

Exactly solvable models for two-sublattice magnets

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Stationary-profile waves are investigated for an isotropic two-sublattice magnet whose free energy is described by a homogeneous quadratic form. An explicit form of solution is found, corresponding to a domain wall whose structure is determined by the ratio of the constants of uniform and of nonuniform exchange. A classification is made of the cases in which it is possible to lower the dimensionality of the phase space of the equation system and to separate the classes of exact solutions.

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1. Investigations of moving domain walls and of solitons in magnetic media have led to the discovery of new, completely integrable models, directly related to the basic equations of the dynamics of magnetically ordered media - the Landau-Lifshitz equations. In a work of Sklyanin¹ there is carried out, essentially, a proof of the integrability of the Landau-Lifshitz equations for a physically meaningful model of a uniaxial ferromagnet, with allowance for magnetic dipole interaction, in the case of x, t geometry. At present there is no doubt of the fact that a new source of completely integrable and physically meaningful models is provided by the Landau-Lifshitz equations for media with several magnetic sublattices, whose free energy is described by a homogeneous quadratic form in the magnetic moments of the sublattices. For example, we have shown² that integrable models can be extracted in the case of weak ferromagnets.

In the present paper, in the case of an isotropic two-sublattice magnet whose free energy is a quadratic form, stationary-profile waves are investigated, and an explicit form of solution is found that determines a topological soliton. The latter is, with respect to the anti-ferromagnetism vector, a moving domain wall, within whose bounds there exists a self-localized distribution of magnetic moment. The characteristic dimension of the region of self-localization of the magnetic moment is of the order of

$$\gamma AM_0^{-1}(2\gamma^2 ADM_0^{-2} - U^2)^{-1/2}. \quad (1.1)$$

Here γ is the gyromagnetic ratio, M_0 is the saturation magnetization, D and A are the constants of uniform and of nonuniform exchange, and U is the velocity of motion of the topological soliton. It is obvious that the limiting velocity

$$U_{max} = \gamma(2AD)^{-1/2} M_0^{-1}$$

determines the boundary that separates solitons and spin waves.

A peculiarity of the domain wall found is the fact that its structure is determined by the ratio of the constants of uniform and of nonuniform exchange. Realization of such a domain wall is probably possible in layered magnetic structures. The problem of stationary-profile waves for the Landau-Lifshitz equations in this situation can be compared to the mechanical problem of the motion of two material points on the surface of a sphere with a bilinear interaction potential.

Furthermore, for the case of a two-sublattice magnet

whose free energy is a general quadratic form, a new form of differential conservation law is found, connected with the translational invariance of the system (the analog of the law of conservation of momentum). The representation found can be generalized to the case of an arbitrary number of magnetic sublattices. Thus the geometric approach to the derivation of certain divergent forms, proposed in Ref. 3 and carried out in the case of an isotropic one-sublattice ferromagnet,² permits generalization to the case of substantially more general systems.

2. We consider an isotropic two-sublattice magnetic material with free energy of the form

$$2F = [(m_1')^2 + (m_2')^2 + 2m_1 m_2] D. \quad (2.1)$$

The spatial coordinate is referred to δ_0 , where δ_0^2 is the ratio of the constants A of nonuniform and D of uniform exchange. For stationary-profile waves,

$$m_i(x, t) = m_i(x - Ut)$$

and the Landau-Lifshitz equations have the form ($f' = df/d\xi$)

$$\begin{aligned} u m_1' + \mu_1' &= [m_1 m_2], \\ u m_2' + \mu_2' &= -[m_1 m_2]. \end{aligned} \quad (2.2)$$

Here

$$\mu_i = [m_i m_i'], \quad m_i^2 = 1, \quad i=1, 2, \quad (2.3)$$

$$\xi = x/\delta_0 - ut/t_0, \quad t_0 = M_0/\gamma D,$$

and the parameter u is connected with the velocity of the stationary-profile wave by the relation

$$U = \gamma(AD)^{1/2} M_0^{-1} u.$$

The dimensionality of the phase space of the system (2.2) is eight; and in contrast to the case of an isotropic one-sublattice magnet, the analysis of stationary-profile waves is not a simple problem. In fact, in the concepts of mechanics, the problem that corresponds to the Landau-Lifshitz equations (2.2) is that of the motion of two interacting material points on the surface of a sphere of unit radius.

The Landau-Lifshitz equations (2.2) lead to the following obvious conservation laws:

$$m_1^2 + m_2^2 - 2m_1 m_2 = \mathcal{H}, \quad (2.4)$$

$$\mu_1 + \mu_2 + u(m_1 + m_2) = M_0, \quad (2.5)$$

which can be related to the invariance of the system with respect to displacement and rotation.

