Kinetic theory of plasma equilibrium in an electromagnetic field

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The kinetic theory of equilibrium of a plasma in an electromagnetic field is studied under the assumption that particles which escape from the dense plasma have a Maxwellian distribution. The conditions for existence of a stationary state are discussed. The spatial distributions of the fields, density, and flow velocity are investigated.

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The problem of equilibrium of a plasma in an electromagnetic field first arose in connection with attempts to use rf fields to confine plasmas.[1,2] Also investigated were equilibrium states in which the electromagnetic field completely displaces the plasma from some region of space. It is obvious that this condition cannot be satisfied for a Maxwellian distribution function, when there are particles with arbitrarily high velocities and infinitely large electromagnetic fields are required to confine them. Therefore, the papers Refs. 1 and 2 used a truncated Maxwellian distribution function with no fast particles. In the review of Ref. 3, it was pointed that the use of such a distribution function presupposes that in the dense plasma into which the field does not penetrate collisions are completely ignored and fast particles leaving the plasma are not replenished in the plasma by collisions.

In the present paper, we consider the equilibrium of a plasma under the assumption that the particles which leave the plasma have a Maxwellian distribution (in contrast to Refs. 1 and 2). We show that the equilibrium state is possible only when there is a definite connection between the amplitude of the incident wave and the concentration of the particles leaving the plasma. At the same time, the particle concentration vanishes nowhere, there is a directed flow of the plasma, and a constant current flows through the plasma. We investigate the laws of variation in the space of the fields, density, and flow velocity.

We note that the considered state can be realized only under the condition that the current flowing in the plasma is short-circuited to the plasma source (as occurs, for example, in experiments on the conversion of microwave radiation into a dc current[4]). This question is discussed in more detail in the Conclusions.

§1. FORMULATION OF THE PROBLEM

We shall assume that a linearly polarized electromagnetic wave of frequency ω_0 is incident on the plasma. The vector of the electric field $\tilde{\mathbf{E}}$ in the wave is directed along the x axis and depends only on the coordinate z:

$$\tilde{\mathbf{E}}(z,t)={}^1/_2\mathbf{e}_x(\tilde{\mathbf{E}}_0(z)e^{-i\omega_0 t}+\tilde{\mathbf{E}}_0^*(z)e^{i\omega_0 t}).$$

In the equilibrium state in the plasma there is in addition to the rf field a time-independent field directed along the z axis which arises from the separation of the charges: $\mathbf{E}=-\mathbf{e}_z[d\varphi(z)/dz]$, where φ is the electrostatic potential.

We shall seek the distribution function of the plasma electrons as a sum of a time-independent term $\bar{f}$ and a rapidly varying term $\tilde{f}$:

$$\tilde{f}(z,v,t)={}^1/_2(f_0(z,v)e^{-i\omega_0 t}+f_0^*(z,v)e^{i\omega_0 t}).$$

After substitution of the distribution function and the rf electric and magnetic fields, and also the charge separation field in the Vlasov equation for $\tilde{f}_0$, we find in the linear approximation

$$\tilde{f}_0(z,v)=-\frac{ie}{m\omega_0}\left[\left(E_0+i\frac{v_z}{\omega_0}\frac{dE_0}{dz}\right)\frac{\partial\bar{f}}{\partial v_x}-i\frac{v_x}{\omega_0}\frac{dE_0}{dz}\frac{\partial\bar{f}}{\partial v_z}\right].\qquad(1)$$

Equation (1) holds only for sufficiently high frequencies ω_0, when an electron traverses a distance in the direction of the z axis during a period of the field that is small compared with the scale of variation L_z of the field in this direction, the change in the electron velocity during this time being small:

$$\frac{1}{m\omega_0}\left|e\frac{d\varphi}{dz}\right|\ll v_z\ll\omega_0 L_z.\qquad(2)$$

After averaging over the time, the Vlasov equation, which describes the stationary part of the electron distribution function, can be represented by means of the expression (1) in the form

$$v_z\frac{\partial\bar{f}}{\partial z}-\frac{e}{m}\frac{d}{dz}(\varphi+U)\frac{\partial\bar{f}}{\partial v_z}=0,\qquad(3)$$

where $U=e|\tilde{E}_0|^2/4m\omega_0^2$ is the rf potential.

