where φ_{mk} and f_{mk} are certain functions of x, y, z on which only the constraints which ensure $\Box k_{km}{}^{TT}=0$ are imposed, namely,

$$\Delta f_{mk}+h_{00,\,mk}=0,\ \Delta\varphi_{mk}=0. \tag{5.11}$$

From the components h_{km}^{TT} we find

$$R_{0klm}^{TT}={}^1/_2(\varphi_{lk,m}-\varphi_{mk,l}).$$

Comparing this expression with (5.9), we obtain the equation

$$\varphi_{lk,\,m}-\varphi_{mk,\,l}=h_{0m,\,lk}-h_{0l,\,mk},$$

which is satisfied by the choice $\varphi_{lk}=-h_{0l.k}$, and, in addition, (5.11) is also satisfied, since $\Delta h_{01}=0$. Thus, no contradictions arise in the values of R_{0klm} and R_{0klm}^{TT}.

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1) The gauge condition is derived in such a form in the Ref. 1 in connection with the equations for weak gravitational waves.

2) This means that although the count of the number of "physical" components of free fields gives the number 2, it is in the general case impossible to localize these degrees of freedom explicitly.

3) Equation (1.6) as the condition of simultaneous fulfillment of a complete set of gauge conditions for weak gravitational waves was obtained for the first time in Ref. 5.

4) The introduction of spinors in a Riemannian space requires the construction of a tetrad field. The covariant derivative here includes the spinor indices.[11]

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Regularization of the energy–momentum tensor and particle production in a strong varying gravitational field

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A generally covariant method is proposed for regularizing the vacuum expectation values of a quantized field interacting with a strong varying classical field (smoothing method). The main types of divergence are found and a simple algorithm given for calculating finite quantities for the case when an explicit expansion of the field operator with respect to quantum modes is given. The smoothing method is used to calculate the energy density and pressure of produced particles for a fermion field and a massless scalar field with minimal coupling in a Friedmann space.

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1. INTRODUCTION

Quantum field theory in a classical curved space-time is a natural first approximation to the construction of a complete quantum theory in which gravitation is also quantized. In such a quasiclassical approach, one also encounters problems which are of independent interest such as the production of particles from an initial vacuum state or vacuum polarization in spaces with non-Euclidean topology,[1-4] these effects leading to nonvanishing vacuum expectation values $\langle 0|T^{\mu\nu}|0\rangle$ of the energy–momentum tensor of the quantized field. The most important applications of these effects are to cosmology[5,6] and to black holes,[7] where one encounters strong gravitational fields that can be treated naturally as classical.

The calculation of definite physical quantities (the concentration n of produced particles, their energy density ε, pressure P, etc.) encounters certain difficulties, since the vacuum expectation values of the corresponding spatial densities diverge, as a rule, at large momenta. Besides the terms associated with the zero-point fluctuations of the vacuum [and proportional, for example, for the energy density to the integral $\int d^3k\,\omega(\mathbf{k})$], which can be eliminated either by direct subtraction[2] or by appropriate definition of the normal product,[3] there are divergences of other types as well. A number of methods have been proposed to eliminate these last (see, for example, the review of Ref. 8 and also the book Ref. 4). The most convenient for calculations are the n-wave method of Zel'dovich and Starobinskiĭ[2] and the analogous method of adiabatic regularization of Parker and Fulling,[6] which give simple algorithms for the necessary subtractions. For example, $\langle 0|T^{\mu\nu}|0\rangle$, regularized by the n-wave method, is[2]

$$\langle 0|T^{\mu\nu}_{\text{reg}}(x)|0\rangle=\int d^3k\,T^{\mu\nu}_{\text{reg}}(\mathbf{k}),$$
$$T^{\mu\nu}_{\text{reg}}(k)=\lim_{n\to\infty}\left[T^{\mu\nu}(m,\mathbf{k})-\sum_{q=0}\frac{1}{q!}\frac{\partial^q}{\partial(1/n^2)^q}\frac{1}{n}T^{\mu\nu}(nm,n\mathbf{k})\right]. \tag{1.1}$$

It is obvious that the conservation of $T^{\mu\nu}$ is preserved, and also the conformal invariance of the theory. In general, the applicability of these methods is restricted by the requirement that there exist an expansion of $T^{\mu\nu}(m,\mathbf{k})$ in a Taylor series in powers of k^{-1} or at least the first terms of such an expansion should exist. In the general case, $T^{\mu\nu}$ contains oscillating terms of the form e^{ikx}. Therefore, the point $k=\infty$ is an essential singularity, and the expansion (1.1) does not exist. The simple example of a massless scalar field with minimal coupling in a Friedmann space leads to the following expression for the probability of pair production in the state k (see Sec. 4 of the present paper):

$$|\beta|^2=\frac{1}{4k^2}\left[\frac{1}{\eta_0^2}+\frac{1}{\eta^2}-\frac{2\cos 2k(\eta-\eta_0)}{\eta_0\eta}\right]-\frac{(\eta-\eta_0)\sin 2k(\eta-\eta_0)}{4k^3\eta_0^2\eta^2}+\frac{1-\cos 2k(\eta-\eta_0)}{8k^4\eta_0^2\eta^2}.$$

It is easy to show that regularization of $\langle n(x)\rangle$ or $\langle T^{\mu\nu}(x)\rangle$ by means of an expansion of the form (1.1) is here impossible. The other methods of regularization, for example, the point-splitting method,[4,8] are also ineffective in this case.

Besides the limited applicability, a shortcoming of the existing methods is, in our view, the nonuniqueness in the determination of $T^{\mu\nu}_{\text{reg}}$. For example, the energy–momentum tensor of a conformally invariant scalar field in a Friedmann space is finite after the terms associated with the zero-point fluctuations of the vacuum have been subtracted.[2,3] One can however, define a different energy–momentum tensor which contains a greater number of subtractions than is needed to ensure convergence and leads to different results (see Ref. 4).

