The symmetry of the center of the gauge group and the problem of quark confinement in quantum chromodynamics

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A lattice gauge model is considered which exhibits a phase transition related to spontaneous symmetry breakdown of the center of the gauge group. It is shown that the phase diagram obtained in the large N (number of colors) limit separates phases with confined and liberated quarks. The correct way of taking the continuum Lorentz-invariant limit in the confined phase is discussed. Consequences of our discussion for the standard lattice gauge theory are listed.

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1. INTRODUCTION

At present quantum chromodynamics (QCD) is regarded as a consistent theory of strong interactions. The interaction of quarks, which under the action of the color gauge group $SU(3)$ transform according to the fundamental (triplet) representation, is mediated in QCD by the octet of gluon fields. It is expected that in addition to the huge number of experimentally verified consequences, QCD will allow one to explain quark confinement.

The brightest successes of attempts to explain quark confinement by means of QCD have been achieved in lattice gauge theories.[1,2] It was shown that in each order of the strong coupling expansion the Wilson "area law" is verified and the quarks are confined. The central problem which remains unsolved in this approach is to show that the only phase transition occurs at zero "temperature" g^2. If this is a phase transition of second order, then the lattice gauge theory has a Lorentz-invariant continuum limit: as the lattice constant decreases, the bare coupling constant must approach the critical value $g_*^2=0$ according to the asymptotic freedom formula, whereas the correlation length—the radius of quark confinement—remains finite.

By means of his approximate recurrence relation A. A. Migdal[3] has shown that such a situation may indeed by realized in QCD, and the stipulation "may" refers to the fact that it is not clear how "crude" the approximations that are used are. More "refined" investigations have shown that the approximate recurrence relation works sufficiently well for a number of problems, but in some cases,[4] such as the $Z(N)$-lattice gauge theories for $N=2, 3, 4$, it leads to a qualitatively false result (a phase transition of the second instead of first order).

The latter models have been investigated by various methods,[5,6] both theoretically, and "experimentally," i.e., by computer integration. Similar investigations of $SU(2)$ gauge theories bear witness[8-11] in favor of the nonexistence of a "finite-temperature" phase transition, although so far a computer "experiment" for the group $SU(3)$ has not been carried out.

The most important characteristic of the mechanism which guarantees quark confinement in lattice gauge theory is the following. It makes sense to talk about confinement only for objects with nonzero triality, i.e., transforming according to a nontrivial representation of the group $Z(3)$—the center of the gauge group $SU(3)$.[12-14] For example, for the quark fields, such a transformation is defined by

$$\Psi(x)\to Z\Psi(x),\quad Z=\exp(2\pi in/3)\in Z(3), \tag{1.1}$$

where n is an integer. If one considers a test charge with zero triality, e.g., transforming according to the octet representation, then in place of Wilson's "area law" criterion we always obtain a "perimeter law," which signals the absence of forces rising linearly with the distance. The meaning of this is quite simple. An object without triality can be screened by gluon excitations which exist in the system. This cannot be done with a quark, of course. The triality of a quark cannot be screened by gluons, if, as is the case in strong coupling expansions, the symmetry of the center of the gauge group is not broken.

Thus it makes sense to talk only of *triality confinement*, in contradistinction of *the screening of the color charge*. This approach differs essentially from those approaches[1)] in which "color confinement" (independently of triality) is related only to the growth of the effective coupling constant in the infrared region, independently of the properties of the gauge group itself. In the symmetry mechanism of quark confinement, a decisive role is played exactly by the properties of the gauge group (and its representations), and the infrared growth of the coupling constant is not directly related to the problem of quark confinement, although it complicates the problem considerably.

The symmetry mechanism of quark confinement can be formulated[12,13] without reference to the lattice model. Polyakov has shown[12] that if the symmetry of the center of the gauge group is not broken, then the "area law" is valid, and the quarks are confined. The problem whether the symmetry of the center of the gauge group is broken and which field fluctuations maintain this symmetry [such as the monopoles in 2+1-dimensional

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$SU(N)$ gauge theory[13]] is dynamical problem and is solved individually for each particular model. Based on the natural assumption of the absence of massless particles in the QCD spectrum, 't Hooft came to the conclusion[13] that if the symmetry of the center of the gauge group is broken in such a manner that the "area law" is valid for a quantity that plays the role of a disorder parameter, then the "perimeter law" must be satisfied for the Wilson criterion.

In the present paper, which is a detailed version of Ref. 17, we continue the study initiated by Polyakov and 't Hooft of the relation between the center of the gauge group and the quark confinement problem.

In Sec. 2 we define a two-charge model (similar models have been investigated in Refs. 18 and 19) and investigate it by perturbative methods. In Sec. 3 we derive a chain of loop equations of motion[20-25] for the two-charge model, and with its aid prove the factorizability of loop averages. In Sec. 4 we calculate the dependence of the free energy on the external field (the dependence of the energy on the external field allows one to determine whether the symmetry of the center of the gauge group holds or whether it is spontaneously broken). It is shown that for a large-N two-charge model the energy depends on the external field in the same manner as for the $U(1)$ gauge theory. In Sec. 5 we discuss the connection of the spontaneous breakdown of the symmetry of the center in two-charge QCD with the quark-confinement problem and the consequences which one can derive from our model for standard QCD on a lattice. In Sec. 6 we construct the phase diagram for two-charge QCD and discuss briefly how to pass to the continuum limit in a phase with confined quarks. And finally, in Sec. 7, we give a brief summary of the results of this paper.

2. TWO-CHARGE QCD AND PERTURBATION THEORY

A systematic method of construction of a quantum theory of gauge fields consists, as is well known, in the following. One first constructs a lattice version of the theory, then one determines the critical points (points of second-order phase transitions). Finally, the last step consists in passing to the continuum limit: the size of the lattice is made to go to zero and at the same time the bare parameters (coupling constants) must approach the appropriate critical values in such a manner that dimensional physical quantities remain finite.

The lattice gauge theory is constructed in the following manner.[1] Into a four-dimensional Euclidean space one builds a hypercubic lattice the sites of which, $\mathbf{x}=n_\mu \mathbf{e}_\mu$, are determined by the set of integers n_μ and $\mathbf{e}_\mu$ denotes the vectors of an orthogonal frame ($\mathbf{e}_\mu\cdot\mathbf{e}_\nu=a^2\delta_{\mu\nu}$). The variables which correspond to the gauge fields are localized on the links (bonds, edges) of the lattice and are denoted by the symbol U_l (the subscript l denotes the link joining the sites x and $x+l$). Under the action of the gauge group G the matrices U_l transform as $U_l \to S_x U_l S_{x+l}^{-1}$. One obtains gauge-invariant quantities if one takes the trace of the product of factors ordered along a closed contour (loop) C in the lattice[2)]

$$U_C=\prod_{l\in C}U_l. \tag{2.1}$$

Everywhere in the sequel we shall consider theories with the gauge group $SU(N)$, and U_l will be the matrices of its fundamental representation.

The most general form of the action $S(U_{\partial p})$ which is invariant under gauge transformations, assuming nearest-neighbor interactions, is:

$$S(U_{\partial p})=\sum_p\sum_n\{\beta_n\chi_n(U_{\partial p})+\text{c.c.}\}, \tag{2.2}$$

where ∂p denotes the boundary of an elementary plaquette (square) p in the lattice; $\chi_n(U_{\partial p})$ is the character of the matrix $U_{\partial p}$ in the n-th representation (n is a comprehensive index of the representation), and β_n are the bare coupling constants of the theory.

