11)

$$q_k^2=2(\alpha-\alpha_0)/u(1-2u)+O((\alpha-\alpha_0)^2) \quad (k=1,2),$$

and inequality (2.20) takes the form

$$\left|\frac{d}{dx}(\alpha-\alpha_0)^{1/2}\right| \ll \frac{\omega}{c}p^2|1-2u|^{1/2}. \tag{2.21}$$

On the other hand, in the vicinity of the point where $\alpha=p^2/4$ and $|q_1|=|q_2|=|q_0|$ (Fig. 1) we find from (2.20) when $u<\frac{1}{2}$[1)]

$$\left|\frac{d\alpha}{dx}\right| \ll \frac{1}{4}\frac{\omega}{c}\frac{p}{u}|p^2-4\alpha|(1-4u^2)^{1/2}. \tag{2.22}$$

It follows from (2.21) and (2.22) that the region of applicability of the WKB solution in the vicinity of the singular points narrows down as $u\to\frac{1}{2}$.

3. REFLECTION FROM THE POINT WHERE $\alpha=p^2/4$

We consider a wave with $u<\frac{1}{2}$ propagating in the direction of increasing density in the vicinity of the singular point where $\alpha=p^2/4$ (Fig. 1a). Without loss of generality we may assume that this point is the point $x=0$ near which

$$\alpha(x)=p^2/4+x/D. \tag{3.1}$$

It follows from (2.11) that for $x<0$ the values of $q_{1,2}^2$ are real and for $x>0$ complex conjugate, and we assume that

$$(p^2-4\alpha)^{1/2}=2(-x/D)^{1/2}>0 (x<0), \quad (p^2-4\alpha)^{1/2}=2i(x/D)^{1/2} \quad (x>0). \tag{3.2}$$

Let $q=q_1$ in the incident wave. We shall look for the WKB solution of the set (2.5) for $x<0$ in the form [see (2.18)]

$$\psi(x)=C_1q_1^{-1/2}(p^2-4\alpha)^{-1/4}\chi(q_1)\exp\left(i\frac{\omega}{c}\int_0^x q_1\,dx\right)$$
$$+C_2q_2^{-1/2}(p^2-4\alpha)^{-1/4}\chi(q_2)\exp\left(i\frac{\omega}{c}\int_0^x q_2\,dx\right), \tag{3.3}$$

where the first term corresponds to the incident wave and the second one to the reflected wave [see the remark after Eq. (2.18)]. The coefficients C_1 and C_2 must be chosen such that when $x>0$ (3.3) changes into the wave which is damped as $x\to\infty$. At sufficiently large positive x the latter must have in agreement with (3.2) the form

$$\psi(x)=Cq_2^{-1/2}(p^2-4\alpha)^{-1/4}\chi(q_2)\exp\left(i\frac{\omega}{c}\int_0^x q_2\,dx\right). \tag{3.4}$$

To determine the connection between the coefficients C_1, C_2, and C we must trace how the function (3.4) changes when we change continuously from $x>0$ to $x<0$. As, however, the expressions (3.3) and (3.4) are not valid for small x, this transition must proceed in the complex plane, as is done in Zwaan's method (see, e.g., Ref. 1).

To simplify the calculations we assume temporarily that the singular point lies in the complex plane, putting in (3.1)

$$x\to x-i\varepsilon, \tag{3.5}$$

where $\varepsilon>0$ is a small quantity; after finishing the calculations we let $\varepsilon\to 0$.

We now introduce a new variable

$$w=(4/p^2D)(x-i\varepsilon) \tag{3.6}$$

and assume that $-\pi<\varphi=\arg w\leq\pi$.

We show at the end of this section that the WKB approximation is valid also when $|w|\ll 1$, provided p^2D is sufficiently large. For small $|w|$ we get from (2.11), using (3.1), (3.5), and (3.6)

$$q_{1,2}\approx\frac{p(1-4u^2)^{1/2}}{2u}\left[1-\frac{w}{2(1-4u^2)}\pm\frac{(e^{i\pi}w)^{1/2}}{1-4u^2}\right]. \tag{3.7}$$

One checks easily that as $\varepsilon\to 0$ the values of $q_{1,2}$ on the real axis will be in exact agreement with the rule for the signs in (3.2).