In the present paper, we propose a method for regularizing the particle number density n, the energy density ε, the pressure P, and the other quantities characterizing the produced particles; it is not associated with expansions of the form (1.1). In Sec. 2, we consider why the divergences arise and give a generally covariant method for eliminating them. In Sec. 3, we formulate rules for calculating divergent integrals for the case when an explicit expansion of the field with respect to quantum modes is given. In Sec. 4, we calculate the energy–momentum tensor for a fermion ($s=\frac{1}{2}$) field and a massless scalar ($s=0$) field with minimal coupling in the Friedmann metric. We use a system of units in which $\hbar=c=1$.

2. REGULARIZATION OF THE DYNAMICAL VARIABLES BY THE SMOOTHING METHOD

We consider in more detail the origin of the divergences in the dynamical variables which characterize the produced particles, for example, n, ε, and P. The most common formulation of the problem of particle production from the vacuum is as follows. Let Φ be some free (linear) field in a curved space-time satisfying the dynamical equations

$$\hat{L}[\Phi]=0, \tag{2.1}$$

where $\hat{L}$ is a self-adjoint differential operator. We take two spacelike hypersurfaces Σ_0 and Σ_1, which can serve as Cauchy hypersurfaces for Eq. (2.1), and assume that Σ_1 lies in the future of Σ_0. The field Φ is quantized by introducing a complete set of positive- and negative-frequency solutions of Eq. (2.1) on Σ and expanding Φ with respect to this set. Denoting, for example, the corresponding solutions by $u_{\mathbf{k}}(x)$, we can write

$$\Phi=\int d^3k[a_{\mathbf{k}}u_{\mathbf{k}}(x)+a_{\mathbf{k}}^+u_{\mathbf{k}}^*(x)]. \tag{2.2}$$

The operators $a_{\mathbf{k}}$ ($a_{\mathbf{k}}^+$) in (2.2) correspond to operators of annihilation (creation) of field quanta and satisfy the standard (anti) commutation relations

$$[a_{\mathbf{k}},a_{\mathbf{k}'}^+]_\pm=\delta_{\mathbf{k}\mathbf{k}'},\quad [a_{\mathbf{k}},a_{\mathbf{k}'}]_\pm=0. \tag{2.3}$$

(By k, we have denoted the set of discrete or continuous indices that define the quantum state of the field.) Formally, this scheme does not differ from quantization of the field Φ in Minkowski space. The difference is that a solution $u_{\mathbf{k}}(\Sigma)$ of Eq. (2.1) having a definite frequency type (positive or negative) on Σ_0 no longer has it on Σ_1. This has the consequence that the initial vacuum state $|0\rangle$ is a many-particle state with respect to the vacuum $|0_1\rangle$ on Σ_1, i.e., particles are produced from the vacuum. The probability of pair production in the state k, $|\beta(\mathbf{k},x)|^2$, can be found either by diagonalizing the instantaneous Hamiltonian $H(\Sigma)$ by means of a canonical Bolgolyubov transformation or by expanding $u_{\mathbf{k}}(\Sigma_0)$ with respect to a new complete set of functions $u_{\mathbf{k}}(\Sigma_1)$ and $u_{\mathbf{k}}^*(\Sigma_1)$. (A detailed discussion of the questions involved here can be found, for example, in Refs. 4 and 8.)

In calculating n,ε, and the other characteristics of the produced particles, we shall assume that the divergences associated with the zero-point fluctuations of the vacuum have already been eliminated either by direct subtraction of the corresponding terms[2] or by introducing a time-dependent operation of normal ordering.[3] This last can be expressed in the covariant form

$$N_\Sigma[A]=A-\langle 0_\Sigma|A|0_\Sigma\rangle. \tag{2.4}$$

We find the physical characteristics associated with A of the particles produced in the time determined by the mutual separation of Σ_0 and Σ_1 by taking the average of (2.4) with respect to the initial vacuum $|0\rangle$. The n and ε determined in this manner can be expressed directly in terms of the probability of pair production in the state k:

$$n \propto \int d^3k |\beta(\mathbf{k})|^2, \quad \varepsilon \propto \int d^3k\, \omega(\mathbf{k}) |\beta(\mathbf{k})|^2. \tag{2.5}$$

The other vacuum expectation values $\langle 0|N_\Sigma[T^{\mu\nu}(x)]|0\rangle$ (for example, the pressure) are given by more complicated constructions, but they do not differ fundamentally from (2.5).

Let t be the timelike coordinate determined by the hypersurface Σ with respect to which the expansion (2.2) of the field Φ with respect to frequency components is given. We shall denote the derivative with respect to t by a dot. The particle production is due to the nonstationarity of the space-time metric $g_{\mu\nu}(x)$ in the four-space region Ω bounded by the hypersurfaces Σ_0 and Σ_1. Thus, the probability amplitude $\beta(\mathbf{k})$ is a functional of $\dot{g}_{\mu\nu}(x)$: $\beta=\beta[\mathbf{k};\dot{g}_{\mu\nu}(x)]$, and $\beta=0$ if $\dot{g}_{\mu\nu}\equiv 0$ in Ω. The derivative $\dot{g}_{\mu\nu}$ determines the "intensity of the interaction" of the field Φ with the classical gravitational field. The correction formulation of the problem in the theory of interacting fields usually contains the hypothesis of adiabatic switching on of the interaction.[9] In the present case, this hypothesis means that $\dot{g}_{\mu\nu}\to 0$ as $t\to-\infty$; in the interval $-\infty<t\leqslant t_0$ there is a slow variation of $\dot{g}_{\mu\nu}$ from zero to the value $\dot{g}_{\mu\nu}(\Sigma_0)$; in the interaction region $t_0\leqslant t\leqslant t_1$, the derivative $\dot{g}_{\mu\nu}$ changes in accordance with the Einstein equations; finally, in the interval $t_1\leqslant t<\infty$ the derivative $\dot{g}_{\mu\nu}$ decreases slowly from the value $\dot{g}_{\mu\nu}(\Sigma_1)$ to zero. It is then possible to define correctly the initial and final vacuum states, and the adiabaticity of the switching on and switching off processes ensures the absence of quantum transitions in the intervals $(-\infty, t_0)$ and (t_1,∞).