The characters $\chi_n(U)$ are representations of the group $Z(N)$, the center of the group $SU(N)$. Under translations $U\to ZU$ of the group element $U\in SU(N)$ by an element of the center $Z\in Z(N)$ the characters $\chi_n(U)$ undergo the transformation

$$\chi_n(ZU)\to Z^{e[n]}\chi_n(U). \tag{2.3}$$

Here $e[n]$ is an integer equal, e.g., to one for the fundamental representation ($e[f]=1$), and to zero for the adjoint representation ($e[a]=0$).

Leaving aside the analysis of the general situation (2.2) we shall investigate the case when only the coefficients of the characters of the fundamental and the adjoint representations are nonvanishing[3)]:

$$S(U_{\partial p})=\sum_p \operatorname{Re}\left\{\frac{\beta}{N}\operatorname{Sp}U_{\partial p}+\frac{1}{2\lambda}|\operatorname{Sp}U_{\partial p}|^2\right\}. \tag{2.4}$$

We recall that the characters are given by the traces: $\chi_f(U)=\operatorname{Sp}U$, $\chi_a(U)=|\operatorname{Sp}U|^2-1$, and the summation is over all the links of the lattice. The first term coincides with the action of the standard lattice gauge theory.[1]

The second term,[4)] which was added for reasons which will become clear from the sequel, is invariant with respect to a larger group than the group of gauge transformations of the second kind. On each of the links of the lattice the matrices U can be transformed according to the law

$$U_l\to Z_lU_l,\quad Z_l=\exp(2\pi i n_l/N)\in Z(N), \tag{2.5}$$

where n_l is an integer. Such gauge transformations, which depend not only on the point but also on the direction in space, have been named gauge transformations of the third kind. The symmetry (2.5) plays an important role in the computation of expectation values in lattice gauge theories. Thus, it is clear that for $\beta=0$ the only nonvanishing expectation values are those of quantities invariant with respect to the transformations (2.5).

We shall first study the two-charge lattice gauge theory (2.4) by means of perturbation theory. It is easy to verify that the matrices

$$U_l=Z_lS_xS_{x+l}^{-1} \tag{2.6}$$

are extremals of the action. Indeed, substitution of the matrix

$$U_l = Z_l S_x u_l S_{x+l}^{-1} \tag{2.7}$$

into the action shows that it is quadratic with respect to deviations from the unit matrix. For the extremals (2.6) the action (2.4) takes the form

$$S(U_{\partial p}) = \beta \sum_p \mathrm{Re} \prod_{l\in\partial p} Z_l + \frac{N^2}{2\lambda}\sum_p 1 \tag{2.8}$$

and agrees apart from a constant with the action for the $Z(N)$ gauge theory.

The extremals we have found, and which we shall call central instantons, play an important role in quantum theory. We shall verify this for the case when the bare parameters $\beta \sim 1$ and $\lambda \ll 1$ are such that one may use perturbation theory with respect to λ.

We consider the Wilson loop average

$$W_1(C) = \int \prod_l d\mu(U_l) \frac{\mathrm{Sp}}{N} U_C \exp S(U_{\partial p}) \left[\int \prod_l d\mu(U_l) \exp S(U_{\partial p})\right]^{-1} \tag{2.9}$$

where $d\mu(U_l)$ is the Haar measure on the group $SU(N)$. When one computes $W_1(C)$ to zeroth order in λ one needs simply to sum over the central instantons. This yields

$$W_1^{(0)}(C) = \Gamma(C;\beta), \tag{2.10}$$

where Γ is the loop average of the $Z(N)$ gauge theory:

$$\Gamma(C,\beta) = \sum_{Z_l\in Z(N)} Z_C \exp\left(\beta\sum_p \mathrm{Re}\, Z_{\partial p}\right)\left[\sum_{Z_l\in Z(N)} \exp\left(\beta\sum_p \mathrm{Re}\, Z_{\partial p}\right)\right]^{-1}; \quad Z_C = \prod_{l\in C} Z_l. \tag{2.11}$$

In order to calculate the corrections in λ we note that for $\lambda \ll 1$ fields U_l of the form (2.7) are important in the functional integral (2.7), where u_l changes as one goes from link to link, i.e.,

$$u_{\partial p} = \exp(iF_p) = 1 + iF_p - F_p^2/2 + \ldots, \tag{2.12}$$

where $\mathrm{Sp}\, F_p^2 \ll 1$. For such smooth distributions the second term in the action (2.4) can be expanded in powers of the field strength F_p, retaining only the first nonvanishing term, which equals $\lambda^{-1} N \mathrm{Sp}\, F_p^2$. This yields for $W_1(C)$ the expression

$$W_1^{(1)}(C) = \sum_Z Z_C \left\langle \frac{\mathrm{Sp}}{N} u_C \exp\left(\beta\sum_p \mathrm{Re}\left(Z_{\partial p}\frac{\mathrm{Sp}}{N}u_{\partial p}\right)\right)\right\rangle \times\left[\sum_Z \left\langle \exp\left(\beta\sum_p \mathrm{Re}\left(Z_{\partial p}\frac{\mathrm{Sp}}{N}u_{\partial p}\right)\right)\right\rangle\right]^{-1}. \tag{2.13}$$

Here $\langle\ldots\rangle$ denotes averaging with the action $\lambda^{-1}N\mathrm{Sp}\, F_p^2$ over the fluctuations of the gauge field in the sense of perturbation theory. When one calculates the first order of perturbation theory in λ using this formula one must, of course, substitute the expansion (2.12) into it.

The computation of the average (2.13) simplifies considerably in the limit of a large number of colors[26]: $N\to\infty$ for fixed β and λ. In this limit the averages $\langle\ldots\rangle$ of gauge-invariant products factorize into products of averages. Making use of this property, we obtain

$$W_1^{(1)}(C) = \Gamma(C;\beta\omega(\partial p))\,\omega(C;\lambda), \tag{2.14}$$

where we have introduced the notation

$$\omega(C,\lambda) = \langle N^{-1}\mathrm{Sp}\, u_C\rangle. \tag{2.15}$$

Thus, for $\lambda \ll 1$ the loop average $W_1^{(1)}(C)$ is obtained by multiplying $\omega(C)$, which is determined to order $O(\lambda^2)$ by two terms of the perturbation series with coupling constant $g^2 = \lambda/N$, by $\Gamma(C)$—the loop average in a $Z(N)$-gauge theory with "renormalized" charge $\beta\omega(\partial p)$.