We shall also consider equilibrium states for which the wave is reflected from the plasma and in the region $z>0$ the field of the wave decreases, while in the region $z<0$ it has the structure of a standing wave. We assume that the distribution function of the electrons which leave the region into which the wave does not penetrate is Maxwellian with temperature T_e and concentration n_0. The solution of Eq. (3) satisfying this condition has the form

$$\bar{f}(\mathbf{v},z)=2n_0\left(\frac{m}{2\pi T_e}\right)^{3/2}\exp\left\{-\frac{1}{T_e}\left[\frac{mv^2}{2}+e(\varphi+U)\right]\right\},\qquad(4)$$

where in the region $z>0$ there are, besides the electrons which leave the region $z\to+\infty$, reflected electrons, so that the velocity component v_z is in the in-

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terval $v_m^{(e)}(z) > v_z > -\infty$, while in the region $z<0$ there are only electrons that have overcome the potential barrier and $-\infty < v_z < -v_m^{(e)}$. Here, $v_m^{(e)}$ is determined by the relation

$$v_m^{(e)}(z) = \{(2e/m)[(\varphi+U)_m - (\varphi+U)]\}^{1/2}, \tag{5}$$

where $(\varphi+U)_m$ is the maximal value of the total potential.

The field has virtually no influence on the ions of the plasma, and their motion is determined solely by the electrostatic field. Under the assumption that the distribution function of the ions which escape from the region $z\to+\infty$ is also Maxwellian with temperature T_i and concentration N_0, we write

$$F(\mathbf{v},z) = 2N_0\left(\frac{m_i}{2\pi T_i}\right)^{3/2}\exp\left\{-\frac{1}{T_i}\left(\frac{m_i v^2}{2}+e_i\varphi\right)\right\}, \tag{6}$$

where in the region $z>0$ we have $v_m^{(i)} > v_z > -\infty$ and in the region $z<0$ we have $-\infty < v_z < -v_m^{(i)}$, where $v_m^{(i)}(z) = [(2e_i/m_i)(\varphi_m - \varphi)]^{1/2}$, φ_m is the maximal value of the electrostatic potential, and $e_i = Z|e|$ and m_i are, respectively, the charge and mass of the ions. The concentration N_0 of ions is related to the electron concentration by $ZN_0 = n_0$, which follows from the conditions of neutrality of the plasma in the limit $z\to+\infty$.

The variation of the rf electrostatic field is determined by the Maxwell equations, in which the current density can be found by means of Eqs. (1) and (4). As a result, we have

$$\frac{d^2E_0}{dz^2}+\frac{\omega_0^2}{c^2}E_0 = \frac{\omega_p^2}{c^2}E_0 e^{-e(\varphi+U)/T_e}(1\pm\Phi(\eta_e)), \tag{7}$$

where the plus sign corresponds to the region $z>0$, and the minus sign to the region $z<0$; $\omega_p^2 = 4\pi n_0 e^2/m$;

$$\Phi(a) = 2\pi^{-1/2}\int_0^a dx e^{-x^2}$$

is the error function; and $\eta_e = v_m^{(e)} m^{1/2}/(2T_e)^{1/2}$.

By means of Eqs. (4) and (6), the Poisson equation for the charge separation potential can be transformed to

$$\frac{d^2\varphi}{dz^2} = 4\pi e n_0[e^{-e(\varphi+U)/T_e}(1\pm\Phi(\eta_e)) - e^{-e_i\varphi/T_i}(1\pm\Phi(\eta_i))], \tag{8}$$

where the plus and minus signs correspond to the same regions as in Eq. (7).

Our further treatment is based on Eqs. (7) and (8), which determine the combined variation in space of the charge separation field and the rf field.

§2. THE QUASINEUTRALITY APPROXIMATION. EQUILIBRIUM CONDITIONS

We consider the approximation in which we can ignore the separation of the charges and set the right-hand side of Eq. (8) equal to zero:

$$\exp\left[-\frac{e}{T_e}(\varphi+U)\right](1\pm\Phi(\eta_e)) = \exp\left[-\frac{e_i\varphi}{T_i}\right](1\pm\Phi(\eta_i)).$$

This relation is satisfied if

$$\varphi = U\frac{eT_i}{e_iT_e - eT_i} = \frac{|e|T_i}{4m\omega_0^2(T_i+ZT_e)}|E_0|^2. \tag{9}$$

We use the expression (9) to eliminate the potential φ from Eq. (7). As a result, for the determination of the rf field we obtain

$$\frac{d^2W}{d\xi^2}+W = nWe^{-W^2}[1\pm\Phi([W_m^2-W^2]^{1/2})], \tag{10}$$

where $\xi = z\omega_0/c$, $n = \omega_p^2/\omega_0^2$, $W = [Ze^2\bar{E}_0^2/4m\omega_0^2(ZT_e+T_i)]^{1/2}$ and W_m corresponds to the maximal value of the rf potential.