The formulation of the problem described at the beginning of this section does not correspond to adiabatic but instantaneous switching on of the interaction at t_0 and switching off at t_1. The space-time metric corresponding to this formulation is

$$g_{\mu\nu}(x)=\begin{cases} g_{\mu\nu}(\Sigma_0), & t<t_0 \\ g_{\mu\nu}(x), & t_0\leqslant t\leqslant t_1. \\ g_{\mu\nu}(\Sigma_1), & t_1<t \end{cases} \tag{2.6}$$

The interaction intensity $\dot{g}_{\mu\nu}$ has discontinuities on Σ_0 and Σ_1. In quantum field theory in flat space-time such a situation has been investigated in considerable detail. In the case of instantaneous switching on of the interaction, the well-known Stückelberg surface divergences arise, whereas the physical quantities remain finite in the case of "smooth" switching on.[9] It is natural to assume that in curved space-time too the divergences at large k in expressions of the type (2.5) are also due to the instantaneous switching on of the interaction, since such switching excites with appreciable probability the high-energy modes of the field Φ. Then to eliminate the divergences from these expressions it is sufficient to smooth the process of switching on and off of the interaction.

It would be very attractive to combine the conditions of smoothness and adiabaticity of the switching on of the interaction. However, it is not clear whether one can define sensibly the metric tensor $g_{\mu\nu}$ in the sections of adiabatic switching on and switching off. We shall restrict ourselves to the fulfillment of the smoothness requirement. For the mathematical expression of the corresponding conditions, it is convenient to use Bogolyubov's device[9] in the derivation of the Tomonaga–Schwinger equation. Namely, we introduce a discontinuous function $\theta_\Sigma(x)$ such that

$$\begin{aligned} \theta_\Sigma(x)&=1 \quad \text{for } t>T_\Sigma(x), \\ \theta_\Sigma(x)&=0 \quad \text{for } t<T_\Sigma(x), \end{aligned} \tag{2.7}$$

where $t=T_\Sigma(x)$ is the equation which defines the hypersurface Σ. Let $g(x)$ be a smooth function that differs from $\theta_\Sigma(x)\equiv\theta[t-T_\Sigma(x)]$ only within the region $|t-T_\Sigma(x)|<\Delta$. We go over from $\dot{g}_{\mu\nu}$ to the smoothed quantities $\dot{\tilde{g}}_{\mu\nu}$:

$$\dot{\tilde{g}}_{\mu\nu}=\dot{g}_{\mu\nu}(x)\,g(x_0)\,g(-x_1). \tag{2.8}$$

For small Δ, the quantity $\dot{\tilde{g}}_{\mu\nu}$ is equal to $\dot{g}_{\mu\nu}$ in the four-region Ω bounded by the hypersurfaces Σ_0 and Σ_1; in a layer of thickness Δ to the left of Σ_0 and to the right of Σ_1 the value of $\dot{\tilde{g}}_{\mu\nu}$ varies smoothly from its values on Σ_0 and Σ_1 to zero, and it vanishes identically in the remaining regions. It is obvious that the corresponding metric can be defined in such a way that its difference from (2.6) does not exceed a small quantity of order Δ. The excitation of the high-energy modes of the field Φ will now be determined by the degree of smoothness of the functions $g(x)$ [for example, for analytic $g(x)$ the excitation probability for such modes is exponentially small]. Taking $g(x)$ sufficiently smooth, we can always ensure convergence of quantities such as (2.5). It is obvious that the proposed covariant regularization procedure does not change the conservation properties of $T^{\mu\nu}$ or the (possible) conformal invariance of the field Φ.

3. RULES FOR CALCULATING DIVERGENT INTEGRALS

Eliminating the divergences by means of the smoothing (2.8), we introduce in general an explicit dependence on the smoothing functions $g(x)$ into the amplitude β. In the region of frequencies $k\sim\Delta^{-1}$ their influence cannot be regarded as small. On the other hand, it is clear that the vacuum expectation values of $T^{\mu\nu}$ must be determined solely by the variation $\dot{g}_{\mu\nu}$ of the physical metric and cannot depend on the method of smoothing (or on the method of regularization). Therefore, the regularization (2.8) is of little use for indicating a method of calculating the expectation values $\langle 0|N[T^{\mu\nu}]|0\rangle$ which does not depend on the specific form of the functions $g(x)$. Since the explicit introduction of quantum states and the expansion (2.2) of the field operator with respect to the frequency components are usually associated with the possibility of separating the variables in Eqs. (2.1), we shall restrict ourselves to this practically important case.

