Scattering of charged particles by a set of Coulomb centers and factorization of the eikonal amplitude

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(Submitted 21 August 1980)
Zh. Eksp. Teor. Fiz. **80**, 1334–1347 (April 1981)

The small-angle scattering of fast (short-wavelength) particles by a harmonic scatterer (one whose interaction potential satisfies Laplace's equation), including the scattering of charged particles by electric and magnetic fields, is considered in the eikonal approximation. On introducing the complex impact parameter (b) and the complex momentum transfer (p), the variables in the eikonal integral separate and the scattering amplitude is expressed as a sum, each term of which is the product of two contour integrals, of which one depends on p and the other on p^*. Scattering by a set of charges, by two charges, and by two like and two opposite charges are examined, as well as the limiting cases of scattering by a point dipole and by a charge and a dipole at the same point. In all these cases the amplitude is expressed in terms of generalized and confluent hypergeometric functions and Bessel functions of various types. The behavior of the scattering amplitude near focal points is also examined. The limiting case of small momentum transfers, in which the scattering reduces to scattering by the total charge (or by the dipole moment if the total charge vanishes) is considered, as well as the case of relatively large momentum transfers, in which the scattering reduces to the sum of the scatterings by the separate charges. In the classical limit, harmonic scattering generates a local conformal mapping of the impact-parameter plane onto the momentum-transfer plane, so the nonlocal transformation of the eikonal for harmonic scattering may be regarded as a quantum generalization of the conformal mapping of a plane. The results may be applied to electron optics, to the scattering of ions by molecules, crystals, and nonspherical nuclei, and to the scattering of electromagnetic and acoustic waves.

PACS numbers: 11.80.Fv, 03.80. + r

1. INTRODUCTION

The eikonal approximation[1,2] for small angle scattering of fast (short-wave) particles not only makes it possible to express the scattering amplitude as an integral, but also preserves many important properties of the exact solution—for example, it preserves the unitarity relation, which the cruder Born approximation does not do. In limiting cases the eikonal approximation reduces to the Born and classical approximations for small-angle scattering. The classical approximation has been discussed in Ref. 3 for the case of harmonic scattering (in which the interaction potential satisfies Laplace's equation), and it was shown that if the impact parameter b and the transverse momentum transfer p are treated as complex variables, the problem simplifies greatly and reduces to a conformal mapping.

Here we shall show that the same basic simplification of the harmonic-scattering problem also arises in the quantum case—in the eikonal approximation. On introducing the same complex variables b and p, the integration variables separate and the scattering amplitude factors, i.e., it can be expressed as a sum, each term of which is the product of two functions that depend on p and p^*, respectively, and can be expressed as contour integrals; this is very convenient both for general studies and for numerical calculations.

In Sec. 2 we examine the separation of variables in the eikonal integral, using the fairly general example of the scattering of a charged particle by a set of charges. In addition to the factoring of the integrand noted in Ref. 3, it is important that one can also ensure separation of the variables by deforming the integration surface.

In Sec. 3 we examine the following special cases, some of which involve passage to a limit: scattering by a set of two charges, equal in magnitude and opposite in sign, located at different points; scattering by a point dipole; and scattering by a charged dipole (i.e., by a charge and a dipole at the same point). For the first time we obtain explicit expressions for the eikonal amplitude in terms of confluent hypergeometric functions and Bessel functions of various types for all these problems.

In Sec. 4 we investigate the focal points, where the classical scattering amplitude becomes infinite, which play an important part in the general theory of diffraction and in catastrophe theory. The diffraction structure of the scattering intensity in the vicinity of such points is the same for all possible harmonic scatterers and can be easily investigated by the methods under discussion.

In Sec. 5 we examine the limiting case of very small momentum transfers, in which the principal part is played by large impact parameters and the scattering is determined by the first nonvanishing multipole of the set of scattering charges, as well as the case of relatively large momentum transfers in which the total scattering amplitude is the result of interference between the amplitudes for scattering by the individual charges.

The fact that passing to the classical limit leads to a local conformal mapping of the b plane onto the p plane permits us to say that the eikonal integral for harmonic scattering provides a nonlocal quantum generalization of an arbitrary conformal mapping of the plane.

Physical problems involving harmonic scattering are discussed in Ref. 3. If the angular resolution $\Delta p/k$ is such that the condition $R\Delta p/\hbar \lesssim 1$ is satisfied (R is the transverse size of the scatterer), a wave treatment must be used and the diffraction structure of the scattering, which is considerably simpler for a harmonic scatterer than for a general scatterer, can be observed.

 0038-5646/81/040681-07$02.40

2. FACTORIZATION OF THE EIKONAL AMPLITUDE

We shall take the charge and mass of the incident particle, as well as Planck's constant $\hbar$, equal to unity. Let us consider N Coulomb centers with charges q_j at the points $r_j = (x_j, y_j, z_j)$, $j = 1, 2, \ldots, N$ and introduce the screened Coulomb interaction potential

$$U(x, y, z) = \sum_j \frac{q_j}{|\mathbf{r}-\mathbf{r}_j|} e^{-\alpha k|\mathbf{r}-\mathbf{r}_j|}, \qquad (1)$$

where k is the wave number of the incident particle and the dimensionless screening parameter α is ultimately to be made to approach zero.