In calculating the next orders in λ there appear corrections to Eq. (2.14) from higher powers of F_p in the expansion of the second term in the action (2.4). It is easy to see that if one takes them into account, this leads simply to the result that after a "renormalization" of the constant $\lambda[\lambda\to\bar\lambda = \lambda/\omega(\partial p)]$ the above formulas remain valid if one only calculates the expectation values according to the formula

$$\langle Q(u_l)\rangle = \int \mathscr{D}\mu(u) Q(u_l)\exp\left(\frac{N}{\bar\lambda}\sum_p \mathrm{Re}\,\mathrm{Sp}\, u_{\partial p}\right) \times\left[\int \mathscr{D}\mu(u)\exp\left(\frac{N}{\bar\lambda}\sum_p \mathrm{Re}\,\mathrm{Sp}\, u_{\partial p}\right)\right]^{-1}, \tag{2.16}$$

which coincides with the expectation value in standard gauge theory with a coupling constant $\bar\lambda$. Thus, the factorized expression (2.14) is replaced, when all orders of λ are taken into account, by

$$W_1(C) = \Gamma(C;\beta\omega(\partial p))\,\omega(C;\lambda/\omega(\partial p)). \tag{2.17}$$

On the basis of a perturbative analysis the expression (2.16) for the expectation value should be taken to mean only an abbreviated form of the series in powers of λ. However, in the next section we shall show, using the loop equations of motion, that the factorized expression (2.17) with $\omega(C)$ defined by the expectation value (2.16) with $Q = N^{-1}\mathrm{Sp}\, u_C$ is valid not merely in each order of perturbation theory with respect to λ, but exactly.

To conclude this section we remark that the factorization occurs only for the expectation values (2.16) and does not extend to the loop functionals

$$W_n(C_1,\ldots,C_n) = \int \mathscr{D}\mu(U)\frac{\mathrm{Sp}}{N}U_{C_1}\ldots\frac{\mathrm{Sp}}{N}U_{C_n}\exp S(U) \times\left[\int \mathscr{D}\mu(U)\exp S(U)\right]^{-1}, \tag{2.18}$$

where $S(U)$ is the action (2.4). In the limit of a large number of colors ($N\to\infty$) we have

$$W_n(C_1,\ldots,C_n) = \Gamma(C_1+\ldots+C_n;\beta\omega(\partial p))\prod_{i=1}^n \omega(C_i;\bar\lambda), \tag{2.19}$$

where $\Gamma(C)$ coincides with the loop average in the $U(1)$ gauge theory [which is the limit of $Z(N)$ gauge theory as $N\to\infty$], which, of course, does not factor into a product of averages of individual loops.

3. THE LOOP EQUATIONS OF MOTION IN TWO-CHARGE QCD

In this section we give a derivation of Eq. (2.19) which is not based on perturbation theory, and consequently we prove its validity for loops of arbitrary size (we remind the reader that in QCD perturbation theory is valid for calculating the loop averages for small size loops, since the renormalized coupling constant increases with increasing distance). Our derivation is based on an analysis of the chain of equations for the loop averages $W_n(C_1, \dots, C_n)$. For the model with a single coupling constant these equations were derived in Refs. 23–25. Here we derive the chain of equations in the case of two-charge QCD of interest to us, and show that the factorized expression (2.19) is indeed a solution of this chain of equations.

In order to derive the loop equations of motion we utilize the standard method of shifting the variable in the functional integral (2.18):

$$U_l \to (1+i\varepsilon_l)U_l,$$

where ε_l is an infinitesimal traceless hermitean matrix. Proceeding as in Refs. 23–25, we obtain the following chain of coupled equations:

$$\frac{1}{2}\sum_p\Big\{\frac{\beta}{N^2}[W_n(C_1+\partial p,\dots,C_n)-W_n(C_1-\partial p,\dots,C_n)] + \frac{1}{\lambda}[W_{n+1}(-\partial p,C_1+\partial p,\dots,C_n)-W_{n+1}(\partial p,C_1-\partial p,\dots,C_n)]\Big\}$$
$$=\sum_{l'\in C_1}\delta_{xx'}\tau_\nu(l')\Big[W_{n+1}(C_{xx'},C_{x'x},\dots,C_n)-\frac{1}{N^2}W_n(C_1,\dots,C_n)\Big]$$
$$+N^{-2}\sum_{j=2}^{n}\sum_{l'\in C_j}\delta_{xx'}\tau_\nu(l')[W_{n-1}(C_1+C_j,\dots,\hat{C}_j,\dots,C_n) - W_n(C_1,\dots,C_j,\dots,C_n)]. \quad (3.2)$$

These equations are derived in the following manner. We open up the loop C_1 at the point $x\in C_1$, i.e., we consider the average containing the ordered product U_{C_1} (the contour C_1 is traversed starting at the point x), rather than its trace. In the integral so obtained we shift the integration variable U_l on some link of the lattice according to Eq. (3.1) and equate to zero the variation of the integral (the coefficient of ε_l). Taking into account the fact that only averages of gauge-invariant quantities are different from zero, we arrive by straight-forward but rather tedious calculations at the equations (3.2). All this was done in Refs. 23–25, and for this reason we give no details here. The only distinction is that the variation of the action in the standard model is different from the case of the two-charge QCD considered here, and this has been taken into account.

The left-hand side of Eq. (3.2) comes from the variation of the action (2.4). The contour $C_1+\partial p$ comes from the loop C_1 by adding to it at the point x the boundary ∂p of the plaquette p; $-\partial p$ denotes that the same boundary is traversed in the opposite direction. The loop $C_1-\partial p$ is also pictured in Fig. 1. The link l on which the shift (3.1) is effected starts at the point x in the direction ν, and the summation over p is over all the directions orthogonal to ν. The first summand in the left-hand side of Eq. (3.2) comes from the variation of the first term in the action (2.4) and coincides with the left-hand side of the equation in the standard lattice gauge model. It is convenient to introduce the notation

$$\frac{1}{2}\sum_p\{W_n(C_1+\partial p,\dots,C_n)-W_n(C_1-\partial p,\dots,C_n)\}\equiv L_\nu(x)W_n(C_1,\dots,C_n). \quad (3.3)$$

The second term appeared from the variation of the second term, which is quadratic in $\mathrm{Sp}\,U_{\partial p}$, of the action

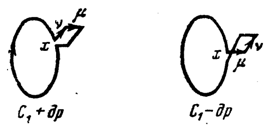


FIG. 1. The loops $C_1+\partial p$ and $C_1-\partial p$.

(2.4). It is for just this reason that $\mathrm{Sp}\,U_{\partial p}$ is added to the product of n traces $\mathrm{Sp}\,U_{C_i}$ after the variation, and in place of the n-loop average we obtain the $(n+1)$-loop average.

The right-hand side of the equation comes from the variation of the expression which is to be averaged. It does not depend on the form of the action and therefore coincides with the right-hand side of the equation in the standard lattice model. The term involving the sum over $l'\in C_1$ comes from the variation of U_{C_1} and the term with the sum over $l'\in C_j$ $(j>1)$ comes from the other loops. The point x' is defined as the origin[5] of the link l', and $\tau_\nu(l')=0,1$ is its projection on the basis vector $\mathbf{e}_\nu$. The notation in the form of a sum of the Kronecker delta $\delta_{xx'}$ over the loop is convenient for loops with arbitrarily many self-intersections. If a loop has at the point x a self-intersection of k-th order, the summation over this loop is easily done and one obtains a sum of k different terms (cf. Ref. 25). The loops $C_{xx'}$, $C_{x'x}$ and C_1+C_j are shown in Fig. 2, where the double line joining the points x and x' represents the Kronecker delta $\delta_{xx'}$. The symbol $\hat{C}_j$ signifies that the corresponding loop is absent. One should keep in mind that the loop C_1 was selected arbitrarily. This means that we have a row of n equations which come from opening up each of the loops C_i.