Thus, suppose the time and spatial variables in the field equations (2.1) have been separated. The time-dependent part $u_{\mathbf{k}}(t)$ of the frequency functions satisfies an ordinary second-order differential equation with t-dependent coefficients. (We recall that Φ is effectively a free field.) The probability of pair production in the state $\mathbf{k}$ can be found in this case from the system of equations

$$\dot{\alpha}=W(\mathbf{k},t)\beta e^{i\theta(\mathbf{k},t)},\quad \dot{\beta}=V(\mathbf{k},t)\alpha e^{-i\theta(\mathbf{k},t)},$$
$$\alpha(-\infty)=1,\quad \beta(-\infty)=0. \tag{3.1}$$

[The case $\dot{\alpha}=f_1\beta+f_2\alpha$, $\dot{\beta}=\varphi_1\alpha+\varphi_2\beta$ can be reduced to (3.1) by the method of variation of an arbitrary constant.] The functions $W(\mathbf{k},t)$ and $V(\mathbf{k},t)$ are proportional to $\dot{g}_{\mu\nu}$ (or $\dot{\tilde{g}}_{\mu\nu}$ for the smoothed metric), and $\theta(\mathbf{k},t)$ for the high-energy modes is $\theta(\mathbf{k},t)\approx kt\gg 1$. The integrals of the type (2.5), in terms of which the vacuum expectation values of the spatial densities $n(x)$, $T^{\mu\nu}(x)$, etc., are expressed, behave as $|\mathbf{k}|\to\infty$ as

$$\int\limits^{\infty} dk\, k^n|\beta(\mathbf{k},t)|^2.$$

Since the divergences are associated with the behavior of $|\beta|^2$ at large k, it is sufficient to determine a method for calculating integrals of the form

$$J=\int\limits_0^{\infty} dk\, k^n|\beta(\mathbf{k},t)|^2, \tag{3.2}$$

where β is a solution of the system of equations

$$\dot{\beta}=V(t)\alpha e^{-ikt},\quad \dot{\alpha}=W(t)\beta e^{ikt};$$
$$\beta(-\infty)=0,\quad \alpha(-\infty)=1. \tag{3.3}$$

[We have introduced the notation $W(k\to\infty,t)=W(t)$, $V(k\to\infty,t)=V(t)$.]

We consider first the simple case when the "potentials" $V(t)$ and $W(t)$ contain a small parameter and the solution of the system (3.3) is to good accuracy

$$\alpha\approx 1,\quad \beta\approx\int\limits_{-\infty}^{\infty} dx\,\tilde{V}(x)e^{-ikx}. \tag{3.4}$$

If the interaction is swtiched on instantaneously at t_0 and off at t_1,

$$\tilde{V}(x)=V(x)\theta(x-t_0)\theta(t_1-x),$$

then $\beta(k\to\infty)\sim k^{-1}$, and the integral (3.2) diverges at the upper limit as k^{n-1}. Regularizing in accordance with (2.8), we replace the θ functions in $\tilde{V}(x)$ by functions $g(x)$ which are sufficiently smooth to ensure convergence of the integrals J. It is necessary to distinguish the values $n=2m$ and $n=2m-1$. For $n=2m$, integrating by parts in (3.4), we find

$$\beta=(ik)^{-m}\int dx\,\tilde{V}^{(m)}(x)e^{-ikx},$$

$$|\beta|^2=k^{-2m}\int dx\,dy\,\tilde{V}^{(m)}(x)\tilde{V}^{(m)}(y)\cos k(x-y)$$

and the required integral is

$$J_{2m}=\int dx\,dy\,\tilde{V}^{(m)}(x)\tilde{V}^{(m)}(y)\int\limits_0^{\infty} dk\cos k(x-y)=\pi\int dx[\tilde{V}^{(m)}(x)]^2.$$

We have used the equation

$$\int\limits_0^{\infty} dk\cos kz=\pi\delta(z).$$

To separate in J_{2m} the contribution from the "physical" region $t_0<t<t_1$, we transform the last integral:

$$J_{2m}=\pi(-)^m\int dx\,\tilde{V}(x)\tilde{V}^{(2m)}(x),$$

and go to the limit $g(x)\to\theta(x)$. The divergences which then arise can be readily separated and can be eliminated in a general form. For this, we arrange the regions of smoothing for $\tilde{V}(x)$ and $\tilde{V}^{(2m)}(x)$ in such a way that the transition regions for $\tilde{V}^{(2m)}(x)$ occur at the values of x where $\tilde{V}(x)\equiv 0$, and we go to the limit $g\to\theta$ first for $\tilde{V}(x)$. As a result, we obtain the regularized value of the integral J_{2m}:

$$J_{2m}=\int\limits_0^{\infty} dk\,k^{2m}\int\limits_{t_0}^{t_1} dx\,dy\,V(x)V(y)\cos k(x-y)=\pi(-)^m\int\limits_{t_0}^{t_1} dx\,V(x)V^{(2m)}(x). \tag{3.5}$$

The case $n=2m-1$ is somewhat more complicated. Integrating by parts, we obtain as in the preceding case

$$J_{2m-1}=\int dx\,dy\,\tilde{V}^{(m)}(x)\tilde{V}^{(m)}(y)\int\limits_0^{\infty} dk\frac{\cos k(x-y)}{k}$$

$$=\int dx\,dy\,\tilde{V}^{(m)}(y)\left[\tilde{V}^{(m-1)}(x)\int\limits_0^{\lambda} dk\sin k(x-y)+\tilde{V}^{(m)}(x)\int\limits_{\lambda}^{\infty} dk\frac{\cos k(x-y)}{k}\right]$$

$$=\int dx\,dy\,\tilde{V}^{(m)}(y)\left[\tilde{V}^{(m-1)}(x)\frac{1-\cos\lambda(x-y)}{x-y}-\tilde{V}^{(m)}(x)\,\mathrm{Ci}\,\lambda|x-y|\right].$$