We change in the solution (3.3) and (3.4) to the variable w and make the transition above the singular point for $|w|=\text{const}\ll 1$ and for increasing argument of w from zero to π (Fig. 2). In that case $(e^{i\pi}/w)^{1/2}\to(|w|e^{2i\pi})^{1/2}=-|w|^{1/2}=-(e^{i\pi}w)^{1/2}$, i.e.,

$$q_1\to q_2, \quad q_2\to q_1, \quad (p^2-4\alpha)^{-1/4}\to e^{-i\pi/2}(p^2-4\alpha)^{-1/4}. \tag{3.8}$$

Correspondingly, expression (3.4) changes into the first term of (3.3) with a coefficient $C_1=C\exp(-i\pi/2)$.

The second term in (3.3) arises as a result of the Stokes phenomenon (see, e.g., Ref. 3). To understand the cause of this phenomenon we determine the Stokes line for our problem, taking into account its specific properties. To do this we introduce the quantity

$$K=\exp\left[\frac{i\omega}{c}\int_{i\varepsilon}^{x}(q_1-q_2)\,dx\right], \tag{3.9}$$

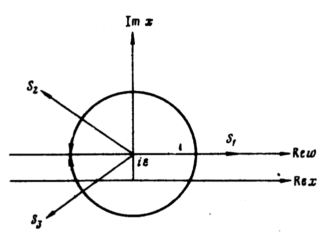

FIG. 2. Direction of contours and Stokes lines in the vicinity of the point where $\alpha=p^2/4$.

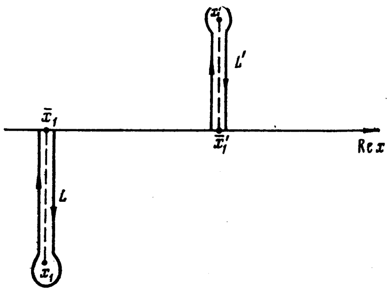

FIG. 3. Integration contours for the evaluation of the tunneling transformation coefficients.

which is the ratio of the exponentials of the corresponding WKB solutions (2.18). For small $|w|$

$$K=\exp\left[\frac{\omega}{c}\frac{p^3 D}{6u(1-4u^2)^{1/2}}w^{3/2}\right]. \tag{3.10}$$

We now define the Stokes line as the lines where K is real, i.e., according to (3.10), where $\sin(3\varphi/2)=0$. For small $|w|$ these lines are three rays S_k ($k=1,2,3$) (Fig. 3) corresponding to the angles $\varphi_1=0$, $\varphi_2=2\pi/3, \varphi_3=-2\pi/3$. It also follows from (3.10) that on the lines S_2 and S_3 the modulus of K is a minimum and on the line S_1 a maximum. According to the meaning of the Stokes phenomenon this means that for the above-indicated motion in the upper semi-circle (Fig. 2) on the line S_2 the solution with $q=q_1$ "appears" and changes into the solution with $q=q_2$ as $\varphi\to\pi$. This explains the appearance of the second term in (3.3). To find its coefficient we shall move along the lower semi-circle in Fig. 2 where φ changes from zero to $-\pi$. In that case $(e^{i\pi}w)^{1/2}\to|w|^{1/2}=(e^{i\pi}w)^{1/2}$, i.e., $q_k\to q_k$ ($k=1,2$), $(p^2-4\alpha)^{-1/4}\to(p^2-4\alpha)^{-1/4}$ and expression (3.4) changes into the second term in (3.3) with coefficient $C_2=C$. However, the first term in (3.3) (the coefficient C_1 of which we have already found) arises when we cross the Stokes line S_3.

Putting now $\varepsilon\to 0$ in (3.5) we obtain for real x the solution given by Eqs. (3.3) and (3.4).

The arguments given above do, of course, not contradict the uniqueness principle: the form of the asymptotic solution (3.3) is independent of the direction in which we go round the singular point, as the latter is only the point where the WKB approximation is inapplicable, but is not a singular point of the initial set of Eqs. (2.5). The only difference between the two transitions is that the first one enables us to determine C_1 and the second to determine C_2.

The energy fluxes in the incident and reflected waves in (3.3) are equal. Indeed, according to (2.17)

$$|\Pi_x(q_1)|=|\Pi_x(q_2)|=\frac{c}{4\pi}|C|^2, \tag{3.11}$$

i.e., in total internal reflection occurs the case considered.