Then the eikonal approximation[2] the scattering amplitude can be expressed as a double integral over the impact-parameter plane [$b = (x, y)$ being the impact parameter]:

$$f = \frac{k}{2\pi i} \int_{-\infty}^{\infty} dx \int_{-\infty e^{i\varepsilon}}^{\infty e^{-i\varepsilon}} dy \exp\left[-i(xp_x + yp_y) + \frac{i}{k} V(x, y)\right], \qquad (2)$$

where we have dropped the α-dependent phase factor

$$\exp\left[\frac{2i}{k} \sum q_j \ln\left(\frac{\alpha e^{\gamma}}{2}\right)\right] \quad (\gamma = 0{,}5772\ldots),$$

which usually arises in this procedure[2] but is not important in the present problem. The displacement of the ends of the y-integration contour into the complex plane ($\varepsilon \to +0$ for $p_y > 0$ and $\varepsilon \to -0$ for $p_y < 0$) ensures convergence of the integral at infinity, $\mathbf{p} = (p_x, p_y)$ is the transverse momentum transfer, and the two-dimensional potential $V(x,y)$ is the electrostatic potential of a set of uniform line charges parallel to the direction of motion:

$$V(x, y) = \sum_j q_j \ln\{k^2[(x - x_j)^2 + (y - y_j)^2]\}, \qquad (3)$$

x_j and y_j being the projections of the coordinates of the Coulomb centers onto the impact-parameter plane.

To pass to the classical limit one may treat the argument of the exponential in (2) as large as evaluate the integral by the saddle point method. Then one obtains the classical small-angle scattering amplitude discussed in Ref. 3. The phase of this amplitude is determined by the value of V at the saddle point.

Because the potential $V(x, y)$ is a harmonic function, it can be expressed as a sum of functions of the variables $b = x + iy$ (the complex impact parameter) and $b^* = x - iy$, which are complex conjugates when x and y are real:

$$\begin{gathered} V(x, y) = V(b) + V^*(b^*), \\ V(b) = \sum_j q_j \ln[k(b - b_j)], \\ V^*(b^*) = \sum_j q_j \ln[k(b^* - b_j^*)], \\ b_j = x_j + iy_j, \quad b_j^* = x_j - iy_j. \end{gathered} \qquad (4)$$

The integrand in (2) is therefore a product of functions of b and b^*. To separate the variables in the integral we transform to an integration over a surface on which b and b^* are real variables that vary in independent intervals, and continue the integrand analytically into the complex y plane containing the vertical cuts $[y_j + i|x - x_j|, y_j + i\infty)$ and $[y_j - i|x - x_j|, y_j - i\infty)$.

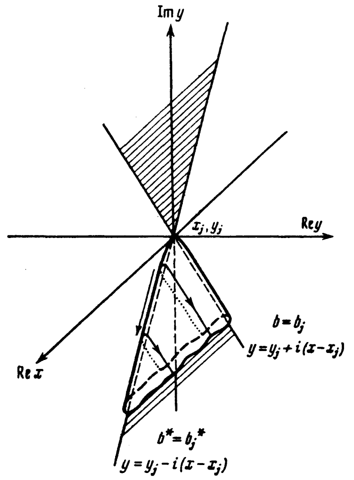

FIG. 1. Integration surface for one of the Coulomb centers. After deformation, the surface is reminiscent of a "pillow-case corner" covering the hatched quadrant, which represents the surface of the cut. When the integration variables are separated one of the integrations (e.g. the one over $b^* - b_j^*$) may be taken along the loops shown in the figure, while the other integration (over $b - b_j$) is taken from zero to infinity.

Assuming for definiteness that p_y is positive, we deform the y integration contour in the lower half plane and reduce it to a sum of contours that go around the lower cuts $[y_j - i|x - x_j|, y_j - i\infty)$ in the clockwise direction. Then the double integral (2) reduces to a sum of integrals over surfaces that can be conveniently imaged in the three-dimensional space whose coordinates are Re(x), Re(y), and Im(y). (In what follows we shall virtually always consider only real values of x.) In Fig. 1, the hatched sectors on the plane Re(y) $= y_j$ are the cuts corresponding to the charge q_j; on the lower cut, the value of the integrand at the left edge of the cut is equal to its value at the right edge multiplied by $\exp(-2\pi q_j/k)$. The integral for the Coulomb center q_j is taken over a surface that encompasses the lower cut, as shown in the figure. On this surface the shifted variables $b - b_j$ and $b^* - b_j^*$ vary in independent real intervals ($b - b_j \in [0, \infty)$ and $b^* - b_j^* \in (-\infty, 0]$), so the transformation from x and y to b and b^* leads to factorization of the integral for the corresponding center q_j and to an expression of the following form for the eikonal amplitude for scattering by a set of Coulomb centers:

$$f = \frac{k}{4\pi} \sum_j I_j, \qquad (5)$$

where I_j is the product of a function of the complex momentum $p = p_x + ip_y$ by a function of the conjugate momentum $p^* = p_x - ip_y$:

$$I_j = \int_{C_{b_j^*}} \exp\left[-\frac{1}{2} ib^* p + \frac{i}{k} V^*(b^*)\right] db^* \int_{C_{b_j}} \exp\left[-\frac{1}{2} ibp^* + \frac{i}{k} V(b)\right] db. \qquad (6)$$