Since

$$\cos\lambda(x-y)=(x-y)\frac{d}{dx}\mathrm{Ci}\,\lambda|x-y|,$$

we obtain

$$J_{2m-1}=-\int dx\,dy\,\tilde{V}^{(m)}(x)\tilde{V}^{(m)}(y)\ln|x-y|$$

$$=(-)^{m+1}\int dx\,dz\ln|z|\tilde{V}(x)\tilde{V}^{(2m)}(x+z).$$

In the last integral, we can go to the limit $g(x)\to\theta(x)$ first for $\tilde{V}(x)$ and then in $\tilde{V}^{(2m)}(x+z)$:

$$J_{2m-1}=(-)^{m+1}\int dz\ln|z|\int\limits_{t_0}^{t_1} dx\,V(x)\frac{d^{2m}}{dx^{2m}}[\theta(x+z-t_0)V(x+z)\theta(t_1-x-z)]$$

$$=(-)^{m+1}\int dz\ln|z|\int\limits_{t_0}^{t_1} dx\,V(x)V^{(2m)}(x+z)\theta(x+z-t_0)\theta(t_1-x-z)$$

$$+(-)^{m+1}\int dz\ln|z|\int\limits_{t_0}^{t_1} dx\,V(x)\sum_{n=1}^{2m}\binom{2m}{n}[\theta(t_1-x-z)\delta^{(n-1)}(x+z-t_0)$$
$$\times V^{(2m-n)}(x+z)-\theta(x+z-t_0)\delta^{(n-1)}(x+z-t_1)V^{(2m-n)}(x+z)].$$

Here, $\binom{2m}{n}$ are binomial coefficients; we have used the equation $\theta'(x)=\delta(x)$, and also the circumstance that

$$\int\limits_{t_0}^{t_1} dx\,\varphi(x)\delta^{(k)}(x+z-t_0)\delta^{(p)}(x+z-t_1)=0;\quad k,p=0,1,2,\ldots.$$

Transferring the derivatives from the δ functions to the potentials, we find the regularized value of J_{2m-1}:

$$J_{2m-1}=(-)^{m+1}\int\limits_{t_0}^{t_1} dx\,dy\ln|x-y|V(x)V^{(2m)}(y)+(-)^{m+1}$$
$$\times\int\limits_{t_0}^{t_1} dx\left\{\sum_{n=1}^{2m}(-)^{n-1}\binom{2m}{n}\frac{d^{n-1}}{dz^{n-1}}[\ln|t_0-x|V(z)V^{(2m-n)}(z-x+t_0)\right.$$
$$\left.-\ln|t_1-x|V(z)V^{(2m-n)}(z-x+t_1)]|_{z=x}\right\}.$$

Calculating the sum over n, we reduce this expression to the simpler form

$$J_{2m-1}=\int_0^\infty dk\,k^{2m-1}\int_{t_0}^{t_1} dx\,dy\,V(x)V(y)\cos k(x-y)$$

$$=(-)^{m+1}\int_{t_0}^{t_1} dx\,dy\ln|x-y|V(x)V^{(2m)}(y)+(-)^{m+1}\sum_{n=0}^{2m-1}(-)^n$$

$$\times\int_{t_0}^{t_1} dx[\ln|t_0-x|V^{(n)}(x)V^{(2m-n-1)}(t_0)-\ln|t_1-x|V^{(n)}(x)V^{(2m-n-1)}(t_1)].\quad(3.6)$$

Equations (3.5) and (3.6) give the rules for calculating the two standard divergent integrals J_{2m} and J_{2m-1}.

In the calculation of the vacuum expectation values, we also encounter divergences of a somewhat different form:

$$i_n=\int_0^\infty dk\,k^n\int dx\,\tilde{V}(x)\cos k(x-t).\quad(3.7)$$

We again distinguish the values $n=2m$ and $n=2m-1$. For $n=2m$, integrating by parts in (3.7), we obtain

$$i_{2m}=(-)^m\int dx\,\tilde{V}^{(2m)}(x)\int_0^\infty dk\cos k(x-t)=\pi(-)^m\int dx\,\tilde{V}^{(2m)}(x)\delta(x-t).$$

Neglecting the integrals over the smoothing regions (i.e., the surface terms that diverge in the limit $\Delta\to 0$), we find

$$i_{2m}=\int_0^\infty dk\,k^{2m}\int_{t_0}^{t_1} dx\,V(x)\cos k(x-t)$$

$$=\pi(-)^m\int_{t_0}^{t_1}dx\,V^{(2m)}(x)\delta(x-t)=\begin{cases}\pi(-)^m V^{(2m)}(t); & t_0<t<t_1\\ 0; & t<t_0,\ t_1<t\end{cases}.\quad(3.8)$$

For $n=2m-1$, as before, we have

$$i_{2m-1}=(-)^m\int dx\,\tilde{V}^{(2m)}(x)\int_0^\infty dk\frac{\cos k(x-t)}{k}=-(-)^m\int dx\,\tilde{V}^{(2m)}(x)\ln|x-t|,$$

i.e., the regularized value of the integral is

$$i_{2m-1}=\int_0^\infty dk\,k^{2m-1}\int_{t_0}^{t_1} dx\,V(x)\cos k(x-t)=(-)^{m+1}\int_{t_0}^{t_1} dx\,V^{(2m)}(x)\ln|x-t|.\quad(3.9)$$

We now consider the case when the potentials $W(t)$ and $V(t)$ are not small. We introduce a parameter λ satisfying the condition $\lambda t\gg 1$ to divide the integral (3.2) into two:

$$J=\int_0^\lambda dk\,k^n|\beta|^2+\int_\lambda^\infty dk\,k^n|\beta|^2=J^{(1)}+J^{(2)}.\quad(3.10)$$