We introduce the generalized angular momentum of the

magnetic sublattice vectors,

$$M_i = \mu_i + u m_i; \quad i=1, 2 \quad (2.6)$$

and we write the conservation laws (2.4) and (2.5) in the form

$$M_1^2 + M_2^2 - 2m_1 m_2 - 2u^2 = \mathcal{H}, \quad (2.7)$$

$$M_1 + M_2 = M_0. \quad (2.8)$$

The Landau-Lifshitz system of equations (2.2) can be written in the form of a system of equations solved with respect to the highest derivatives,

$$m_1'' - m_2 + \lambda_1 m_1 = u \mu_1, \quad (2.9)$$

$$m_2'' - m_1 + \lambda_2 m_2 = u \mu_2.$$

Here

$$\lambda_i^2 = \mu_i^2 + m_i m_2 = M_i^2 + m_i m_2 - u^2, \quad (2.10)$$

λ_i are Lagrange multipliers.

We shall show that the problem of stationary-profile waves can be investigated completely at least in the case when one of the integrals of motion, namely the total angular momentum (2.5) of the magnetic sublattice vectors, is zero. In fact, because of the vanishing of the constant vector M_0 , according to the conservation law (2.8) the moduli of the generalized angular momenta (2.6) of the magnetic sublattice vectors are equal. The latter fact implies equality of the Lagrangian multipliers (2.10). Thus when $M_0 = 0$,

$$\lambda_1 = \lambda_2 = \Lambda. \quad (2.11)$$

For the common Lagrangian multiplier (2.11) of the system of equations (2.9) we get, using the conservation law (2.7), the expression

$$2\Lambda = M_1^2 + M_2^2 + 2m_1 m_2 - 2u^2 = \mathcal{H} + 4m_1 m_2. \quad (2.12)$$

We transform from the unit vectors of the magnetic moments of the sublattices to the representation of the antiferromagnetism and total-magnetic-moment vectors

$$2l = m_1 - m_2, \quad 2m = m_1 + m_2. \quad (2.13)$$

It is obvious that

$$m^2 + l^2 = 1, \quad ml = 0. \quad (2.14)$$

The system of equations (2.9), with use of the relations (2.11) and (2.12), takes the form

$$m'' + (\mathcal{H}/2 - 3 + u^2 + 4m^2)m = 0, \quad (2.15)$$

$$l'' + (\mathcal{H}/2 + 3 - 4l^2)l = u \{ [ml'] + [lm'] \}. \quad (2.16)$$

We note that the separation of the degrees of freedom connected with the motion of the total magnetic moment m of the two-sublattice magnet is due to the vanishing of the constant vector M_0 in the conservation law (2.8). In the new representation, the conservation laws (2.7) and (2.8) take the form

$$(m')^2 + (l')^2 + l^2 - m^2 = \mathcal{H}/2, \quad (2.17)$$

$$[mm'] + [ll'] + um = M_0 = 0. \quad (2.18)$$

Equations (2.15) may be regarded as the equations of motion of a material point in the field of a central force with potential energy

$$V(m) = \frac{1}{2}(\mathcal{H}/2 - 3 + u^2)m^2 + m^4. \quad (2.19)$$

An obvious consequence of equations (2.15) is the conservation laws

$$\frac{1}{2}(m')^2 + V(m) = \mathcal{E}_m, \quad (2.20)$$

$$[mm'] = C_m, \quad (2.21)$$

where \mathcal{E}_m and C_m are the constants of the first integrals. Furthermore, equations (2.16) permit a first integral of the form

$$\frac{1}{2}(l')^2 + V(l) = \frac{1}{2}u^2(1 - l^2) + \mathcal{E}_l. \quad (2.22)$$

Here

$$V(l) = \frac{1}{2}(\mathcal{H}/2 + 3)l^2 - l^4 \quad (2.23)$$

and \mathcal{E}_l is a new constant.