The cuts in the complex impact-parameter plane (the b plane) and in the conjugate plane (the b^* plane) are drawn parallel to the real axis from $-\infty$ to each of the points b_j and b_j^*. For $p_y > 0$, the contour $C_{b_j^*}$ goes from $-\infty$ along the upper edge of the j cut and, going clockwise around the point b_j^*, returns to $-\infty$ along the lower edge; the contour C_{b_j} goes from the point b_j to $+\infty$. For $p_y < 0$ the contour $C_{b_j^*}$ goes from b_j^* to $+\infty$, while C_{b_j} goes around the cut in the clockwise direction. All the logarithms in (4) take their principal values on the integration path.

For a single charge q located at $b = x + iy$, Eqs. (5) and (6) lead to the exact value of the eikonal Coulomb amplitude with the phase factor $\exp[-i\,\mathrm{Re}(bp^*)]$ associated with the displacement of the origin of coordinates:

$$f=-\frac{2q}{|p|^2}\left(\frac{2k}{|p|}\right)^{2iq/k}\exp\{2i\delta_0-i\,\mathrm{Re}(bp^*)\}=f^c\exp\{-i\,\mathrm{Re}(bp^*)\},\qquad \delta_0=\arg\Gamma(1+iq/k). \tag{7}$$

In this case the substitution $|p| = 2k\cdot\sin(\theta/2)$, where θ is the scattering angle, reduces the eikonal amplitude to the exact one.

In the general case of N Coulomb centers the integrals in (6) can be expressed in terms of confluent hypergeometric functions of $N-1$ variables.[5] The expression for the amplitude for scattering by a set of multipoles can be obtained from Eqs. (5) and (6) by passing to the appropriate limit.

The above described simplification of the general problem of scattering by an electric field remains valid, just as in the classical approximation,[3] even when a weak magnetic field is applied. If the gauge is so chosen that the divergence of the vector potential $\mathbf{A}$ vanishes and the quadratic term in $\mathbf{A}$ is neglected, the Hamiltonian for a particle with unit mass and charge moving in an electromagnetic field with the vector potential $\mathbf{A}$ and scalar potential U takes the form

$$H=-\frac{1}{2}\Delta+\frac{i}{c}\mathbf{A}\nabla+U.$$

In the eikonal approximation, the wave function for a particle with initial momentum $\mathbf{k}$ can be expressed[2] as the product of a rapidly oscillating exponential by a slowly varying function $F(\mathbf{r})$:

$$\psi=e^{i\mathbf{kr}}F(\mathbf{r}).$$

Since $|kF| \gg |\nabla F|$, acting on ψ with the operator $i\mathbf{A}\nabla/c$ is equivalent to multiplying by $-\mathbf{A}\cdot\mathbf{k}/c \equiv -A_z k/c$. In this case, therefore, the application of a magnetic field is equivalent to adding the term $-A_z k/c$ to the scalar potential, and as the added term also satisfies Laplace's equation, one can separate the variables just as before. The choice of the integration contours, however, will depend on the specific forms of $\mathbf{A}$ and U.

Integrals closely related to (2) are encountered in optics, and the same methods can be used to calculate them. Thus, the authors of Ref. 6, in describing the elliptical umbilical catastrophe in diffraction theory (see below), actually factor integral (2) with $V(x, y) = kx(x^2 - 3y^2)$, using much the same method as we did. The variables can also be separated in this case since $V(x, y)$ is a harmonic function. Since the behavior at infinity of the potential $V(x, y)$ used in Ref. 6 differs substantially from that of the potential for a set of charges that we considered, the integration contours used in Ref. 6 differ considerably from ours.

We note that in the case of a harmonic scatterer the variables can be separated even when the problem is treated in the first Born approximation, since then the integrand in the expression for the amplitude reduces to a sum, each term of which is the product of a function of b by a function of b^*.

3. SPECIAL CASES

The method presented above makes it relatively simple to calculate the amplitude for scattering by two Coulomb centers in the eikonal approximation. Let the complex numbers b_1 and b_2 represent the projections of the charges q_1 and q_2 onto the impact-parameter plane in accordance with formulas (4). Introducing the notation

$$\tau=\frac{i}{2}p(b_2-b_1)^*,\quad \varkappa=\frac{i}{2k}(q_2-q_1),\quad \mu=\frac{1}{2}+\frac{i}{2k}(q_1+q_2), \tag{8}$$

and performing some calculations, we obtain the following expression for the scattering amplitude in terms of the Whittaker function $M_{\varkappa,\mu}(z)$:

$$f(p,b_1,b_2)=\frac{i}{2\pi k}\exp\left\{-\frac{i}{2}\mathrm{Re}[(b_2+b_2)p^*]\right\}|k(b_2-b_1)|^{4\mu}\times\sin(2\pi\mu)|\tau|^{-2\mu-1}\Gamma\left(\frac{1}{2}-\varkappa+\mu\right)\Gamma\left(\frac{1}{2}+\varkappa+\mu\right)\times\left\{\frac{\Gamma^2(-2\mu)}{\Gamma(1/2+\varkappa-\mu)\Gamma(1/2-\varkappa-\mu)}M_{-\varkappa,\mu}(\tau)M_{\varkappa,\mu}(\tau^*)-(\mu\to-\mu)\right\}, \tag{9}$$

where the second term in the curly brackets is obtained from the first one by the substitution $\mu \to -\mu$. The obvious symmetry of the problem—the symmetry of the scattering pattern with respect to the projection of the segment joining the scattering centers onto the impact-parameter plane—is reflected in the invariance (up to a phase factor) of formula (9) under the substitution $\tau \to -\tau^*$.

The formula for the eikonal amplitude for scattering by two identical Coulomb centers ($q_1 = q_2 = q$, $\varkappa = 0$, and $\mu = \frac{1}{2} + iq/k$) simplifies to

$$f(p,b_1,b_2)=\frac{i}{4k}\exp\left\{-\frac{i}{2}\mathrm{Re}[(b_1+b_2)p^*]\right\}\Gamma^2\left(\mu+\frac{1}{2}\right)\mathrm{ctg}(\pi\mu)\times\left(\frac{2k^2|b_2-b_1|}{|p|}\right)^{2\mu}\left\{I_\mu\left(\frac{\tau}{2}\right)I_\mu\left(\frac{\tau^*}{2}\right)-I_{-\mu}\left(\frac{\tau}{2}\right)I_{-\mu}\left(\frac{\tau^*}{2}\right)\right\}, \tag{10}$$

in which $I_\mu(z)$ is the modified Bessel function.[5] This expression is invariant under the substitutions $\tau \to -\tau$ and $\tau \to \tau^*$; this reflects the symmetry of the scattering pattern with respect to the projection of the segment joining the centers and its perpendicular bisector.

Formulas (9) and (10) can also be obtained by integrating in the complex momentum-transfer plane, rather than in the complex impact-parameter plane, as was done above. Since according to formula (2) the eikonal amplitude (with the appropriate regularization) is the Fourier transform of the function $\exp[iV(x,y)/k]$, which reduces to the product of two factors when the argument is the sum of the potentials, we can use the convolution theorem to express the amplitude for scattering by two centers in the form

$$f(\mathbf{p},\mathbf{b}_1,\mathbf{b}_2)=\frac{i}{2\pi k}e^{-i\mathbf{pb}_1}\iint d^2p'e^{-i\mathbf{p}'(\mathbf{b}_2-\mathbf{b}_1)}f_1(\mathbf{p}-\mathbf{p}')f_2(\mathbf{p}'), \tag{11}$$

where f_1 and f_2 are the eikonal amplitudes for scattering by the first and second centers, respectively. For Coulomb centers, f_1 and f_2 are known and are power functions of the magnitude of the momentum transfer alone [see Eq. (7)]. From this it follows that the Coulomb scattering amplitude can be expressed as an exponential with a harmonic function of p_x and p_y in the argument (with a complex quasicharge).[1)] On introducing the complex momentum transfer and the complex impact parameter in accordance with formulas (4), therefore, the integrand in (11) takes the form of the product of a function of the complex variables p and p' by a function of the complex conjugates p^* and p'^*. By regularizing the integral (11) at the points $p'=0, p$ (this can be done, for example, by adding small imaginary quantities to the charges) and carrying through a procedure analogous to the one described in the preceding section, we succeed in separating the integration variables and finally reach formulas (9) and (10).

We see that the use of the convolution formula to calculate the eikonal amplitude for the case of two Coulomb centers leads to the same results as the method presented in this paper and that the calculations required by the two methods are about equally difficult. In the case of many Coulomb centers, however, the convolution formula leads to an integration over many (four, six,...) variables, whereas in our method we still have to integrate over only two variables, and the variables can be separated at once.

For a finite dipole ($q_1=-q_2=q$) formula (9) yields

$$f(p, b_1, b_2)=-\frac{iq^2}{k^3}\exp\left\{-\frac{i}{2}\operatorname{Re}[(b_1+b_2)p^*]\right\}\left(\frac{2k}{|p|}\right)^2 \times\operatorname{Re}\left\{\Gamma\left(i\frac{q}{k}\right)M_{iq/k,1/2}(\tau)W_{-iq/k,1/2}(\tau^*)\right\}, \qquad (12)$$

where $W_{\mu,\varkappa}(z)$ is the second Whittaker function.[5]

It is evident from Eq. (12) that if $b_1=-b_2$ the eikonal amplitude for scattering by the field of a finite dipole will be purely imaginary. The eikonal amplitude for scattering by any potential that is antisymmetric under the inversion $x\to -x, y\to -y$ has this property, as is immediately evident from the initial formula (2).