Finally we point out that the condition for the applicability of the WKB approximation (2.22) at $|w|\ll 1$ takes in this case the form

$$|w|\gg 4\frac{c}{\omega}\frac{u}{D}(1-4u^2)^{-1/2}p^{-3}. \tag{3.12}$$

For small values of the difference $\frac{1}{2}-u$ this condition is very stringent.

We can similarly solve the problem of incidence of a wave with $q=-q_2$ upon the point where $\alpha=p^2/4$. In that case $q=-q_1$ for the reflected wave and for the field beyond the point of reflection [i.e., in Eq. (3.4)] $q=-q_1$. Here, too, total internal reflection occurs.

In both cases the change in q in the vicinity of the reflection point takes place continuously in agreement with Fig. 1a.

4. TRANSFORMATION OF WAVES CAUSED BY THE TUNNELING TRANSITION

The process considered in the preceding section can be considered to be a complete transformation of the branches $q_k\to q_i$ ($i\neq k$) at the point where $\alpha=p^2/4$. It is clear from the method described above that the transformation of q_k into q_i occurs in those cases when the branch point lies in the complex plane. The process $q_k\to q_i$ can then be considered to be the result of the passage of a ray through the complex plane, similar to the classical trajectory of a particle when passing through a potential barrier. We can therefore say that the transformation of the modes in the case considered is caused by a tunneling transition.

We consider, for instance, the tunneling transformation of a wave with $q=q_1$ and $u<\frac{1}{2}$ propagating in the positive direction of the x-axis. The asymptotic behavior of the wave field as $x\to\infty$ then has the form

$$Cq_1^{-1/2}(p^2-4\alpha)^{-1/4}\chi(q_1)\exp\left(i\frac{\omega}{c}\int_{x_0}^{x}q_1\,dx\right), \tag{4.1}$$

where x_0 is an arbitrary point. We shall look for the asymptotic behavior as $x\to-\infty$ in the form

$$C_1q_1^{-1/2}(p^2-4\alpha)^{-1/4}\chi(q_1)\exp\left(i\frac{\omega}{c}\int_{x_0}^{x}q_1\,dx\right)$$
$$+C_2q_2^{-1/2}(p^2-4\alpha)^{-1/4}\chi(q_2)\exp\left(i\frac{\omega}{c}\int_{x_0}^{x}q_2\,dx\right). \tag{4.2}$$

The first term in this expression corresponds to a wave propagating in the positive direction while the second term according to the remark following (2.18) describes a wave propagating in the negative direction.

To find the coefficients in (4.2) we continue (4.1) into the lower half-plane and afterwards go to $x\to-\infty$ along a contour enclosing the point x_1 closest to the real axis, where $\alpha(x_1)=p^2/4$. After going along the contour indicated the function $q_1(x)$ changes into $q_2(x)$ and (4.1) goes over into the second term of (4.2). Taking for the sake of convenience $x_0=\bar{x}_1=\mathrm{Re}\,x_1$ we find that

$$C_2=C\exp\left(i\frac{\pi}{2}+i\frac{\omega}{c}\int_L q_1\,dx\right), \tag{4.3}$$

where the contour L is drawn in Fig. 3. When making this transition we certainly intersect a Stokes line, as a result of which the first term in (4.2) arises when a coefficient C_1. We can determine the latter by moving along the real axis. This gives

$$C_1=C. \tag{4.4}$$

The transformation coefficient, i.e., the ratio of the energy fluxes in the reflected and incident waves, is determined by means of (2.17). After simple transformations we get

$$T=\left|\frac{C_2}{C}\right|^2=\exp\left[-2\frac{\omega}{c}\operatorname{Im}\int_{\bar{x}_1}^{x_1}(q_1-q_2)\,dx\right]. \qquad (4.5)$$

We now show that $T<1$. To do this we consider the function

$$\psi(y)=\operatorname{Im}\int_{\bar{x}_1}^{\bar{x}_1+iy}(q_1-q_2)\,dx.$$

It is clear that $\psi'(y)=\operatorname{Re}(q_1-q_2)$. It then follows from the definition of x_1 that $\psi(y)$ reaches an extremum in the point $y=\operatorname{Im}x_1$ while in the range $\operatorname{Im}x_1\leqslant y<0$ the function $\psi(y)$ is monotonic. As $\psi(0)=0$ and $\psi'(0)=q_1-q_2<0$ (this is clear from Fig. 1a), we have $\psi(y)>0$ for $\operatorname{Im}x_1\leqslant y<0$. Hence, the argument of the exponential in (4.5) is negative, i.e., $T<1$, i.e., the second term in (4.2) is exponentially small compared to the first one.[2)]