On summing (2.20) and (2.22) we find, after comparing with the conservation law (2.17), that the constant \mathcal{E}_m and \mathcal{E}_l are connected by the relation

$$\mathcal{E}_m + \mathcal{E}_l = \mathcal{H}/2. \quad (2.24)$$

We shall show that the relations obtained enable us to write explicit expressions for solitary waves. A state of equilibrium of the Landau-Lifshitz system of equations,

$$m = 0, \quad |l| = 1 \quad (2.25)$$

corresponds to the following values of the constants of the first integrals:

$$\mathcal{H} = 2, \quad C_m = 0, \quad \mathcal{E}_m = 0. \quad (2.26)$$

According to (2.21), the motion of the magnetic-moment vector m in configuration 3-space occurs along a straight line that passes through the origin of coordinates. On choosing a reference system in which

$$m(0, 0, m_z) \quad (2.27)$$

and integrating (2.20), we find

$$m_z(\xi) = 2^{-1/2} \frac{(2 - u^2)^{1/2}}{\text{ch}[(2 - u^2)^{1/2} \xi]}. \quad (2.28)$$

Since l and m are orthogonal, the motion of the antiferromagnetism vector occurs in a plane orthogonal to the z axis; that is,

$$l(l_x, l_y, 0). \quad (2.29)$$

In polar coordinates

$$l_x = l \cos \varphi, \quad l_y = l \sin \varphi \quad (2.30)$$

the conservation law (2.18) leads to the relation

$$\varphi' = -u m_z / l^2 = -u m_z / (1 - m_z^2), \quad (2.31)$$

which determines the rotation of the antiferromagnetism vector in the $l_x l_y$ plane during variation of the total magnetic moment in accordance with (2.28). The total angle of rotation is π .

Thus the solution found represents a moving domain wall with respect to the antiferromagnetism vector l , with which is connected a self-localized perturbation of the total magnetic moment m with a maximum amplitude

$$\max m_z = (1 - u^2/2)^{1/2} = [1 - (U/U_{\max})^2]^{1/2}. \quad (2.32)$$

We note that the rotation of the antiferromagnetism vector is connected also with a variation of its modulus. The solution found belongs to the class of topological solitons and is the unique (to within trivial symmetry transformations) separatrix solution of the Landau-Lifshitz equations for vanishing total angular momentum M_0 .

The solutions found above correspond to a separatrix that goes out from the equilibrium position (2.25). But the system of equations (2.15) has still another equilibrium position

$$l = 0, \quad |m| = 1. \quad (2.33)$$

Here we get directly from (2.17) that $\mathcal{K} = -2$. Consequently

$$(\mathbf{m}_1')^2 + (\mathbf{m}_2')^2 = 2\{m_1 m_2 - 1\} \leq 0, \quad (2.34)$$

since $m_1 \cdot m_2 \leq 1$; the equality is attained only in the case $m_1 = m_2$. In this case the equilibrium position (2.33) is isolated, and for $\mathbf{M}_0 = 0$ there are no other separatrixes.

3. We consider a two-sublattice magnetic material characterized by a free energy of the form

$$2F = (\mathbf{m}_1')^2 + (\mathbf{m}_2')^2 - m_1 \hat{A} m_1 - 2m_2 \hat{B} m_1 - m_2 \hat{C} m_2. \quad (3.1)$$

All energies are referred to the anisotropy energy K , and the coordinates are measured in units $(A/K)^{1/2}$. Here the symmetric matrices \hat{A} and \hat{C} take account of the anisotropy energies of the magnetic sublattices, the matrix \hat{B} of the energies of uniform exchange (the symmetric part of the matrix \hat{B}) and of the Dzyaloshinskii (the skew-symmetric part of \hat{B}).

The Landau-Lifshitz equations, which determine the temporal evolution of the magnetic moments of the sublattices, have the form

$$\dot{\mathbf{m}}_i = \left[\mathbf{m}_i \times \frac{\delta F}{\delta \mathbf{m}_i} \right]; \quad i=1, 2. \quad (3.2)$$

Following Lakshmanan *et al.*,³ we shall regard the unit vectors \mathbf{m}_i of the sublattices as tangent vectors to two three-dimensional curves characterized by curvature

$$\mu_i^2 = (\mathbf{m}_i')^2 \quad (3.3)$$

and torsion

$$\tau_i = \frac{\mathbf{m}_i [\mathbf{m}_i' \times \mathbf{m}_i'']}{\mu_i^3}. \quad (3.4)$$

We consider the pair of Frenet trihedra⁴ connected with these spatial curves:

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{m}_1; \quad \mathbf{e}_2 = \mathbf{m}_1' / \mu_1; \quad \mathbf{e}_3 = [\mathbf{e}_1 \times \mathbf{e}_2], \\ \mathbf{f}_1 &= \mathbf{m}_2; \quad \mathbf{f}_2 = \mathbf{m}_2' / \mu_2; \quad \mathbf{f}_3 = [\mathbf{f}_1 \times \mathbf{f}_2]. \end{aligned} \quad (3.5)$$

Here $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ are the orthonormal basis vectors corresponding to the Frenet trihedra.