For a finite dipole, just as for any system whose total charge is zero, the integrand is not altered by passing around all the cuts. The integration surface may therefore be closed with respect to one of the variables and may be given the shape of a "tube" encompassing the edge of all the quadrants. In the case of a finite dipole, the integration surface should be a "gable roof" having endless gables at 45° angles and whose "ridge" is equal in length to the distance between the centers. Such a deformation of the contour is useful in passing to the limiting case of a point multipole.

Let us write

$$q_1={}^1/_2(Q_0+2|Q_1|/R),\quad q_2={}^1/_2(Q_0-2|Q_1|/R),\quad b_1=\mathrm{Re}^{i\varphi},\quad b_2=0,$$

in the two-center problem. Then by taking the limit as $R\to 0$ of formula (9) we obtain the answer for an important special case—the eikonal amplitude for scattering by the field due to a point charge Q_0 and a point dipole $Q_1=|Q_1|e^{i\varphi}$, both lying at the same point. The result is obtained as a sum of products of Bessel functions:

$$f(p)=-\frac{\pi}{2k\,\mathrm{sh}(\pi Q_0/k)}\left|\frac{2kQ_1}{p}\right|^{\mu}\left\{J_\mu\left(\left[\frac{2pQ_1^*}{k}\right]^{1/2}\right)\times J_\mu\left(\left[\frac{2p^*Q_1}{k}\right]^{1/2}\right)-J_{-\mu}\left(\left[\frac{2pQ_1^*}{k}\right]^{1/2}\right)J_{-\mu}\left(\left[\frac{2p^*Q_1}{k}\right]^{1/2}\right)\right\}, \qquad (13)$$

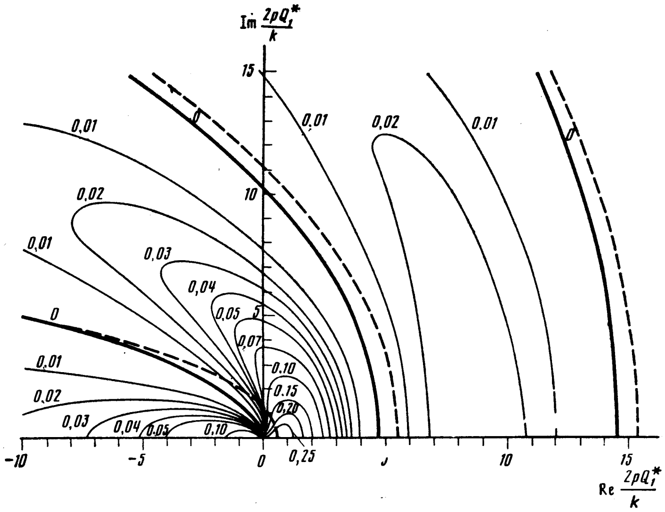

FIG. 2. Contours of the quantity $|pf(p)/4Q_1|^2$ for scattering by a point dipole. The zero contours are drawn heavy. The dashed curves are the zero contours in the semiclassical approximation.

where $\mu=1+iQ_0/k$. Taking the limit of this formula as $Q_1\to 0$, we obtain the eikonal amplitude (7) for scattering by a single Coulomb center of charge Q_0.

The expression for the eikonal-approximation amplitude for scattering by a point follows at once from formula (13). Setting $Q_0=0$, we have

$$f(p)=-\frac{4i|Q_1|}{|p|}\operatorname{Re}\left\{I_1\left(i\left[\frac{2pQ_1^*}{k}\right]^{1/2}\right)K_1\left(-i\left[\frac{2p^*Q_1}{k}\right]^{1/2}\right)\right\}, \qquad (14)$$

where $K_1(z)$ is the Macdonald function.

Figure 2 shows contour lines for $|pf(p)/4Q_1|^2$ in the plane of the complex variable $2pQ_1^*/k$, which is the only parameter in the problem. The pattern is symmetric about the real axis, so only the upper half plane is shown; however, the axial symmetry of the classical differential cross section for scattering by a multipole[3] is clearly broken here.

When $|2pQ_1^*/k|\gg 1$, we reach the quasiclassical region where

$$f(p)=-i|2kQ_1/p^3|^{1/2}\cos(2\operatorname{Re}[2pQ_1^*/k]^{1/2}).$$

The quasiclassical amplitude for scattering by a point dipole vanishes on the parabolas

$$y^2=\pi^2(n+{}^1/_2)^2x+{}^1/_4\pi^4(n+{}^1/_2)^4,\quad n=0, 1, 2,\ldots,\quad x+iy=2pQ_1^*/k,$$

which are shown in Fig. 2 by dashed curves. The true null contours approach these parabolas for large values of $2pQ_1^*/k$.

It follows from Eq. (14) that as $p\to 0$, we have $f(p)\to 2i\operatorname{Re}(pQ_1^*)/|p|^2$, and this is just the Born-approximation amplitude[7] for scattering by the field of a point dipole.