We note that one can give a reason for a shift of the contour into the lower half-plane to find C_2 by considering the transition from (4.2) to (4.1). When moving into the lower half-plane the ratio of the first term in (4.2) to the second, which is proportional to

$$\exp\left[i\frac{\omega}{c}\int_{x_0}^{x}(q_1-q_2)\,dx\right],$$

decreases and this makes it possible to "notice" the second term, notwithstanding the smallness of C_2 (cf. Ref. 1, §47).

One can similarly also consider other cases, for instance, the transformation of a wave with $q=-q_1$, incident in the negative direction, into a wave with $q=-q_2$ where, due to (2.17) the energy flux is positive. In this case the appropriate asymptotic behavior of the wave as $x\to-\infty$ is obtained from (4.1) by replacing i by $-i$. However, in that case one must continue the passing wave into the upper half-plane along a contour enclosing the singular point x_1' closest to the real axis in the upper half-plane where $q_1=q_2$. As a result we are led to an asymptotic behavior as $x\to\infty$ obtained from (4.2) by replacing i by $-i$. The transformation coefficient now has the form

$$T=\left|\frac{C_2}{C_1}\right|^2=\exp\left[2\frac{\omega}{c}\operatorname{Im}\int_{\bar{x}_1'}^{x_1'}(q_1-q_2)\,dx\right], \qquad (4.6)$$

where $\bar{x}_1'=\operatorname{Re}x_1'$. The smallness of the coefficient T is proved in the same way as before.

One can similarly consider the case of the singular point where $q_1(x)=0$. If that point lies on the real axis, total internal reflection of the wave with $q=q_1$ occurs in it (Fig. 1a). If, however, it lies in the upper half-plane, we are, by analogy of the earlier considerations, led to an incomplete $q_1\to-q_1$ transformation, i.e., if the wave has the asymptotic behavior (4.1) as $x\to\infty$, in the asymptotic expression as $x\to-\infty$ there occurs an additional term proportional to

$$\exp\left(-i\frac{\omega}{c}\int_{x_0}^{x}q_1\,dx\right).$$

This process is completely analogous to above-barrier reflection[1,4,5] and the transformation (reflection) coefficient now has the form

$$T=\exp\left(-4\frac{\omega}{c}\operatorname{Im}\int_{\bar{x}_1}^{x_1}q_1\,dx\right),$$
$$\bar{x}_1=\operatorname{Re}x_1,\quad \operatorname{Im}x_1>0,\quad q(x_1)=0. \qquad (4.7)$$

In particular, if $q_1(x)$ vanishes on the real axis, $x_1=\bar{x}_1$ and $T=1$. In that case, the region bordering the point x_1 can also be studied using Airy functions.

If, however, the point where $q_1(x)=0$ lies in the lower half-plane, it corresponds to the transformation $-q_1\to q_1$. The corresponding coefficient T is given by a formula similar to (4.7).

In a similar fashion one considers the case[3)] when $q_2(x)=0$ for $u>\frac{1}{2}$.

When there are other singularities present of the functions $q_k(x)$ (such as poles) it is necessary to take them also into account provided they lie no farther from the real axis than the singularities considered above.

5. REFLECTION AND REFRACTION OF A WAVE INCIDENT ON A SMOOTH TRANSITION LAYER

An interesting and important example in which the above results can be used is the incidence of a wave upon a smooth transition layer with a decreasing density, specified, for the sake of argument, by the expression

$$\alpha(x)=A_1-A_2\operatorname{th}(x/D),$$

where $A_1<A_2<0$ [$A_1-A_2\gg1$ because of (2.3)]. We shall assume that in the incident wave $q=q_1$ and $u<\frac{1}{2}$, with $\alpha(x)\neq p^2/4$ for all $-\infty<x<\infty$. We write

$$\operatorname{th}(x/D)=\tau,\quad \operatorname{th}(x_0/D)=\tau_0,\quad \operatorname{th}(x_1/D)=\tau_1, \qquad (5.2)$$