The changes of orientation of the trihedra with change of the parameter of the curves, namely of the spatial variable x , at the instant of time t are determined by Frenet's formulas⁴

$$\begin{aligned} \dot{\mathbf{e}}_1' &= \mu_1 \mathbf{e}_2; \quad \dot{\mathbf{e}}_2' = -\mu_1 \mathbf{e}_1 + \tau_1 \mathbf{e}_3; \quad \dot{\mathbf{e}}_3' = -\tau_1 \mathbf{e}_2', \\ \dot{\mathbf{f}}_1' &= \mu_2 \mathbf{f}_2; \quad \dot{\mathbf{f}}_2' = -\mu_2 \mathbf{f}_1 + \tau_2 \mathbf{f}_3; \quad \dot{\mathbf{f}}_3' = -\tau_2 \mathbf{f}_2'. \end{aligned} \quad (3.6)$$

On the other hand, the temporal evolution of the Frenet trihedra (3.5) is determined by the Landau-Lifshitz equations (3.2). For example, for the basis vectors \mathbf{e}_1 and \mathbf{f}_1 the equations of temporal evolution have the form

$$\begin{aligned} \dot{\mathbf{e}}_1 &= -\mu_1 \tau_1 \mathbf{e}_3 + \mu_1' \mathbf{e}_3 + [\mathbf{e}_1 \times \hat{A} \mathbf{e}_1] + [\mathbf{e}_1 \times \hat{B} \mathbf{f}_1], \\ \dot{\mathbf{f}}_1 &= -\mu_2 \tau_2 \mathbf{f}_3 + \mu_2' \mathbf{f}_3 + [\mathbf{f}_1 \times \hat{B} \mathbf{e}_1] + [\mathbf{f}_1 \times \hat{C} \mathbf{f}_1] \end{aligned} \quad (3.7)$$

and represent a new form of writing the Landau-Lifshitz equations, which uses the basic geometric invariants. The equations of temporal evolution of the vectors \mathbf{e}_2 , \mathbf{f}_2 , and \mathbf{e}_3 , \mathbf{f}_3 , determined by the relations (3.5) and by the Landau-Lifshitz equations (3.2), have a rather complicated form, and we shall not write them.

The conditions for compatibility of the Frenet system of equations (3.6) with the system of equations of temporal evolution of the basis vectors of the Frenet trihedra,

$$(\mathbf{e}_\alpha')' = (\dot{\mathbf{e}}_\alpha)', \quad (\mathbf{f}_\alpha')' = (\dot{\mathbf{f}}_\alpha)', \quad \alpha=1, 2, 3, \quad (3.8)$$

lead to the following divergent forms:

$$\begin{aligned} \dot{F} + \{\mathbf{m}_1' \dot{\mathbf{m}}_1 + \mathbf{m}_2' \dot{\mathbf{m}}_2\}' &= 0, \\ (\tau_1 + \tau_2)' + \{\tau_1^2 + \tau_2^2 - \mu_1'' / \mu_1 - \mu_2'' / \mu_2 - F - \mu_1^{-2} (\mathbf{m}_1' \cdot \hat{A} \mathbf{m}_1' + \hat{B} \mathbf{m}_2') - \mu_2^{-2} (\mathbf{m}_2 \cdot \hat{B} \mathbf{m}_1' + \hat{C} \mathbf{m}_2')\}' &= 0. \end{aligned} \quad (3.9)$$

The divergent form (3.9) corresponds to the differential form of the law of conservation of energy. The divergent form (3.10) corresponds to the differential conservation law (3.2) related to translational invariance. The divergent forms (3.9) and (3.10) permit simple generalization to the case of magnetically ordered media characterized by an arbitrary number of magnetic sublattices (under the condition that the condition that the free energy is a homogeneous quadratic form).

Such a geometric approach to the derivation of differential conservation laws was first used by Lakshmanan *et al.*³ for the case of an isotropic one-sublattice magnetic material. In this case the divergent forms contain only the basic geometric invariants (the curvature and torsion). We showed² that the derivation of the divergent forms can be completed with allowance for uniaxial anisotropy and magnetic-dipole interaction. Finally, in the present paper we have achieved a generalization to the case of media with several magnetic sublattices.

4. We now consider a weak ferromagnet with anisotropy of the "axis of easy magnetization" type, whose free energy has the form

$$\begin{aligned} 2F &= A[(\mathbf{m}_1')^2 + (\mathbf{m}_2')^2] + 2D \mathbf{m}_1 \mathbf{m}_2 \\ &\quad - K[(\mathbf{m}_1 \mathbf{n})^2 + (\mathbf{m}_2 \mathbf{n})^2] + 2d \nu [\mathbf{m}_1 \mathbf{m}_2]. \end{aligned} \quad (4.1)$$

Here D and A are the constants of uniform and of non-uniform exchange energy, K is the anisotropy-energy constant, d is the Dzyaloshinskii-interaction constant, and \mathbf{n} and ν are the unit vectors of the axes of anisotropy and of the Dzyaloshinskii interaction.