We use the condition that the field vanishes as $\xi\to\infty$ $(dW\to 0, (dW/d\xi)\to 0)$ and write down the first integral of Eq. (10), this characterizing the constancy of the total pressure at each point of the plasma:

$$(dW/d\xi)^2+W^2 = -n\{e^{-W^2}[1\pm\Phi([W_m^2-W^2]^{1/2})]-1 -\Phi(W_m)+2\pi^{-1/2}W_m e^{-W_m^2}\mp 2\pi^{-1/2}e^{-W_m^2}[W_m^2-W^2]^{1/2}\}. \tag{11}$$

The relation (11) makes it possible to investigate the function $W(\xi)$ for definite values of n and W_m. However, these quantities are not independent, since at the point $\xi=0$ we must satisfy the conditions $W=W_m$ and $dW/d\xi=0$. In accordance with (11), these conditions are satisfied only when W_m and n are related by

$$W_m^2 = n[1+\Phi(W_m) - e^{-W_m^2}(2\pi^{-1/2}W_m+1)]. \tag{12}$$

As we have already noted, in the region $\xi<0$ the rf field has the structure of a standing wave, and the value of W_m is determined by the amplitude of the wave incident on the plasma. One can therefore say that the relation (12) connects the amplitude of the incident wave to the concentration of the particles of the plasma leaving the region into which the field does not penetrate.

Equations (11) and (12) make it possible to find the function $W(\xi)$, in terms of which the other quantities characterizing the equilibrium state of the plasma are expressed. Thus, in accordance with Eq. (4) the electron concentration $n_e(\xi)$ is

$$n_e(\xi) = n_0 e^{-W^2}(1\pm\Phi([W_m^2-W^2]^{1/2})). \tag{13}$$

In the considered state, the plasma particles are not completely confined by the rf field and there is a flux of plasma directed in the direction of negative values of ξ. In accordance with (4) and (6), the mean velocity of the mass transport is

$$v = \frac{2}{(2\pi)^{1/2}}\frac{(T_em)^{1/2}+(T_im_i)^{1/2}}{m+m_i}\frac{e^{W^2-W_m^2}}{1\pm\Phi([W_m^2-W^2]^{1/2})}. \tag{14}$$

Besides the mass flux, there is also a charge flux in the equilibrium state. This is due to the circumstance that the velocity of the electrons having the plasma exceeds the velocity of the ions, and in the plasma a dc current arises with density

$$j_0 = e\int d\mathbf{v}\, v_z f + e_i\int d\mathbf{v}\, v_z F = \frac{2|e|n_0}{(2\pi)^{1/2}}\left(\frac{T_e^{1/2}}{m^{1/2}} - \frac{T_i^{1/2}}{m_i^{1/2}}\right)e^{-W_m^2}, \tag{15}$$

where n_0 is related to W_m by Eq. (12).

§3. ANALYSIS OF THE EQUILIBRIUM STATE

The relation (12), which determines the equilibrium condition, simplifies in two limiting cases.

For strong fields $(W_m \gg 1)$, it follows from (12) that

$$W_m^2 \approx 2n. \tag{16}$$

In this case, virtually all the plasma particles are re-

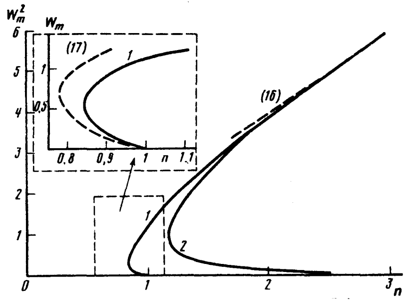

FIG. 1. Square of the maximal intensity W_m^2 of the electric field in the wave as a function of the concentration n of the particles leaving the plasma. Curve 1 is obtained in accordance with Eq. (12), and curve 2 corresponds to the results of Ref. 1. The broken curve shows the results of calculation in accordance with Eqs. (16) and (17).

flected from the rf barrier and Eq. (16) shows that the total pressure $2n_0(T_e+T_i/Z)$ of the particles is equal to the field pressure $\bar{E}^2_{0,\max}/16\pi$.