In the interval $0\leqslant k\leqslant\lambda$, we find the amplitude $\beta(k,t)$ by the usual methods of the theory of differential equations from the system (3.3), using the unsmoothed potentials V and W. To calculate $J^{(2)}$, we note that for sufficiently large λ the solution (3.4) of the system (3.3) is a good approximation to the exact solution irrespective of assumptions about the smallness of V and W. Therefore, in the frequency interval $\lambda<k<\infty$ we use the value

$$\beta=\int dx\,\tilde{V}(x)e^{-ikx},$$

where $\tilde{V}$ is the smoothed potential. We transform the integral $J^{(2)}$ as follows:

$$J^{(2)}=\int_\lambda^\infty dk\,k^n|\beta|^2=-\int_0^\lambda dk\,k^n\int_{t_0}^{t_1} dx\,dy\,V(x)V(y)\cos k(x-y)$$

$$+\int_0^\infty dk\,k^n\int_{t_0}^{t_1} dx\,dy\,V(x)V(y)\cos k(x-y).$$

We have separated in $J^{(2)}$ the standard divergent part, whose values are given by the expressions (3.5) or (3.6), and taken the smoothing from the potentials in the finite integral over the interval $(0,\lambda)$. Substituting this value of $J^{(2)}$ in (3.10), we obtain

$$J=\int_0^\infty dk\,k^n\int_{t_0}^{t_1} dx\,dy\,V(x)V(y)\cos k(x-y)$$

$$+\lim_{\lambda\to\infty}\left\{\int_0^\lambda dk\,k^n\left[|\beta|^2-\int_{t_0}^{t_1} dx\,dy\,V(x)V(y)\cos k(x-y)\right]\right\}.\quad(3.11)$$

Similar calculations can be readily performed for the expressions that diverge in accordance with (3.7).

In the general case of arbitrary $V(k,t)$ and $W(k,t)$, the scheme of calculations is still preserved. In the interval $0\leqslant k\leqslant\lambda$, we find the solution of the system (3.1) by using the known unsmoothed values of V and W. For values $\lambda<k$, the functions $V(k,t)$ and $W(k,t)$ in (3.1) can be represented as series in powers of k^{-1}, and we can use the asymptotic expansion of the solution of the system of equations (3.1) with the smoothed potentials $\tilde{V}$ and $\tilde{W}$ in series in powers of $(kt)^{-1}$. Separating the standard divergent integrals, which can be calculated in accordance with Eqs. (3.5)–(3.9), we obtain expressions of the form (3.11) for the regularized values of the required vacuum expectation values.

4. PRODUCTION OF SCALAR AND SPINOR PARTICLES IN A FRIEDMANN SPACE

To illustrate the regularization method developed in the previous sections, we consider the production of scalar and spinor particles in the space with metric

$$ds^2=a^2(\eta)(d\eta^2-dx^2-dy^2-dz^2),\quad a(\eta)=a_0\eta\quad(4.1)$$

1. *Fermions*. The quantum theory of a soubir ($s=\frac{1}{2}$) field in homogeneous and isotropic spaces has been considered on many occasions (see, for example, Ref. 4). We shall use the results of Mamaev, Mostepanenko, and Frolov.[10] In accordance with Ref. 10, the energy density and pressure of the produced fermions are

$$\varepsilon=\frac{2}{\pi^2a^4}\int_0^\infty dk\,k^2\omega s,\quad P=\frac{2}{3\pi^2a^4}\int_0^\infty dk\frac{k^3}{\omega}\left[ks-\frac{ma}{2}u\right],$$

$$s=|\beta|^2,\quad u=-2\,\mathrm{Re}(\alpha\beta e^{-2i\theta}),\quad \theta=\int_0^\eta\omega(x)dx,\quad(4.2)$$

$$\omega^2=k^2+m^2a^2;$$

here, α and β satisfy the system of equations

$$\alpha^{*\prime}=\frac{kma'}{2\omega^2}\beta e^{-2i\theta},$$

$$\beta'=\frac{-kma'}{2\omega^2}\alpha^*e^{2i\theta},\quad\alpha(0)=1,\quad\beta(0)=0.\quad(4.3)$$

(We have specified the initial conditions at the cosmological singularity.)

We restrict ourselves to the most interesting case $ma(\eta)\eta\ll 1$. Introducing the parameter $\lambda, ma\ll\lambda\ll 1/\eta$, we find for $k<\lambda$

$$s(k<\lambda)={}^1/_2-k/2\omega,\quad u(k<\lambda)=ma/\omega.$$

The contributions from this region to ε and P are

$$\varepsilon(k<\lambda)=\frac{m^2\lambda^2}{4\pi^2a^2}-\frac{m^4}{8\pi^2}\left[\ln 2\lambda-\ln ma-\frac{1}{4}\right],$$
$$P(k<\lambda)=-\frac{m^2\lambda^2}{12\pi^2a^2}+\frac{m^4}{8\pi^2}\left[\ln 2\lambda-\ln ma-\frac{7}{12}\right]. \quad (4.4)$$

For $\lambda<k$, the coefficients in (4.3) can be expanded in a series with respect to the parameter ma/k [the same expansion can be made in the integrals (4.2)]. Calculating α and β to terms $\sim(ma/k)^3$, we find

$$\alpha=1-\frac{m^2a^2}{8ik^3\eta}+\frac{m^2a^2}{16k^4\eta^2}(e^{-2ik\eta}-1),$$
$$\beta=-\frac{m}{2k}\int dx\,a'(x)\,e^{2ikx}$$
$$+\frac{m^3a^3e^{2ik\eta}}{4k^3}\left[-\frac{1}{3}+\frac{3}{2ik\eta}+\frac{11-e^{-2ik\eta}}{8k^2\eta^2}-\frac{5(1-e^{-2ik\eta})}{8ik^3\eta^3}\right]. \quad (4.5)$$