The formulas obtained are directly related to the eikonal amplitude[8] for the spherically symmetric potential a/r^2, which also factors.[8] Then by calculating $V(x,y)$ for this potential, choosing the y axis in (2) in the direction of the momentum transfer (this is possible because of the axial symmetry of the problem), and adding a term that is odd in x and therefore vanishes on integration, we obtain

$$f=\frac{k}{2\pi i}\int_{-\infty}^{\infty}dx\int_{-\infty e^{i\varepsilon}}^{\infty e^{-i\varepsilon}}dy\left(1+\frac{x}{(x^2+y^2)^{1/2}}\right)\exp\left\{-i|p|y-i\frac{\pi a}{k(x^2+y^2)^{1/2}}\right\}.$$

Making the change of variable $x=(u^2-y^2)/2u$, we reach the integral

$$f=\frac{k}{2\pi i}\int_{-\infty e^{i\varepsilon}}^{\infty e^{-i\varepsilon}}dy\int_{0}^{\infty}du\exp\left\{-i|p|y-i\frac{2\pi au}{k(u^2+y^2)}\right\},$$

which determines the eikonal amplitude for scattering by a point dipole in the special case in which the momentum transfer is perpendicular to the dipole axis (for classical scattering this corresponds to a 45° angle between the dipole axis and the impact parameter b); in this case, however, the integration is taken over a half plane. After transforming to the variables b and b^*, therefore, the integration region is no longer two tubes, but is a single tube surrounding the line singularity $u-iy=0$. We obtain

$$f=-2\pi a|p|^{-1}J_1[(2\pi ia|p|/k]^{1/2})K_1([2\pi ia|p|/k]^{1/2}),$$

in agreement with the result obtained earlier in Ref. 8.[2)]

4. FOCAL POINTS

The classical differential cross section for the small-angle scattering of a classical particle by a set of Coulomb centers has singularities[3,4]: it diverges at certain values of the momentum transfer p (we shall denote these values by p^f). A quantum mechanical treatment removes the divergences of the differential cross section, and in the small-angle region the eikonal approximation is a quantum mechanical treatment. Under quasiclassical conditions, however, the amplitude, although it remains finite, has prominent maxima near the rainbow and focal singularities, and these maxima are the more prominent, the more nearly classical are the scattering conditions. It turns out that the harmonic character of the scattering potential and the consequent possibility of factoring the eikonal amplitude substantially simplify the interference pattern near the singularities.

It has been shown[3] that in the case of N Coulomb centers, the mapping of the complex impact-parameter plane (the b plane) onto the complex momentum transfer plane (the p plane) given by the classical equations of motion in the small-angle approximation is a conformal mapping of N sheets. Then the classical differential cross section has only point singularities (at the focal points p^f), and this is the case for any harmonic potential.

To the focal points in the p plane there correspond points b^f in the impact-parameter plane; these are points at which two or more values of the impact parameter are confluent, leading to scattering with the given momentum transfer p.

Near such points the quasiclassical asymptotic behavior of the amplitude (2) can no longer be obtained by the usual stationary phase method, but it can be described with the aid of canonical integrals. The Airy function is such a canonical integral in the case of rainbow scattering by a spherically symmetric potential.[12] More complicated canonical integrals arise in more complicated situations.

The confluence of two values of the impact parameter in the case of $N\geq 2$ Coulomb centers is typical, and there may be $2N-2$ such points.[3] In the vicinity of such a point the complex potential (4) behaves as $V(b)\sim(b-b^f)^3$, and the scattering amplitude near the corresponding momentum transfer p^f is described by the canonical integral discussed in Ref. 6. From the point of view of catastrophe theory,[6,11] this singularity of the amplitude has come to be called an "elliptic umbilicus." The intensity pattern for this diffraction catastrophe[6] is shown schematically in Fig. 3. The central maximum is the point where three straight ridges come together at 120° angles, while the valleys (shown by heavy bands on Fig. 3) and the curved ridges that they separate are cubic hyperbolas. Thus, the elliptic umbilicus is a typical singularity of the amplitude for small-angle scattering by a set of Coulomb centers.

The confluence of three or more values of the impact parameter takes place only when the scattering centers are disposed special ways. One such disposition of the scattering centers was discussed in Ref. 3: N identical Coulomb centers lying in a plane at the vertices of a regular polygon. The center of the polygon is the point of confluence of $N-2$ values of the impact parameter.

Let us examine the "elliptic umbilical" singularity of the amplitude in more detail, using the scattering by the field of two identical charges q at the points $b=\pm R$ as an example. The values of the momentum transfer corresponding to the rainbow singularities are[3,4] $p^f=\pm 2iq/kR$. The condition $kR\gg 1$ that the scattering be quasiclassical means that the wavelength of the particle is much smaller than the characteristic size of the scatterer. In addition, for the focal singularity to be observed, the focal angle $\theta^f=|p^f|/k$ must be much larger than the quantum mechanical indeterminancy of the scattering angle, and this yields $q/k^2R=O(1)$, or $q/k\gg 1$.