where the points x_0 and x_1 are defined by the relations

$$\alpha(x_0)=\alpha_0,\quad \alpha(x_1)=p^2/4, \qquad (5.3)$$

and α_0 is given in (2.19). Thus, $q_1(x_0)=0$ and $q_1(x_1)=q_2(x_1)$. We shall assume that $|\tau_0|<1$ and $|\tau_1|>1$. The former means that the point x_0 where total reflection of the first branch occurs lies on the real axis. The second condition means that the point x_1 where the first branch changes into the second one lies in the complex plane. The conditions $|\tau_0|<1$ and $|\tau_1|>1$ lead to the following limits on the quantity p^2:

$$\max\left[\frac{A_1-A_2}{u(1-u)},\,4(A_1+A_2)\right]<p^2<\frac{A_1+A_2}{u(1-u)} \qquad (5.4)$$

[we recall that $u(1-u)\leqslant\frac{1}{4}$ everywhere].

We now introduce instead of A_2 a new constant $b=b(p,u,A_2)$:

$$A_2=\frac{p^2}{4}(1-2u)^2b \qquad (5.5)$$

and express A_1 in terms of b and $\tau_1=(4A_1-p^2)/4A_2$. We then get from (2.11) and (5.1)

$$q_{1,2}^2=p^2\frac{(1-4u^2)}{4u^2}\left[1+\frac{(1-2u)b}{1+2u}(\tau-\tau_1)\mp\frac{2(b(\tau-\tau_1))^{1/2}}{1+2u}\right]. \qquad (5.6)$$

From what we have said earlier and Fig. 1 it follows that the graph of $q_{1,2}(\tau)$ must have the form shown in

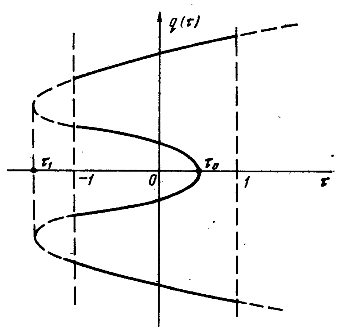

FIG. 4. Transverse wave number q as function of $\tau = \tanh(x/D)$ for the density profile (5.1).

Fig. 4. The region of propagation of waves with $q = \pm q_1$ is $-\infty < x < x_0$ $(-1 < \tau < \tau_0)$, and waves with $q = \pm q_2$ can propagate at $-\infty < x < \infty$ which corresponds to $-1 < \tau < 1$. The region $\tau_1 < \tau < -1$ is not realized on the real x-axis. Equation (5.2) for x_1 has an innumerable set of complex roots:

$$\frac{x_1}{D} = \frac{1}{2}\ln\frac{|\tau_1|-1}{|\tau_1|+1} + i\frac{\pi}{2} + i\pi n; \quad n = 0;\ \pm 1,\ \pm 2, \ldots \tag{5.7}$$

Apart from (5.7) there are also other singularities of $q_{1,2}$ in the complex plane, viz., the points where $\tau = \infty$. These occur when $x/D = i\pi/2 + i\pi n$, where $n = 0, \pm 1, \pm 2, \ldots$ The analysis of the behavior of the function (5.6) in the vicinity of these points shows that they are not important for transformation processes.

There remain thus the two points (5.7) which are closest to the real axis, for which $\mathrm{Im}\, x_1/D = \mp\pi/2$. According to the results of the preceding section they are responsible, respectively, for the tunneling transformations $q_1 \to q_2$ and $-q_1 \to -q_2$. On the other hand, in the point $x_0 = \mathrm{artanh}\,\tau_0$ (one verifies easily that $\tau_0 = \tau_1 + 1/b$) there occurs total reflection: $q_1 \to -q_1$. When $x > x_0$ the quantity q_1 becomes pure imaginary.