In the limit of a large number of colors ($N\to\infty$ for β and λ fixed) one may neglect terms of the order $O(N^{-2})$,[6] and the equation (3.2) simplifies:

$$\frac{1}{2\lambda}\sum_p\{W_{n+1}(-\partial p,C_1+\partial p,\dots,C_n)-W_{n+1}(\partial p,C_1-\partial p,\dots,C_n)\}$$
$$=\sum_{l'\in C_1}\delta_{xx'}\tau_\nu(l')W_{n+1}(C_{xx'},C_{x'x},\dots,C_n)+O(N^{-2}). \quad (3.4)$$

It is easy to note that this equation has a factorized solution of the form (2.19). To see this we substitute the Ansatz (2.19) into Eq. (3.4). It will be satisfied for any $\Gamma(C_1+\dots+C_n)$ if and only if $\omega(C)$ satisfies the equation

$$L_\nu(x)\,\omega(C)=\bar{\lambda}\sum_{l'\in C}\delta_{xx'}\tau_\nu(l')\,\omega(C_{xx'})\,\omega(C_{x'x}), \quad (3.5)$$

where

$$\bar{\lambda}=\lambda/\omega(\partial p). \quad (3.6)$$

Strictly speaking this proof is incomplete, since we have not shown so far that Eq. (3.4) has no other fac-

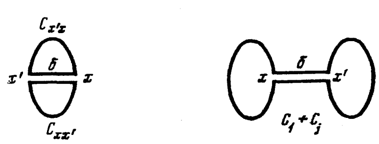


FIG. 2. The loops $C_{xx'}$, $C_{x'x}$, and C_1+C_j. The double line designates $\delta_{xx'}$.

torized solutions. In order to complete the proof it is necessary to derive an equation for the difference

$$G_n(C_1,\ldots,C_n)=W_n(C_1,\ldots,C_n)-\Gamma(C_1+\ldots+C_n)\prod_{i=1}^{n}\omega(C_i). \quad (3.7)$$

Such an equation is easily derived from Eq. (3.2), account being taken of (3.5), and is an inhomogeneous equation. Therefore it fixes the order of magnitude of the quantity G_n for $N\to\infty$. The inhomogeneous term is proportional to N^{-2}, yielding $G_n=O(N^{-2})$. Thus, the factorized solution is valid up to terms of order $O(N^{-1})$.

As was already noted, $\Gamma(C)$ is not determined by the equation (3.4). This ambiguity[7)] reflects the fact that the second term in the action (2.4) is invariant with respect to the transformations (2.5). The equation which determines $\Gamma(C)$ is obtained by shifting the integration variable by an element from the center of the group. We consider directly the large-N limit, when the discrete group $Z(N)$ goes over into the continuous group $U(1)$, i.e., the transformation (2.5) becomes

$$U_l\to\exp(i\eta_l)U_l, \quad (3.8)$$

where η_l is an infinitesimal real number. Noting that only the first term in the action (2.4) changes under the transformations (3.8), we obtain the equation

$$\frac{\beta}{2}\sum_p\{W_{n+1}(\partial p,C_1,\ldots,C_n)-W_{n+1}(-\partial p,C_1,\ldots,C_n)\}$$
$$=\sum_{l'\in C_1}\delta_{xx'}\tau_\nu(l')W_n(C_1,\ldots,C_n). \quad (3.9)$$

Substituting into this equation the factorized Ansatz (2.19) it is easy to see that it is a solution if $\Gamma(C)$ satisfies the equation

$$L_\nu(x)\Gamma(C)=e^2\sum_{l'\in C}\delta_{xx'}\tau_\nu(l')\Gamma(C), \quad (3.10)$$

where

$$e^2=1/\beta\omega(\partial p). \quad (3.11)$$

The equations (3.5) and (3.10) coincide, respectively, with the equations for the loop averages in the standard $SU(N)$ and $U(1)$ gauge theories. Consequently, to order $O(N^{-1})$, Eq. (3.5) has a solution in the form of the functional integral:

$$\omega(C)=\int\mathscr{D}\mu(U)\frac{\mathrm{Sp}}{N}U_C\exp\Big(\frac{N}{\lambda}\sum_p\mathrm{Re\,Sp}\,U_{\partial p}\Big)$$
$$\times\Big[\int\mathscr{D}\mu(U)\exp\Big(\frac{N}{\lambda}\sum_p\mathrm{Re\,Sp}\,U_{\partial p}\Big)\Big]^{-1}, \quad (3.12)$$

where the integration is over the group $SU(N)$, and the equation (3.10) has the solution

$$\Gamma(C)=\int\mathscr{D}\mu(Z)Z_C\exp\Big(\frac{1}{e^2}\sum_p\mathrm{Re}\,Z_{\partial p}\Big)$$
$$\times\Big[\int\mathscr{D}\mu(Z)\exp\Big(\frac{1}{e^2}\sum_p\mathrm{Re}\,Z_{\partial p}\Big)\Big]^{-1}, \quad (3.13)$$

which coincides with the expression of the loop average in a compact abelian gauge theory.

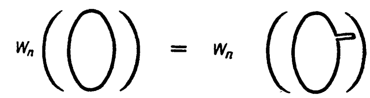
In the limit $\beta\to 0$ the expression (3.13) differs from zero only for a loop of zero minimal area[8)]:

$$\Gamma(C;\beta=0)=\delta_{0A_{min}} \quad (3.14)$$

This result has a natural explanation from the point of view of the symmetry (2.5). As was discussed in Sec. 2, the average (2.18) at $\beta=0$ is different from zero only in the case when the expression to be averaged is invariant with respect to the transformations (2.5). This is so, for instance, for $n=2$ if $C_2=-C_1$, i.e., under the average sign there appears $|N^{-1}\mathrm{Sp}\,U_{C_1}|^2$. The loop C_1+C_2 has a zero minimal surface. One should also note that at $\beta=0$ the equation (3.9) is simply the condition of invariance relative to the transformations (2.5), which is guaranteed by the fact that the equation (3.10) has the unique solution (3.14).

The expressions (3.12) and (3.13) generalize the corresponding formulas of Sec. 2 to the case of loops of arbitrary size and for values of λ which are not small, when it is impossible to use perturbation theory. These formulas are exact in accord with their derivation. In the following sections they will be used to determine the phase equilibrium curve in the λ,β plane corresponding to spontaneous breakdown of the symmetry of the center of the gauge group.

To conclude this section we show that the Bianchi identity, which $W_n(C_1,\ldots,C_n)$ must satisfy[23] in addition to the equations of motion and the boundary conditions, is satisfied for the factorized expression (2.19). In the lattice theory it is convenient to use Bianchi identities of the following form:



(3.15)

The identity (3.15) is satisfied for the factorized expression (2.19), since both $\omega(C)$ and $\Gamma(C)$, defined respectively by expressions (3.12) and (3.13), satisfy the identity (3.15) separately.