We have separated the term $(m/2k)\int dxa'(x)e^{2ikx}$, which leads to divergences at large k. To accuracy m^4a^4/k^4 the integrands in (4.2) are

$$k^2\omega|\beta|^2\doteq\frac{km^2}{4}\int dx\,dy\,a'(x)a'(y)\cos 2k(x-y)+\frac{m^4a^4}{16k^2}\left[\frac{2\sin 2k\eta}{3\eta}\right.$$
$$\left.-\frac{2(1-\cos 2k\eta)}{k\eta^2}-\frac{5\sin 2k\eta}{2k^2\eta^3}+\frac{5(1-\cos 2k\eta)}{2k^3\eta^4}\right];$$

$$\frac{k^4|\beta|^2}{\omega}-\frac{k^3ma}{2\omega}u=\frac{1}{2}(k^2\omega|\beta|^2)-\frac{m^2ak}{2}\int dx\,a'(x)\cos 2k(x-\eta)$$
$$+\frac{m^4a^4}{8k}\left[\frac{2\cos 2k\eta}{3}+\frac{\sin 2k\eta}{k\eta}+\frac{4+\cos 2k\eta}{2k^2\eta^2}-\frac{5\sin 2k\eta}{4k^3\eta^3}\right].$$

Thus, the divergent terms in (4.2) are $\propto m^2a^2$, and the regular terms are $\propto m^4a^4$.

We calculate the contributions of the divergent integrals. The first of them is

$$\frac{m^2}{4}\int_\lambda^\infty dk\int dx\,dy\,a'(x)a'(y)\,k\cos 2k(x-y)$$
$$=-\frac{m^2}{4}\int_0^\lambda dk\,k\int_0^\eta dx\,dy\,a_0^2\cos 2k(x-y)$$
$$+\frac{m^2}{4}\int_0^\infty dk\,k\int_0^\eta dx\,dy\,a'(x)a'(y)\cos 2k(x-y)=-\frac{m^2a^2\lambda^2}{8}$$

[we have used the expressions (3.6) and (3.11), and also the fact that $a(x)=a_0x$ on the section of physical variation of the metric]. For the second integral, we find from (3.9)

$$-\frac{m^2a}{2}\int_\lambda^\infty dk\,k\int dx\,a'(x)\cos 2k(x-\eta)=\frac{m^2a}{2}\left[\int_0^\lambda dk\,k\right.$$
$$\left.\times\int_0^\eta dx\,a_0\cos 2k(x-\eta)-\int_0^\infty dk\,k\int_0^\eta dx\,a'(x)\cos 2k(x-\eta)\right]=\frac{m^2a^2\lambda^2}{4}.$$

Calculating also the integrals of the terms proportional to m^4a^4, we find the contribution to ε and P from the region $\lambda<k$:

$$\varepsilon(\lambda<k)=-\frac{m^2\lambda^2}{4\pi^2a^2}+\frac{m^4}{8\pi^2}\left[\gamma+\ln 2\lambda\eta-\frac{73}{36}\right],$$
$$P(\lambda<k)=\frac{m^2\lambda^2}{12\pi^2a^2}+\frac{m^4}{8\pi^2}\left[-\gamma-\ln 2\lambda\eta+\frac{61}{36}\right]. \quad (4.6)$$

Here, γ is Euler's constant. Adding (4.4) and (4.6), we obtain

$$\varepsilon\approx\frac{m^4}{8\pi^2}\left[-\ln\frac{1}{ma_0\eta^2}+\gamma-\frac{16}{9}\right],$$
$$P\approx\frac{m^4}{8\pi^2}\left[\ln\frac{1}{ma_0\eta^2}-\gamma+\frac{10}{9}\right]. \quad (4.7)$$

The region of applicability of Eqs. (4.7) is bounded by the inequality $ma_0\eta^2<1$. The produced particles satisfy approximately the "vacuum-like" equation of state $P\approx-\varepsilon$, and

$$\varepsilon+P\approx-m^4/12\pi^2<0,\ \text{but}\ \varepsilon+3P\approx(m^4/4\pi^2)\ln(1/ma_0\eta^2)>0.$$

In other words, the weak dominant energy condition is violated and the strong energy condition is satisfied. In order of magnitude, the ε and P of the produced fermions are equal to the energy density and pressure of conformal scalar particles,[3] but they have the opposite signs: $\varepsilon^{(1/2)}\approx-2\varepsilon^{(0)}, P^{(1/2)}\approx-2P^{(0)}$ (naturally, these equations hold only if the fermions and scalar particles have the same masses). The values of ε and P found here differ strongly from the result obtained in Ref. 10. In this connection, we note that the choice made in Ref. 10 of the subtracted terms in accordance with the scheme (1.1),

$$s_2=m^2a_0^2k^2/16\omega^6,\quad u_2=-3m^3a_0^2ak/2\omega^6,$$

does not satisfy the initial conditions $s(\eta_0)=u(\eta_0)=0$. Therefore, the ε and P calculated in Ref. 10 correspond to a nonvacuum initial state.

2. *Scalar field with minimal coupling.* We consider a massless scalar field with Lagrangian

$$2L=\Phi_{,\nu}\Phi^{,\nu}. \quad (4.8)$$

In the metric (4.1), the expansion of the field operator Φ with respect to the positive- and negative-frequency parts is

$$\Phi=(2\pi)^{-3/2}\int d^3k[a_\mathbf{k}u_\mathbf{k}(\eta)e^{i\mathbf{kx}}+a_\mathbf{k}^+u_\mathbf{k}^*(\eta)e^{-i\mathbf{kx}}], \quad (4.9)$$

where the operators $a_\mathbf{k}$ and $a_\mathbf{k}^+$ satisfy the usual Bose commutation relations, and the functions $u_\mathbf{k}(\eta)$ satisfy the equation

$$u''+\frac{2}{\eta}u'+k^2u=0. \quad (4.10)$$

The Hamiltonian $H=\int d^3x(-g)^{1/2}T_0^0$ of the field can be readily represented in the form

$$H=\frac{a^2}{2}\int d^3k[E(\eta)(a_\mathbf{k}^+a_\mathbf{k}+a_\mathbf{k}a_\mathbf{k}^+)+F(\eta)a_\mathbf{k}a_{-\mathbf{k}}+F^*(\eta)a_\mathbf{k}^+a_{-\mathbf{k}}^+],$$
$$E(\eta)=|u'|^2+k^2|u|^2,\quad F(\eta)=u'^2+k^2u^2. \quad (4.11)$$