In the opposite limiting case of weak fields ($W_m<1$), we obtain from Eq. (12)

$$W_m=\frac{4}{3\pi^{1/2}}\pm\left[\frac{16}{9\pi}+2-\frac{2}{n}\right]^{1/2}. \tag{17}$$

In this case, the function W_m is two-valued in the interval $1>n>n_{\min}=[1+8/9\pi]^{-1}$. The quantity $n_{\min}$ is the minimal value of the concentration of the particles leaving the plasma for which equilibrium is possible. This value corresponds to field intensity given by $W_m=8/3\pi$.

Figure 1 shows the results of numerical solution of Eq. (12) and the results obtained from Eqs. (16) and (17). The two-valued nature of the function $W_m^2(n)$ indicates that two different equilibrium states correspond to the same value of the concentration of the particles which leave the region $\xi\to+\infty$. Different laws of variation of the field and the density in space correspond to these states. Figures 2 and 3 show the results of calculations of the functions $W(\xi)$, $n_e(\xi)$, and $v(\xi)$ made using Eqs. (11), (13) and (14) for two values of W_m and corresponding to the same value $n=0.94$ (see Fig. 1). It can be seen that for small $W_m\approx0.1$ (Fig. 2) the change in the concentration on the transition from the region $\xi>0$ to the region $\xi<0$ is small. At the same time, although the concentration of the particles incident on the barrier is less than the critical value (i.e., $n<1$), the total concentration in the region $\xi>0$ exceeds the critical value because of the reflection of a small fraction of the particles from the barrier. Under these conditions, the electromagnetic field penetrates fairly deeply into the plasma.

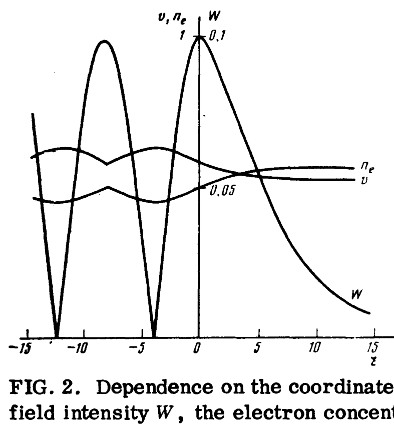

FIG. 2. Dependence on the coordinate $\xi=\omega_0 z/c$ of the electric field intensity W, the electron concentration n_e, and the plasma flow velocity v for $n=0.94$ and $W_m\approx0.1$.

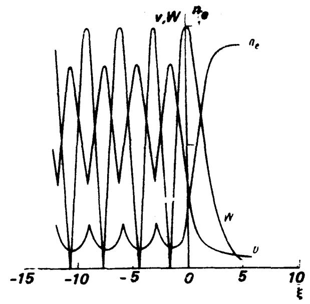

FIG. 3. Dependence on the coordinate $\xi=\omega_0 z/c$ of W, n_e, and v for $n=0.94$ and $W_m\approx1.0$.

For a large value $W_m\approx1.0$ (Fig. 3), more particles are reflected from the rf barrier. The change in the function $n_e(\xi)$ is then appreciable, and in the region $\xi>0$ the concentration appreciably exceeds the critical concentration, and the field penetrates into this region a comparatively short distance.

The function $W_m^2(n)$ is also two valued in the case of an equilibrium plasma with truncated Maxwellian distribution function.[1] However, in this case in the limit $W_m\to0$ the concentration n_0 to which the distribution function is normalized tends to infinity (see curve 2 in Fig. 1). This is explained by the circumstance that when W_m decreases only particles with low velocities are confined in the plasma. If their concentration is to exceed the critical value, the constant n_0 must be increased.

Figure 4 shows the functions $W(\xi)$ and $n_e(\xi)$ for the value $W_m=3$, which in accordance with formula (12) (see also 16) corresponds to $n=4.5$. In this case, virtually all the particles of the plasma are reflected by the rf barrier, and there is no significant difference between the results of the present paper and the results of Ref. 1.