We showed earlier² that in special cases of the mutual orientation of the vectors \mathbf{n} and ν , it is possible to distinguish exact classes of solutions. Here the magnetic-moment vector \mathbf{m} executes a motion along a straight line, and the antiferromagnetism vector \mathbf{l} moves in a plane orthogonal to it. The dimensionality of the phase space of the Landau-Lifshitz system of equations is lowered from eight to four.

We shall give a complete classification of the cases of lowering of the order of the Landau-Lifshitz system of equations for magnetic materials with the free energy (4.1). Let the vector \mathbf{m} execute a one-dimensional motion along some straight line L : then \mathbf{l} lies in a plane P orthogonal to the straight line L . It is easy to show that such a motion is possible only for the following orientations of the vectors \mathbf{n} and ν :

$$\mathbf{n} \perp \mathbf{n}_L, \quad \nu \perp \mathbf{n}_L \quad (4.2)$$

or

$$\mathbf{n} \parallel \mathbf{n}_L, \quad \nu \perp \mathbf{n}_L.$$

The inverse situation, when the vector l moves along the straight line L while m lies in the plane P , is possible only for zero velocity of motion of the stationary-profile wave; the permissible orientations of n and ν are also given by the relations (4.2).

Thus for any orientation of n and ν , lowering of the order of the Landau-Lifshitz system of equations is possible. For this purpose it is sufficient, for example, to choose the direction

$$n_L = [n\nu].$$

Analysis of these exact classes of solutions reduces (by choice of an appropriate system of coordinates and by renormalization of the independent variables x and t) to the completely integrable problem of a uniaxial one-sublattice ferromagnet.

We note that separation of exact classes of solutions and lowering of the order of the Landau-Lifshitz system of equations is possible also for the case when the free energy of the magnetic material has the more general form

$$2F = A[(m_1')^2 + (m_2')^2] + A_1 m_1' m_2' - K[(m_1 n)^2 + (m_2 n)^2] - K_1(m_1 n)(m_2 n) + 2d\nu[m_1 m_2].$$

In this case, however, the resulting system of equations with a four-dimensional phase space may prove unintegrable. (The system can obviously be integrated in the case $n \parallel \nu$: but we know of no magnet with such an orientation of the axes of anisotropy and of Dzyaloshinskii interaction.)

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NMR investigation of the effect of hydrostatic pressure on magnetization of the intermetallic compound YFe₂

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It is shown by the spin echo method that the local fields at the Fe⁵⁷ nuclei decrease on application of hydrostatic pressure, while the local fields of the nuclei of the nonmagnetic yttrium ions increase. This change in the local fields is attributed to occupation of the d band by yttrium valence electrons, resulting in a decrease in the magnetic moment of the iron atom.

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INTRODUCTION

There is very little information on the effect of high hydrostatic pressure on the magnetic moment of iron in alloys. This is the result of technical difficulties: the accuracy of traditional magnetic measurement methods as well as the magnitude of the expected effect often require high pressures that cannot be achieved under laboratory conditions. Accordingly, a method is increasingly widely used which is based on comparison of magnetic characteristics of compounds having different lattice parameters. From such a comparison of the intermetallic compounds YFe₂ and LuFe₂, Buschow and Van Stapele¹ concluded that the magnetic moment of iron μ_{Fe} increases on compression of the lattice. In our opinion, this conclusion is somewhat premature, since the observed change in the magnitude of μ_{Fe} on going from one compound to another may be connected

not only with the change in the distances between the ions, but also with the change in electronic structure. The electronic structure factor is significant if it is recognized that in alloys with iron, cobalt, and nickel, the valence electrons of rare-earth elements and yttrium, making a transition to the localized $3d$ states or occupying the d band, contribute to the magnetic moment of the $3d$ metals.² Within the framework of this idea, the valence electrons of yttrium and lutecium can make different contributions to μ_{Fe} of the compounds YFe₂ and LuFe₂. Thus, to explain the effect of the distance between ions and the electronic structure on μ_{Fe} in the indicated alloys we need experimental results obtained by hydrostatic compression of the lattice under the action of pressure.

Resonance methods and especially NMR at the Fe⁵⁷ nuclei are suitable for such precision measurements.