Thus, in order to distinguish the characteristic behavior of the scattering amplitude in the vicinity of the singularity, we must examine the asymptotic behavior of the function $I_\mu(z)$ in formula (10) in the limits $|\mu|\to\infty$ and $|z|\to\infty$. The following asymptotic formulas are valid[10] for constant z and large $|\mu|$:

$$J_\mu(\mu+z\mu^{1/3})=(2/\mu)^{1/3}\,\mathrm{Ai}\,(-2^{1/3}z)\,(1+O(\mu^{-2/3}))$$
$$N_\mu(\mu+z\mu^{1/3})=-(2/\mu)^{1/3}\,\mathrm{Bi}\,(-2^{1/3}z)\,(1+O(\mu^{-2/3})),$$

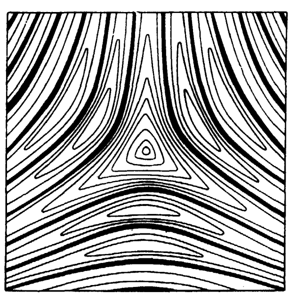

FIG. 3. Qualitative pattern of the intensity in the vicinity of the simplest third-order focal point, or the focal plane of an "elliptic umbilical" catastrophe. The zero contours are drawn heavy. The maximum occurs at the center, and from it there radiate three "spurs."

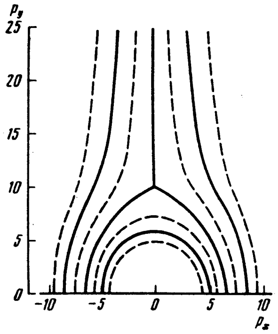

FIG. 4. Qualitative depiction of the diffraction pattern. The in-phase contours (ridges) are shown by full curves, and the out-of-phase contours (valleys), by dashed curves. The pattern is symmetric about the p_x axis. One of the two focal points can be seen.

where $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ are Airy functions.[5]

Going to the limit mentioned above in Eq. (10) with the aid of these expressions, we obtain the following formula, valid in the vicinity of $p^f = i2q/kR$:

$$f(p)\approx -\frac{\pi}{k}\left(\frac{q}{2k}\right)^{1/3}\left(\frac{4k^2R}{|p|}\right)^{1+i2q/k} e^{2i\delta_0}\,\mathrm{Re}\{\mathrm{Ai}(-\zeta^*)\,\mathrm{Bi}(-\zeta)\},$$

where $\zeta=\frac{1}{2}e^{i\pi/6}(2k/q)^{1/3}R(p-p^f)$ and $\delta_0=\arg\Gamma(1+iq/k)$. This formula contains a characteristic expression involving a sum of products of Airy functions which coincides with formula (3.4) of Ref. 6 and yields the diffraction pattern illustrated in Fig. 3.

Far from the focal points one can use the ordinary quasiclassical approximation in which

$$f(p)=[\sigma_1(p)]^{1/2}e^{iS_1}+[\sigma_2(p)]^{1/2}e^{iS_2},$$

where $\sigma_{1,2}(p)$ are the classical cross sections and $S_{1,2}$ are the increments of the classical action on the trajectories with the impact parameters b_1 and b_2 that lead to scattering with the given momentum transfer p. On approaching a focal point we have $\sigma_1\approx\sigma_2\to\infty$ and this approximation is no longer valid. However, the phase shift S_1-S_2 between the amplitudes provides a qualitatively correct description of the diffraction pattern near the singularity. Figure 4 shows contour lines on which the amplitudes have the same phase (full curves) and opposite phases (dashed curves) for the case $R=1$ and $q/k=5$, when the singularity has the coordinates $p_x^f=0$ and $p_y^f=\pm 10$. At this point three lines of equal phase, i.e., three ridges, come together at 120° angles, as is characteristic of elliptic umbilical singularities. The entire pattern is symmetric about the p_x axis. As $p\to 0$, these lines approximate concentric circles which, becoming smaller, contract to a point, as is characteristic of scattering by a single Coulomb center.

In the harmonic character of the potential $V(x,y)$ is disturbed, for example by adding the term $\alpha(x^2+y^2)$ to the cubic form $V(x,y)=x^3-3xy^2$ (which is a harmonic function[8]) or by screening the Coulomb centers,[9] then in the classical approximation the focal point p^f becomes a rainbow line having the form of a triangular hypocycloid, while the interference pattern becomes substantially more complicated.

5. LARGE AND SMALL MOMENTUM TRANSFERS

If we denote the transverse dimension of the scattered by R, the characteristic charge of a single center by q, the distance at which the potential energy in the field of the center q is equal to the total energy by $a(a=2q/k^2)$, and the wavelength of the incident particle by λ, then it is evident from Eqs. (2) and (6) that the amplitude depends on the dimensionless parameters $R/\lambda=kR$ and $a/\lambda=2q/k$, on the product $R|p|\approx kR\theta$ (θ is the scattering angle), and on the parameters x_j/R, y_j/R, and q_j/q, which are of the order of unity. For large or small momentum transfers, formula (6) for the amplitude can be simplified by calculating the corresponding asymptotic behaviors in the parameters λ/R, λ/a, and θ.