4. THE PHASE TRANSITION WITH RESPECT TO THE CENTER OF THE GAUGE GROUP

In the preceding section we have shown that for a large number of colors[9)] the factorized formula (2.19) becomes valid for the loop average (2.18), with $\Gamma(C)$ and $\omega(C)$ defined by Eqs. (3.13) and (3.12), respectively, and e^2 and $\bar{\lambda}$ given by the expressions (3.11) and (3.6). It is known[28] (for a mathematically rigorous proof, cf. Ref. 29), that in an abelian theory with gauge group $U(1)$ there is a phase transition, and the asymptotic behaviors of $\Gamma(C;e^{-2})$ in the two phases are different. On the basis of this information one can reach the conclusion that there exists a phase transition in two-charge QCD with the action (2.4). To verify that this is indeed so it suffices to consider the irreducible correlation function

$$K_2(C_1,C_2)=W_2(C_1,C_2)-W_1(C_1)W_1(C_2), \quad (4.1)$$

which expresses the van der Waals forces between colorless objects formed by a quark and antiquark. Substituting the expression (2.19) into (4.1) we obtain

$$K_2(C_1,C_2)=\omega(C_1)\omega(C_2)[\Gamma(C_1+C_2)-\Gamma(C_1)\Gamma(C_2)], \quad (4.2)$$

i.e., $K_2(C_1,C_2)$ is proportional to the irreducible correlation function for the $U(1)$ gauge theory. Considering the case when the distance between the loops C_1 and C_2

is large compared to their sizes, we verify that in our theory, as well as in the $U(1)$ gauge theory, there are long-range van der Waals forces for $\beta \gg 1$, and consequently there exist massless excitations (Goldstone particles) which are absent at $\beta \ll 1$.[28,30]

Another method of checking that a phase transition occurs in two-charge QCD as β increases, and at the same time of determining the symmetry breaking related to this phase transition, is to study the correlation functions of the disorder parameter.[13] However, in the case under consideration it is more convenient not to use the disorder parameter, but the closely related criterion based on a phenomenon[10)] similar to the Higgs effect in abelian chiral theories.[11)] We recall that in abelian chiral theories one studies the spontaneous breakdown of the global symmetry group of a Lagrangian. One of the ways of answering the question whether a symmetry is broken or not is to include in the Lagrangian an external guage field that makes this symmetry local, and then calculate the dependence of the free energy on the external field. Depending on the phase in which the system is situated, the free energy depends differently on the slowly-varying weak external field.

In the case of a gauge theory we shall proceed completely similarly and switch on an external field which reduces to the symmetry of the second kind of the center of the gauge group to a symmetry of the third kind, i.e., we switch on a field which is capable of absorbing the change of the action under the transformation (2.5) of the fields:

$$S(G_p, U_{\partial p}) = \sum_p \left\{ \frac{\beta}{N} e^{iG_p} \operatorname{Sp} U_{\partial p} + \frac{1}{2\lambda} |\operatorname{Sp} U_{\partial p}|^2 \right\}. \tag{4.3}$$

The field G_p is attached to the oriented plaquettes of the lattice and changes sign when the orientation is changed. The action (4.3) is invariant under gauge transformation of the third kind:

$$U_l \to e^{i\eta_l} U_l, \quad G_p \to G_p - \sum_{l \in \partial p} \eta_l. \tag{4.4}$$

The dependence of the free energy on the external field G_p is given by

$$\exp(-F(G_p)) = \Omega(G_p)/\Omega(0), \tag{4.5}$$

where

$$\Omega(G_p) = \int \mathcal{D}_\mu(U) \exp\{S(G_p, U_{\partial p})\}. \tag{4.6}$$

From the definitions (4.5) and (4.6) it is obvious that the free energy $F(G_p)$ remains unchanged under a shift

$$G_p \to G_p - \sum_{l \in \partial p} \eta_l.$$

The free energy we have introduced can serve as a "litmus paper" for the identification of the phase in which the system finds itself. The dependence of $F(G_p)$ on the slowly varying field G_p is different in a phase with broken symmetry of the center of the gauge group than in the symmetric phase. In the symmetric phase the expansion of the free energy density in powers of G_p starts with a term proportional to $(\nabla_\mu^* G_{[\mathbf{x},\mu\nu]})^2$, where $^*G_{[\mathbf{x},\mu\nu]}$ is the field dual to $G_{[\mathbf{x},\mu\nu]}$, ∇_μ is the covariant difference operator, and $[\mathbf{x}, \mu\nu]$ denotes the plaquette p. In the phase with spontaneously broken symmetry of the center of the gauge group the expansion starts with $(G^\perp_{[\mathbf{x},\mu\nu]})^2$ (here $G^\perp_{[\mathbf{x},\mu\nu]}$ denotes the transverse part of the field G: $\nabla_\mu G^\perp_{[\mathbf{x},\mu\nu]} = 0$).

The free energy $F(G_p)$ can be calculated in our case of two-charge QCD. In order to determine $\Omega(G_p)$ we expand the exponential of $S(G_p, U_{\partial p})$ in powers of β:

$$\Omega(G_p;\beta) = \Omega(\beta=0) \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \left\langle \left\{ \sum_p \operatorname{Re}\left(e^{iG_p} \frac{\operatorname{Sp}}{N} U_{\partial p} \right) \right\}^n \right\rangle_{\beta=0}, \tag{4.7}$$

where $\langle\{\ldots\}\rangle_{\beta=0}$ denotes averaging with the action (2.4) at $\beta=0$. Calculating the loop averages in the expression (4.7) according to Eq. (2.19) we obtain

$$\Omega(G_p;\beta) = \Omega(\beta=0) \sum_{n=0}^{\infty} \frac{(\beta\omega(\partial p))^n}{n!} \times \sum_{p_1 \ldots p_n} \cos G_{p_1} \ldots \cos G_{p_n} \Gamma(\partial p_1 + \ldots + \partial p_n; \beta=0), \tag{4.8}$$

where $\Gamma(\partial p_1 + \ldots + \partial p_n; \beta = 0)$ is the loop average in the $U(1)$ gauge theory at $\beta=0$, and is given by Eq. (3.14). The series (4.8) is easily summed:

$$\Omega(G_p) = \Omega(\beta=0) \int_{-\pi}^{\pi} \prod_l \frac{dA_l}{2\pi} \exp\left\{ \beta\omega(\partial p) \sum_p \cos\left(\sum_{l \in \partial p} A_l + G_p \right) \right\}, \tag{4.9}$$

and this agrees, apart from a constant, with the energy in the $U(1)$ gauge theory with the coupling constant $e^{-2} = \beta\omega(\partial p)$. Returning to the definition (4.5) we obtain the result that the dependence of the free energy on the external field G_p is the same for the two-charge QCD with an infinite number of colors as for the $U(1)$ gauge theory:

$$F(G_p; \beta, \lambda) = F_{U(1)}(G_p; e^2). \tag{4.10}$$

Since in the $U(1)$ gauge theory the phase transition occurs for a value of the charge e_c^2, it follows from this equation that in two-charge QCD a phase transition occurs for

$$\beta_c^{-1}(\lambda) = e_c^2 \omega(\partial p; \lambda). \tag{4.11}$$

Since in $U(1)$ gauge theory the dependence of the free energy on the external field below and above the critical point starts respectively with the terms $(\nabla_\mu {}^*G_{[\mathbf{x},\mu\nu]})^2$ and $(G^\perp_{[\mathbf{x},\mu\nu]})^2$, the same dependence is valid in two-charge QCD above and below the phase transition point, respectively. Consequently, *at the phase transition point there occurs a spontaneous breakdown of the symmetry of the center of the gauge group.*

5. THE PROBLEM OF QUARK CONFINEMENT

In the preceding section we have made it clear that in two-charge QCD there exists a phase transition, such that in the "low-temperature" ($\beta > \beta_c(\lambda)$) phase the symmetry of the center of the gauge group is spontaneously broken. In this section we discuss the consequences one can draw from the information already available to us with regards to the problem of confinement of quarks within hadrons.