As a result it is clear that when a whistler wave with $q = q_1$ is incident upon the layer (5.1) the asymptotic behavior of the complete solution $\psi(x) = \{\Phi(x), F(x)\}$ as $x - x_0 \to \infty$ has the form

$$\psi(x) = C|q_1|^{-1/2}(p^2-4\alpha)^{-1/4}\chi(q_1)\exp\left(-\frac{\omega}{c}\left|\int_{x_0}^{x} q_1\,dx\right|\right) + C_2 q_2^{-1/2}(p^2-4\alpha)^{-1/4}\chi(q_2)\exp\left(-i\frac{\omega}{c}\int_{x_0}^{x} q_2\,dx\right), \tag{5.8}$$

while as $x - x_0 \to -\infty$

$$\psi(x) = C_1 q_1^{-1/2}(p^2-4\alpha)^{-1/4}\chi(q_1)\exp\left(i\frac{\omega}{c}\int_{x_0}^{x} q_1\,dx\right) + C_1' q_1^{-1/2}(p^2-4\alpha)^{-1/4}\chi(q_1)\exp\left(-i\frac{\omega}{c}\int_{x_0}^{x} q_1\,dx\right) + C_2' q_1^{-1/2}(p^2-4\alpha)^{-1/4}\chi(q_2)\exp\left(i\frac{\omega}{c}\int_{x_0}^{x} q_2\,dx\right). \tag{5.9}$$

Here C_1 is the amplitude of the incident wave $(q = q_1)$, C_1' that of the reflected first branch $(q = -q_1)$, and C_2' that of the second branch $(q = q_2)$ which is formed as a result of the tunneling transition $q_1 \to q_2$ and which is moving in the negative direction. Correspondingly, in (5.8) C_2 is the amplitude of the wave formed as a result of the tunneling transition $-q_1 \to -q_2$ and moving in the positive direction.

The coefficients in (5.8) and (5.9) can be expressed in terms of C_1. When determining C and C_1' it is sufficient to repeat verbatim the traditional arguments used in the case of total reflection; this gives[1]

$$C = C_1 e^{-i\pi/4}, \quad C_1' = C_1 e^{-i\pi/2}. \tag{5.10}$$

The coefficients C_2 and C_2' can be expressed, respectively, in terms of C_1' and C_1 by using Eq. (4.3), making in it the necessary transformations. The coefficient for the transformation of the branch $-q_1$ into the branch $-q_2$, propagating "forward" is calculated using Eq. (4.6) and can easily be changed to the form

$$T = \left|\frac{C_2}{C_1'}\right|^2 = \exp\left[\frac{\omega_H u D}{c}(I_1 - I_2)\right], \tag{5.11}$$

where

$$I_k = 2\mathrm{Re}\int_0^{\pi/2} q_k\,dy\,\big|_{\mathrm{Re}\,x=\bar{x}_1'}, \quad y = \mathrm{Im}(x/D), \quad k = 1, 2. \tag{5.12}$$

When evaluating the integrals in (5.12) we use the relations

$$\mathrm{th}\,(\bar{x}_1/D) = 1/\tau_1, \quad \tau - \tau_1 = \mathrm{th}\,(\bar{x}_1/D + iy) - \tau_1 = (1-\tau_1^2)(\tau_1 + i\,\mathrm{tg}\,y)^{-1} \tag{5.13}$$

and change to a new integration variable $\tan y = s$. As a result we get

$$I_k = \frac{p(1-4u^2)^{1/2}}{2u}\int_{-\infty}^{\infty}\left\{1 + \frac{1-2u}{1+2u}\,\frac{ib(\tau_1^2-1)}{s+i|\tau_1|} + \frac{2(-1)^k}{1+2u}\left[\frac{ib(\tau_1^2-1)}{s+i|\tau_1|}\right]^{1/2}\right\}^{1/2}\frac{ds}{1+s^2} \tag{5.14}$$

(we recall that $\tau_1 < -1$). It is convenient to assume firstly that $k = 2$. The integrand in (5.14) has then only one singularity in the upper half-plane—a pole at $s = i$. This makes it possible to close the integration contour in the upper half-plane and to get the answer at once. Afterwards we can extend the result also to the case $k = 1$. We write the final expression in the form

$$I_k = \frac{\pi}{2u}\{(1-4u^2)p^2 + [p^2 - 4(A_1+A_2)] + 2(-1)^k p[p^2 - 4(A_1+A_2)]^{1/2}\}^{1/2}, k = 1, 2. \tag{5.15}$$

Similarly we evaluate the coefficient for the transformation of the branch q_1 into the branch q_2 propagating "backwards," $T_1 = |C_2'/C_1|^2$. Starting from Eq. (4.5) we find that $T' = T$.

Equations (5.11) and (5.15) thus give the complete solution of the problem of the transformation of the first branch into the second one for the layer (5.1).