In the investigation of equilibrium plasma states

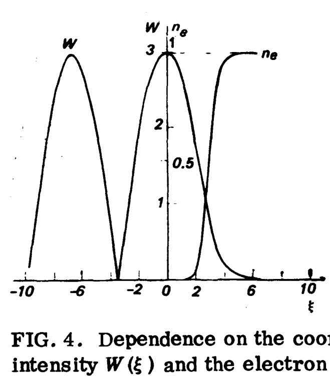

FIG. 4. Dependence on the coordinate ξ of the electric field intensity $W(\xi)$ and the electron concentration $n_e(\xi)$ for $W_m=3$ and $n=4.5$.

without flux, the maximum of the field intensity occurs at the minimum of the plasma density (see, for example, Ref. 5). In contrast, in our case (as in Ref. 4, in which the hydrodynamic theory of the equilibrium of a moving plasma in an electromagnetic field is considered) the maximum of the plasma density occurs at the maximum of the field (see Figs. 2 and 3). This occurs because the particles lose the velocity of the directed motion in climbing the rf barrier. It follows from the condition of flux conservation that the particle concentration then increases.

Near the maxima of the field, analytic expressions can be obtained for the functions W, n_e, and v. In accordance with Eq. (11),

$$W\approx W_m\left[1-\frac{(\xi-\xi_m)^2}{2}(1-ne^{-W_m^2})\right], \tag{18}$$

where ξ_m is the coordinate of the maximum of the function W. From Eqs. (13) and (14) we obtain for $T_e \gtrsim T_i$

$$\begin{gathered} n_e=ne^{-W_m^2}[1-2W_m\pi^{-1/2}|\xi-\xi_m|(1-ne^{-W_m^2})^{1/2}], \\ v\approx 2(2\pi)^{-1/2}(T_i/m_i)^{1/2}[1-2\pi^{-1/2}|\xi-\xi_m|(1-ne^{-W_m^2})^{1/2}]^{-1}. \end{gathered} \tag{19}$$

One can show by means of Eq. (12) that the quantity in the radicand in the expressions (19) is positive.

§4. CONCLUSIONS

In the considered equilibrium state, we have assumed that the particles leaving the plasma have a Maxwellian velocity distribution. This notion of a "one-sided" Maxwellian particle distribution is used to describe the current which flows in the discharge between an emitting cathode and an anode.[7] In accordance with our calculations [see Eq. (15)], the introduction of an electromagnetic field into such a discharge must decrease the current and in this sense play the part of a regulating potential.

A similar state can evidently be realized in a waveguide in which a plasma source is created and maintained at one end and an electromagnetic wave is introduced through the other end (such conditions are realized in many experiments[8]). Being reflected by the dense plasma, the electromagnetic wave produces in the waveguide a standing wave after a time $\lesssim 2L/c$ (L is the length of the waveguide). As a result, high-frequency barriers are formed, and these can be overcome only by sufficiently fast particles. Moving along the waveguide, such particles produce a state similar to the one we have considered. The time of establishment of this state depends on the height of the barrier, but it is certainly less than L/v_{Ti}, where v_{Ti} is the thermal velocity of the ions. If a current is to flow through the plasma to the end through which the electromagnetic field is introduced, it is necessary to place there a collector and short circuit it to the source of the plasma. It should be noted that the electromagnetic field must be switched on at a time when the plasma has flowed along the waveguide a distance L_N short compared with c/ω_0. If this condition is not satisfied, particles will be trapped in the resulting system of high-frequency barriers, and the equilibrium state will depend strongly on their distribution function.

A direct comparison of our conclusions with the experiments on the conversion of microwave energy into a dc current[4] cannot be made, since the microwave radiation itself produces the plasma. However, it can be assumed that in the planar geometry we consider an increase in the microwave power can lead to a decrease in the current.

We now make some comments about the conditions of applicability of the treatment. In investigating the equilibrium, we used the condition of quasineutrality of the plasma. Finding $\varphi(z)$ under this assumption, we can calculate $(d^2\varphi/dz^2)$ and write down the condition for this quantity to be small compared with any of the terms on the right-hand side of Eq. (8). As a result, we find that our treatment is valid if

$$e^{W_m^2}\ll c^2T_e/v_{Te}^2T_i.$$

It follows from this inequality that at large amplitudes of the incident wave it is necessary to take into account the charge separation field, which, moreover, is manifested most strongly, not in a dense plasma, but in a low-density plasma, in which the mean kinetic energy of the particles is maximal.

The question of the stability of the considered state of a plasma with asymmetric distribution functions of the particle velocities requires special study. On the basis of the known results, it can be concluded that current and two-stream instabilities will not develop in the plasma. This follows from the fact that under the conditions of validity of our treatment the spread of the electron velocities is of the same order as the velocity of their directed motion.[9]

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