First of all we note that in the symmetric phase ($\beta < \beta_c(\lambda)$) the quarks are no doubt confined, since the Wilson criterion[1] is satisfied. Indeed, the loop average contains the factor $\Gamma(C; e^{-2})$ which in the symmetric phase has the asymptotic behavior

$$\Gamma(C;\, e^{-2}) \propto \exp\{-r_c^{-2}(e^2) A_{min}\},$$

where A_{min} is the area of a minimal surface spanned by the loop C and $r_c^2(e^2)$ is the correlation length. The "area law" is valid in the symmetric phase independently of the asymptotic behavior of the factor $\omega(C)$ in the factorized expression (2.17).

In the "low-temperature" phase the asymptotic behavior of $\Gamma(C, e^{-2})$ is different ($\Gamma \propto \exp\{-\text{length}\}$), and the "area law" could be valid only on account of $\omega(C)$. However, we cannot calculate $\omega(C)$ for a loop of large size, and symmetry considerations do not play a role in the case of the Wilson criterion. Therefore this criterion does not allow one to decide whether the quarks are or are not confined in the "low-temperature" phase.

Polyakov[12] has proposed for quark confinement another criterion that relates the problem with that of the breaking of the symmetry of the center of the gauge group. This criterion is based on the study of loop correlation functions in a system with boundary conditions which are periodic in "time." Topologically such a system is homeomorphic to a cylinder, and consequently one may consider the loop averages for loops which surround the cylinder and cannot be contracted to a point (see Fig. 3). As was shown in Ref. 12, such averages are directly related to the interaction energy of static quarks.

The assertion that the symmetry of the center of the gauge group is related to the quark confinement problem comes about in the following way. The action (2.4) and the integration measure are invariant with respect to the transformations of the type (2.5) for which U_l depends only on the "time" x_0, i.e.,[12)]

$$n_l = \begin{cases} \pm n(x_0) & \text{for} \quad \tau_0(l) = \pm 1 \\ 0 & \text{for} \quad \tau_0(l) = 0 \end{cases}. \tag{5.1}$$

Indeed, under such a transformation only the fields U_l localized on the links directed along the time direction are subjected to a shift by an element of $Z(N)$, in such a manner that for a given spacelike leaf the shift is independent of the spatial coordinate. For such a shift the product $U_{\partial p}$ [cf. (2.1)] does not change since in addition to the link having the direction $\tau_0(l) = 1$ there must be for each loop ∂p a link having the opposite sense $\tau_0(l') = -1$, so that the shifts cancel mutually. Consequently the action (2.4) remains unchanged.

Similarly, the loop product U_C does not change when the loop C is contractible (to a point). If a loop $\tilde{C}$ surrounds the cylinder then $U_{\tilde{C}}$ is shifted by an element of $Z(N)$. Thus, for the loop $\tilde{C}_1$ in Fig. 3

$$\frac{\text{Sp}}{N} U_{\tilde{C}_1} \to \exp\Big\{ i \frac{2\pi}{N} \sum_{l \in \tilde{C}_1} n(x_0)\, \tau_0(l) \Big\} \frac{\text{Sp}}{N} U_{\tilde{C}_1}, \tag{5.2}$$

where x_0 is the "time" coordinate of the origin of the

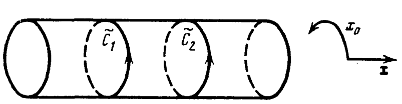


FIG. 3. A lattice with periodic boundary conditions in x_0. The loops $\tilde{C}_1$ and $\tilde{C}_2$ surround the cylinder.

link l. As can be seen from this equation the symmetry with respect to the transformations (2.5), (5.1) is nothing but the symmetry of the center of the gauge group, discussed earlier,[12,13] in a slightly different presentation.

The existence of this symmetry leads, according to Eq. (5.2), to the result $W_1(\tilde{C}_1) = 0$. If the symmetry of the center is spontaneously broken one must expect[13)] the order parameter not to vanish: $W_1(\tilde{C}_1) \neq 0$. As was shown in Ref. 12, for nonvanishing $W_1(\tilde{C}_1)$ the quarks are liberated. Together with the results of the preceding section on the spontaneous breakdown of the symmetry of the center of the gauge group in two-charge QCD, these results lead to the conclusion that in the *"low-temperature" phase the quarks can be emitted.*

Although this assertion is made for the case of two-charge QCD with the action (2.4), it also leads to conclusions for the standard Wilson lattice gauge theory. With this in mind we calculate in the two-charge QCD the order parameter $W_1(\tilde{C}_1)$ which, as is usually done in the theory of phase transitions, must be determined from the two-loop average $W_2(\tilde{C}_1, C_2)$ (see Fig. 3) by moving the loops $\tilde{C}_1$ and $\tilde{C}_2$ apart. In order to calculate W_2 we make use of the factorized expression (2.19), the validity of which for the lattice gauge theory was proved in Sec. 3 for arbitrary boundary conditions in "time," and is valid in our case. Considering the limit of large distance between the loops $\tilde{C}_1$ and $\tilde{C}_2$, we find

$$W_1(\tilde{C}_1;\, \beta,\, \lambda) = \Gamma(\tilde{C}_1;\, e^{-2})\, \omega(\tilde{C}_1;\, \bar{\lambda}), \tag{5.5}$$

where $\omega(\tilde{C}_1; \bar{\lambda})$ coincides with the order parameter in standard lattice gauge theory with coupling constant $\bar{\lambda}$. Since in the "low-temperature" phase the effective coupling constant of the $U(1)$ gauge theory is smaller than its critical value, $e^2(\lambda) < e_c^2$, and therefore $\Gamma(\tilde{C}_1; e^{-2}) \neq 0$, then we have $\omega(\tilde{C}_1) \neq 0$, because $W_1(\tilde{C}_1) \neq 0$. This means that in the standard lattice gauge theory there exists a phase transition related to quark liberation.

6. THE PHASE DIAGRAM AND THE SELF-CONSISTENCY CONDITION

As already noted in Sec. 2, to construct a Lorentz-invariant continuum limit of the lattice gauge theory it is first of all necessary to determine the positions of the second-order phase transition points (where the correlation lengths become large compared to the lattice spacing a). In order to solve this problem we consider the phase diagram defined by Eq. (4.11).

The right-hand side of Eq. (4.11) contains the loop average $\omega(\partial p; \bar{\lambda})$ of the standard model with coupling constant $\bar{\lambda}$ defined by the expression (3.12). The coupling constant $\bar{\lambda}$ itself depends on $\omega(\partial p; \bar{\lambda})$ according to Eq. (3.6); thus, there arises the self-consistency requirement

$$\omega(C;\, \bar{\lambda})\,|_{C=\partial p} = \lambda / \bar{\lambda}. \tag{6.1}$$

If the functional dependence of ω on $\bar{\lambda}$ is known, i.e., $\omega(\partial p; \bar{\lambda}) \equiv f(\bar{\lambda})$, where f is a function determined by the dynamics of the standard model. It will be seen below that the self-consistency condition plays an important role in the dynamics of the two-charge model.