We see that the transformation coefficient T is expressed in terms of four dimensionless parameters: $\omega_H D/c, u, \alpha_{\max} = \alpha(-\infty) = A_1 + A_2$, and the longitudinal wave number p which determines the value $\alpha(x_0) = \alpha_0$ in the point of total reflection of the ray q_1 [see (2.19)]. We can consider the quantity $\alpha(x_0)$ as the minimum dimensionless density in the region where the branch q_1 propagates (Fig. 1).

One checks easily that T is a monotonically decreasing function of p. This makes it possible to simply make clear the behavior of T in the admissible range (5.4) of values of p. Using the fact that

$$4(A_1+A_2)\leq\frac{A_1-A_2}{u(1-u)}\quad\text{when}\quad u\leq u_{cr}=\frac{1}{2}\left[1-\left(\frac{2A_2}{A_1+A_2}\right)^{1/2}\right],$$

we find that

$$T_{max}=T\left\{p^2=\frac{A_1-A_2}{u(1-u)}\right\},\quad u<u_{cr};\tag{5.16}$$

$$T_{max}=T\{p^2=4(A_2+A_1)\}=1,\quad u>u_{cr};\tag{5.17}$$

$$T_{min}=T\left\{p^2=\frac{A_1+A_2}{u(1-u)}\right\}=\exp\left\{-\frac{\pi\omega_H D}{c}(1-2u)^{1/2}\left[\frac{A_1+A_2}{u(1-u)}\right]^{1/2}\right\}.\tag{5.18}$$

As $u\to\frac{1}{2}$ the width of the interval (5.4) tends to zero, and T_{min} increases, approaching unity, according to (5.18).

6. CONCLUDING REMARKS

We have studied above effects not been taken into account in whistler-wave-propagation theories based upon geometric optics or on parabolic types of equations (linear and nonlinear Schrödinger equations). From the latter it follows, in particular, that total internal reflection takes place when a whistler wave q_1 with $\omega<\omega_H/2$ is incident upon a plasma layer parallel to a magnetic field and having a decreasing density. We verified that, indeed, part of the energy of the incident wave passes into the region of lower density thanks to a tunneling transformation into the second branch.[4)]

As one application of the results we dwell upon the confinement of whistlers in plasma waveguides, assuming the latter to be oriented along the magnetic field. In such waveguides the density may be either higher or lower than that of the surrounding plasma (we can call them, respectively, "compression" or "rarefaction" waveguides). The first, in particular, are realized in the so-called magnetospheric ducts—plasma tubes with an increased density in which according to existing ideas whistler atmospherics with $\omega<\omega_H/2$ are channeled.[2] From what has been said above it follows that the guiding of such waves in ducts is incomplete. It is clear from (5.18) that the loss of energy due to the tunneling transformation must increase as $\omega\to\omega_H/2$. Similar conclusions can be reached also for compression waveguides formed as the result of non-linear self-focusing of a whistler wave with $\omega<\omega_H/2$. The self-focusing effect must be cut off the sooner, the closer the frequency is to $\omega_H/2$. When $\omega>\omega_H/2$ whistlers cannot be confined in plasma tubes with increased density. As far as rarefaction waveguides are concerned, in principle, they channel ideally both when $\omega<\omega_H/2$ and when $\omega>\omega_H/2$. We shall consider separately in more detail the propagation of whistlers in plasma waveguides.

In conclusion we note that the effects of a tunneling transformation can occur not only for whistlers, but also for other kinds of waves in inhomogeneous media.

1)The propagation of waves in the vicinity of the point where $\alpha=p^2/4$ is impossible when $u>1/2$ (see Fig. 1b).

2)From energy conservation it follows that $|C_1|>|C|$. However, because T is small, the difference between $|C_1|$ and $|C|$ is beyond the limits of the first WKB approximation. We are thus led to (4.4). We note also that the second term in (4.2) should be taken into account only when it exceeds the error in the determination of the first term in (4.2). This is just the reason why Eq. (4.2) makes sense only for sufficiently large negative x.

3)One checks easily that $q^2(x)$ has no zeroes when $u<1/2$.

4)The tunneling transformation is impossible as a matter of principle in geometric optics; on the other hand, in the Schrödinger-equation approximation the branch q_2 is lost.

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