We shall examine two cases for large momentum transfer when θ is fixed (and small) while $k\to\infty$: 1) $\lambda/a=O(1)$ and $\lambda/R\to 0$; and 2) $\lambda/a\to\infty$ and $\lambda/R\to 0$. In case 1) the wavelength of the incident particle is much smaller than the distance between any two centers, but it is not excessively small: $q/k=O(1)$. Because of the presence of the rapidly decreasing exponential $\exp(-1bp^*/2)$ [$\exp(-ib^*p/2)$], the main contribution to the integrals (6) determining the functions I_j comes from the vicinity of the complex coordinates b_j (b_j^*) of the centers. These integrals can be calculated by the usual asymptotic methods.[13] As a result we obtain an answer whose principal term is the sum of the Coulomb amplitudes f_j^c as given by Eq. (7) for scattering by the individual centers with phase factors that take account of the position of the center and the effect of the other centers:

$$f_{\lambda/R\to 0}=\sum_j f_j^c \exp\Big[-i\,\mathrm{Re}(b_jp^*)+2i\sum_{m\neq j}(q_m/k)\ln(k|b_m-b_j|)\Big]$$
$$\times\Big\{1-4(1+iq_j/k)\,\mathrm{Re}\sum_{m\neq j}q_m/(kp^*(b_m-b_j))+O((\lambda/R)^2)\Big\}.$$

On averaging the differential cross section over the azimuthal angle φ (the argument of the momentum transfer p), the principal term contains the sum of the cross sections for all the centers and the interference terms are of the following order in the small parameter λ/R:

$$\langle\sigma\rangle_\varphi=\frac{1}{2\pi}\int_0^{2\pi}|f|^2\,d\varphi=\Big\{\frac{4}{|p|^4}\sum_j q_j^2+\frac{16}{|p|^4(2\pi|p|)^{1/2}}\sum_{l<j}\frac{q_jq_l}{|b_j-b_l|^{1/2}}$$
$$\times\cos\Big[|p(b_j-b_l)|-\frac{\pi}{4}\Big]\cos\Big[2\delta_j+\frac{2q_j}{k}\ln\Big(\frac{2k}{|p|}\Big)$$
$$+2\sum_{m\neq j}\frac{q_m}{k}\ln(k|b_m-b_j|)-(j\to l)\Big]\Big\}\Big[1+O\Big(\Big(\frac{\lambda}{R}\Big)^{3/2}\Big)\Big],\quad \delta_j=\arg\Gamma\Big(1+i\frac{q_j}{k}\Big).$$

As the incident-particle energy increases, the characteristic length $a=2q/k^2$ tends to zero more rapidly than does $\lambda=1/k$, and we come to case 2); here the leading term of the amplitude is the well-known Born approximation

$$f=-\frac{2}{|p|^2}\sum_j q_j\exp[-i\,\mathrm{Re}(b_jp^*)]\Big\{1+O\Big(\frac{a}{\lambda}\ln\frac{\lambda}{R}\Big)\Big\}.$$

The formula for the averaged differential cross section then becomes

$$\langle\sigma\rangle_\varphi=\Big\{\frac{4}{|p|^4}\sum_j q_j^2+\frac{16}{|p|^4(2\pi|p|)^{1/2}}\sum_{l<j}\frac{q_jq_l}{|b_j-b_l|^{1/2}}$$
$$\times\cos\Big[|p(b_j-b_l)|-\frac{\pi}{4}\Big]\Big\}\Big\{1+O\Big[\Big(\frac{\lambda}{R}\Big)^{1/2}\frac{a}{\lambda}\ln\frac{\lambda}{R}\Big]\Big\}.$$

For small momentum transfers one must calculate the amplitude in the limit $\theta\to 0$, the remaining parameters being of the order of unity. This can be easily done by starting with the double integral (2) and expanding for small $|p|$. The main contribution to the integral comes from the region of large impact parameters, where the multipole expansion of the potential is valid; then the leading terms of the multipole expansion determine the asymptotic behavior as $|p|\to 0$. The calculations lead to the result

$$f=f^c[1-(i/Q_0)\,\mathrm{Re}\,(pQ_1^*)+o(p)],$$

where $Q_0=\Sigma q_j$ is the total charge, and $Q_1=\Sigma q_j b_j$ is the total complex dipole moment of the target. Here the leading term is the Coulomb amplitude for the total charge Q_0; if $Q_0=0$, the second term, the eikonal amplitude for scattering by a dipole, becomes the leading term.

6. CONCLUSION

The approach to harmonic scattering considered here opens up broad prospects for further research and generalization. Here, for example, we have examined only elastic scattering. In the case of a fast incident charged particle, however, an atom or a molecule can be treated as a set of stationary Coulomb centers, provided the velocity of the incident particle is high as compared with those of the particles in the atom. Then by integrating the resulting amplitude over the configurations, weighted as determined by the wave function of the target,[2] we can calculate elastic and inelastic scattering by an atom in the eikonal approximation. Such calculations have been carried through by other methods[14] for the scattering of electrons by hydrogen.

The results of the present work, together with those of Ref. 3, permit us to speak of the discovery of a new class of small-angle harmonic-scattering problems that are close to realistic physical problems, have unique properties, and will permit us to make great progress in the analytic study of the scattering amplitude.

1) We note that Φ, the logarithm of the exact Coulomb scattering amplitude, satisfies the Poisson equation with a constant on the right: $\nabla^2\Phi=\mathrm{const}$; the property is apparently associated with the Fock symmetry of the Coulomb field.

2) There is a misprint in formula (22) of Ref. 8; the angle θ was omitted from the denominator on the right.

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Translated by E. Brunner