The function $f(\bar\lambda)$ can be calculated for small values of $\bar\lambda$ by using perturbation theory, and for large values of $\bar\lambda$ by using strong-coupling expansions, yielding

$$f(\bar\lambda)=1-A\bar\lambda+O(\bar\lambda^2) \quad \text{for} \quad \bar\lambda\ll 1,$$
$$f(\bar\lambda)=\frac{1}{2\bar\lambda}+\frac{1}{8\bar\lambda^5}+O(\bar\lambda^{-9}) \quad \text{for} \quad \bar\lambda\gg 1. \tag{6.2}$$

In the intermediate region "thermodynamic" inequalities impose important restrictions on the form of the function f.

We consider a system for which the partition function $Z_1(\bar\lambda)$ is defined by the denominator of Eq. (3.12). Then the mean energy density will be

$$\langle\varepsilon(\bar\lambda)\rangle=\frac{\bar\lambda^2}{N^2V}\frac{d\ln Z_1(\bar\lambda)}{d\bar\lambda}=-f(\bar\lambda). \tag{6.3}$$

Here V denotes the "volume" of the system, equal to the total number of plaquettes of the lattice. The stability condition consists in requiring the "heat capacity," which is the derivative of $\langle\varepsilon(\bar\lambda)\rangle$ with respect to the "temperature" $\bar\lambda$, to be nonnegative. This yields

$$df(\bar\lambda)/d\bar\lambda\leqslant 0, \tag{6.4}$$

i.e., $f(\bar\lambda)$ is a monotonically nondecreasing function of $\bar\lambda$ [in agreement with Eq. (6.2)].

Another inequality comes from considering a system for which the partition function $Z_2(\lambda)$ is given by the denominator of Eq. (2.18). In analogy with (6.3) we obtain

$$\langle\varepsilon(\lambda)\rangle=\frac{\lambda^2}{N^2V}\frac{d\ln Z_2(\lambda)}{d\lambda}=-\frac{f^2(\bar\lambda(\lambda))}{2}, \tag{6.5}$$

where the factorized expression (2.19) has been used, which is valid for large N. The condition of nonnegativity of the "heat capacity" $d\langle\varepsilon(\lambda)\rangle/d\lambda$, with account taken of Eq. (6.4), yields the stability condition:

$$d\bar\lambda(\lambda)/d\lambda\geqslant 0. \tag{6.6}$$

This, in turn, means that if $\bar\lambda(\lambda)$ is a decreasing function, the corresponding phase is absolutely unstable.

Substituting the asymptotic forms (6.2) into the self-consistency condition (6.1) we obtain

$$\bar\lambda=\lambda+A\lambda^2+\ldots \quad \text{for} \quad \lambda\ll 1, \tag{6.7a}$$

$$\bar\lambda=[4(2\lambda-1)]^{-1/2}+\ldots \quad \text{for} \quad \lambda-{}^1/_2\ll 1. \tag{6.7b}$$

Finally, for arbitrary λ there is yet another solution, namely

$$\bar\lambda(\lambda)=\infty \text{ for arbitrary } \lambda. \tag{6.7c}$$

The solutions (6.7a) and (6.7b) are computable limits of the function $\bar\lambda(\lambda)$ determined by the function $f(\bar\lambda)$. However, the sign of the derivative $\bar\lambda'(\lambda)$ and the monotonicity of $f(\bar\lambda)$ allow one to reach the conclusion that for $\frac{1}{2}<\lambda<\lambda_c$, where λ_c is some number of order unity, there are at least two solutions, with the solution (6.7b) having $\bar\lambda'(\lambda)<0$ and consequently unstable.

The solution (6.7a) has $\bar\lambda'(\lambda)\geqslant 0$ for $0<\lambda<\lambda_c$, i.e., is stable. In this region there exists another stable solution (6.7c). It is clear that physically the solution with the largest free energy density $F(\lambda)=-N^{-2}V^{-1}\ln Z_2(\lambda)$ for given λ is realized. From (6.5) we find that $F(\lambda)$ takes on a maximal value if for $0<\lambda<\lambda_c$ one chooses the solution (6.7a), and for $\lambda>\lambda_c$ one chooses the solution (6.7c):

$$F(\lambda)=-\int_\lambda^{\lambda_c}\frac{d\lambda}{2\bar\lambda^2(\lambda)} \quad \text{for} \quad 0<\lambda<\lambda_c,$$
$$F(\lambda)=0 \quad \text{for} \quad \lambda>\lambda_c. \tag{6.8}$$

The general rule is such that for a given λ the solution with the least $\bar\lambda(\lambda)$ is realized.

Figure 4 shows the two-charge QCD phase diagram determined by the equation $\beta_c^{-1}=e_c^2\lambda/\bar\lambda(\lambda)$. The solid line represents the solution which is realized, and the hatched line represents the unstable branch (6.7b). According to Sec. 5, below the critical point, where the symmetry of the center of the gauge group is spontaneously broken, the quarks are not confined. The region above the critical point corresponds to the phase with unbroken symmetry, and the quarks are confined. However, for $\lambda>\lambda_c$ the effective "temperature" of the $U(1)$-model vanishes: $e^2(\lambda)=0$. Therefore the factor $\Gamma(C)$ which enters into the loop averages is nonzero only for loops of vanishing minimal surface area [cf. Eq. (3.14)]. This means that not only are the quarks confined, but that they do not propagate at all inside the hadrons, i.e., only such field configurations are possible, for which the quark and the antiquark are localized at the same lattice site. In other words, the correlation length vanishes for $\lambda>\lambda_c$. The phase transition, which is denoted by the vertical dotted line, is fictitious, similar to the phase transition discussed by Gross and Witten.[31] If we had considered the case of finite values of N, we would have convinced ourselves that the correlation length simply becomes N-dependent for $\lambda>\lambda_c$, and vanishes in the limit $N\to\infty$. This situation has been called the absence of quarks.

As was discussed in Sec. 5, the existence of a "low-temperature" phase in two-charge QCD implies that in the standard Wilson lattice gauge theory there must be a phase transition, with a critical value $\gamma_*\geqslant\bar\lambda(\lambda_c)$. We note that the position of γ_* is bounded from the right by the point γ_c of the Gross-Witten phase transition in the standard model: $\bar\lambda(\lambda_c)\leqslant\gamma_*\leqslant\gamma_c$.

It follows from the obtained phase diagram that for the existence of a Lorentz-invariant continuum limit in the phase with confined quarks it is necessary to let λ go to zero as $a\to 0$ according to the asymptotic freedom formula, and to let β approach its critical value

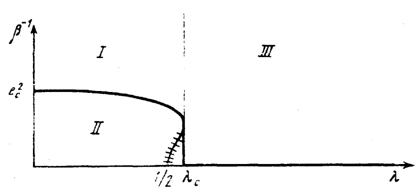


FIG. 4. The phase diagram corresponding to spontaneous breakdown of the symmetry of the center of the gauge group $SU(N)$ for large N. The region I corresponds to confined quarks, the region II corresponds to liberated quarks, and in the region III the quarks are absent.

e_c^2 in such a manner that the correlation lengths in $\Gamma(C)$ and $\omega(C)$ remain fixed. The Lorentz invariance of the limit is guaranteed by the fact that in the $U(1)$-lattice theory the phase transition is of second order.[7,10] The ratio of the correlation lengths in $\Gamma(C)$ and $\omega(C)$ is an arbitrary quantity. As was discussed in Sec. 1, the meaning of this is that in the center confinement mechanism the confinement radius of the quarks and the screening radius of the color charge are, in general, not related to one another.

With this limiting procedure the Lorentz invariance of the continuum limit is guaranteed both for $\Gamma(C)$ and for $\omega(C)$ [in contradistinction from the case $\beta^{-1}\to e^2(\lambda)$ for finite $\lambda<\lambda_c$, when a continuum limit exists for $\Gamma(C)$ but there is no reason to expect the existence of a Lorentz-invariant continuum limit for $\omega(C)$]. Moreover, with respect to λ there will be asymptotic freedom in the confined-quark phase.

At first glance one might think that there exists another method of obtaining asymptotic freedom in the confined phase, by considering the case $\gamma=N^2\beta^{-1}\to 0$ for $\lambda<\lambda_c$ (we fix γ rather than β^{-1} in the $N\to\infty$ limit in order that the correlation length be nonvanishing). However, according to footnote[7]), at $\gamma\ll\lambda$ the two-charge model reduces to the standard model, where, as we have shown above, there is a phase transition in γ at the point $\gamma_*\neq 0$. Therefore, if we choose the bare value $\gamma\to 0$, which is required for asymptotic freedom, we end up in the "low-temperature" phase, where quarks are not confined. Consequently, the only possibility of obtaining a Lorentz-invariant continuum theory with confined quarks and asymptotic freedom (in λ) is to let λ go to zero and β go to e_c^2-0 as $a\to 0$.

7. CONCLUSION

In conclusion we discuss once again the adopted hypotheses and the consequences stemming from them.

We have assumed that in guage theory there exists an only reason for quark confinement, namely the breakdown of the symmetry of the center of the gauge group. It was assumed that in the symmetric phase the quarks are confined within the hadron, and in the phase with spontaneously broken symmetry of the center they are liberated, and therefore the order parameter $W_1(\tilde{C}_1)$ is different from zero.

We have shown that in the large-N limit the order parameter in a guage theory with the standard action[1] is related to the order parameter of the two-charge model by Eq. (5.5), and that for $e^2(\lambda)<e_c^2$ the proportionality factor is different from zero. In Sec. 4 it was made clear that this parameter region corresponds to a phase with spontaneously broken symmetry of the center of the gauge group, and consequently the order parameter of the two-charge QCD must be different from zero, implying a nonvanishing value of that parameter for the standard theory also. Thus, in the standard model there must be a phase transition related to the liberation of the quarks.

It was shown that for the two-charge QCD which contains as a limiting case the standard model, there exists a unique possibility of combining the confinement of quarks in the lattice approach with the requirement that the continuum theory which is obtained in the limit $a\to 0$ be Lorentz invariant. For this it is necessary to let λ go to zero according to the asymptotic freedom formula, simultaneously with $a\to 0$, while β approaches the critical value $e_c^{-2}-0$, all in such a manner that the correlation lengths remain fixed.

The continuum theory resulting from this limiting procedure will be asymptotically free with respect to λ. Unfortunately, until now the renormalization properties of the $Z(N)$ lattice gauge theory have not been studied for $\beta\to e_c^{-2}-0$. However, there are reasons to believe that they are such that the nonperturbative effects contained in $\Gamma(C)$ at small distances are unimportant. There is no doubt that this question is of great interest and deserves a detailed investigation.

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1) Recent reviews on lattice gauge theories[15] and on quark confinement[16] contain additional references to original papers.

2) We note that in the continuum limit $a\to 0$, the operator U_C goes over into the well-known holonomy operator

$$U_C\to P\exp\left(i\int_C A_\mu\,dx_\mu\right).$$

3) In the cases where N is a prime number the action (2.4) is practically the most general, since then the group $Z(N)$ has no subgroups.

4) In the formal local limit ($a\to 0$ at fixed λ,β) both terms go over into the usual continuum action of the Yang-Mills field:

$$S(U)\to -\frac{1}{2}\left(\frac{\beta}{N}+\frac{N}{\lambda}\right)\int d^4x\,\mathrm{Sp}\,F_{\mu\nu}^2(x).$$

This limit is easily derived if one utilizes the formula $U_{\partial p}\to\exp(iF_{\mu\nu}n_{\mu\nu}a^2)$, where $n_{\mu\nu}$ is the unit orientation 2-form of the plaquette p [cf. (2.12)].

5) We recall that if the link l has negative orientation then its endpoint is to be considered as its origin.

6) The fact that $W_n(C_1,\ldots,C_n)\sim 1$ follows from the boundary condition $W_n(0,\ldots,0)=1$ and from equation (3.2) where $C=0$ denotes a loop contracted to a point.

7) This ambiguity is, of course, absent in the standard model, which is obtained from our action (2.4) at $\beta=N^2\gamma^{-1}$ and $\lambda\to\infty$. In this case there is no additional "almost symmetry" relative to the transformations (2.5), and as a consequence of this only $\Gamma=1$ is possible.

8) A loop with zero area minimal surface contracts to a point with the help of the Bianchi identity (3.15).

9) In this section we consider the limiting theory with $N\to\infty$ for fixed β and λ, and speaking of the center of the gauge group we always have in mind the limit $Z(N\to\infty)=U(1)$.

10) This phenomenon was first studied in superconductivity theory and is known there as the Meissner effect.

11) The idea of such an approach was first proposed by A. M. Polyakov (1976, private communication) in connection with the investigation of the phase transition in a $U(1)$ gauge theory.

12) We recall that $\tau_0(l)$ denotes the projection of the direction of the link l onto the basis vector $\mathbf{e}_0$.

13) It is assumed, essentially, that in addition to the symmetry of the center there are no other dynamical reasons for $W_1(\tilde{C}_1)=0$, and hence for quark confinement (see Section 1).

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The Hanle effect in a strong electromagnetic field

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A theory of the Hanle effect in a strong electromagnetic field, when perturbation theory is inapplicable, is developed. A two-level system with terms of which one has zero and the other unity angular momentum is considered. The probability for transition to a third level with zero angular momentum under conditions when the first two terms are at resonance with the strong field is computed. It is shown that the dependence of the probability changes from quadratic to linear as the field intensity is increased. The limits of very weak and very strong fields and the case of a very strong constant magnetic field that splits the term with unity angular momentum are analytically investigated. The intermediate cases are investigated with the aid of a computer calculation. The self-similar character of the problem is pointed out. It is concluded that the resonant character of the probability as a function of the magnetic field vanishes as the electric-field intensity increases.

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§1. FORMULATION OF THE PROBLEM

The optical phenomena connected with interference, due to the presence of adjacent levels, in the radiation of atoms are currently being intensively investigated.[1] One of these effects is the Hanle effect, which consists in the fact that in a magnetic field the intensity of spontaneous radiation with a given polarization depends on the distance, determined by the magnetic-field strength, between the adjacent Zeeman sublevels. The Hanle effect is explained by the fact that the probability of emission of radiation with some definite polarization for an atomic state that is a superposition of energetically close states is determined by the square of the modulus of the sum of the occupation amplitudes of these states. The dependence of the probability on the level spacing is due to the presence of an interference term in the square of the modulus of the sum.

The Hanle effect is normally observed in resonance excitations by radiation with a broad spectral line. Furthermore, it occurs in resonance excitations by

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