(Note that in this case the metrical and canonical Hamiltonians are identical.) At the initial time $\eta=\eta_0$, the Hamiltonian H is diagonal if $F(\eta_0)=0, E(\eta_0)=k/a^2$, which corresponds to the following choice of the initial conditions for Eq. (4.10):

$$u(\eta_0)=(2ka^2(\eta_0))^{-1/2},\quad u'(\eta_0)=-iku(\eta_0).$$

At an arbitrary time η, the Hamiltonian H can be diagonalized by a canonical Bogolyubov transformation:

$$a_\mathbf{k}=\alpha(\mathbf{k},\eta)b_\mathbf{k}(\eta)+\beta(\mathbf{k},\eta)b_{-\mathbf{k}}^+(\eta).$$

The coefficients α and β of this transformation can be expressed as follows in terms of the solutions of Eq. (4.10):

$$\alpha=a\left(\frac{k}{2}\right)^{1/2}\left(u^*-\frac{i}{k}u^{*\prime}\right)e^{-ik(\eta-\eta_0)},$$
$$\beta=-a\left(\frac{k}{2}\right)^{1/2}\left(u^*+\frac{i}{k}u^{*\prime}\right)e^{ik(\eta-\eta_0)}. \quad (4.12)$$

Independently of (4.12), α and β can be found from the system of first-order equations

$$\alpha'=-\frac{a'}{a}\beta e^{-2ik(\eta-\eta_0)},\quad \beta'=-\frac{a'}{a}\alpha e^{2ik(\eta-\eta_0)},$$
$$\alpha(\eta_0)=1,\ \beta(\eta_0)=0. \tag{4.13}$$

The energy density $\varepsilon=\langle 0|N_\eta(T_0^0)|0\rangle$ of the produced particles and the pressure $P=-\langle 0|N_\eta(T_n^n)|0\rangle$ which they produce can be readily expressed in terms of α and β:

$$\varepsilon=\frac{1}{2\pi^2a^4}\int_0^\infty dkk^3|\beta|^2,\quad P=\frac{1}{6\pi^2a^4}\int_0^\infty dkk^3$$
$$\times[|\beta|^2+2\,\mathrm{Re}(\alpha\beta^*e^{2ik(\eta-\eta_0)})]. \tag{4.14}$$

The solution of Eq. (4.10) satisfying the initial conditions is

$$u=\frac{1}{a(2k)^{1/2}}\left[\frac{e^{ik(\eta-\eta_0)}}{2ik\eta_0}+e^{-ik(\eta-\eta_0)}\left(1-\frac{1}{2ik\eta_0}\right)\right]. \tag{4.15}$$

Using (4.12), we find the integrands in (4.14):

$$|\beta|^2=\frac{1}{4k^2}\left[\frac{1}{\eta_0^2}+\frac{1}{\eta^2}-\frac{2\cos 2k(\eta-\eta_0)}{\eta_0\eta}\right]$$
$$+\frac{\sin 2k(\eta-\eta_0)}{4k^3\eta_0\eta}\left(\frac{1}{\eta}-\frac{1}{\eta_0}\right)+\frac{1-\cos 2k(\eta-\eta_0)}{8k^4\eta_0^2\eta^2},$$
$$2\,\mathrm{Re}[\alpha\beta^*e^{2ik(\eta-\eta_0)}]=2|\beta|^2-\frac{\sin 2k(\eta-\eta_0)}{k\eta_0}$$
$$+\frac{\cos 2k(\eta-\eta_0)-1}{2k^2\eta_0^2}. \tag{4.16}$$

In accordance with the smoothing method, we use (4.16) to calculate the contributions to ε and P from the region $0\leqslant k\leqslant\lambda$:

$$2\pi^2a^4(k<\lambda)=\frac{\lambda^2}{8}\left(\frac{1}{\eta^2}+\frac{1}{\eta_0^2}\right)$$
$$-\frac{\lambda\sin 2\lambda(\eta-\eta_0)}{4\eta_0\eta(\eta-\eta_0)}+\frac{\gamma+\ln 2\lambda(\eta-\eta_0)}{8\eta_0^2\eta^2}$$
$$+(1-\cos 2\lambda(\eta-\eta_0))\left[\frac{1}{8\eta_0\eta(\eta-\eta_0)^2}-\frac{1}{8\eta_0^2\eta^2}\right]+o(1/\lambda),$$
$$2\pi^2a^4P(k<\lambda)=2\pi^2a^4\varepsilon(k<\lambda) \tag{4.17}$$
$$+\frac{1}{6}\left[\frac{\lambda^2\cos 2\lambda(\eta-\eta_0)}{\eta_0(\eta-\eta_0)}-\frac{\lambda^2}{2\eta_0^2}\right.$$
$$+\frac{\lambda\sin 2\lambda(\eta-\eta_0)}{2\eta_0(\eta-\eta_0)}$$
$$\left.\times\left(\frac{1}{\eta_0}-\frac{1}{\eta-\eta_0}\right)-\frac{1-\cos 2\lambda(\eta-\eta_0)}{4\eta_0(\eta-\eta_0)^2}\left(\frac{1}{\eta_0}-\frac{1}{\eta-\eta_0}\right)\right].$$

To calculate $\varepsilon(\lambda<k)$ and $P(\lambda<k)$, we find α and β directly from (4.13), using the smoothed potential. To terms $(k\eta)^{-3}$, we obtain for β

$$e^{2ik\eta_0}\beta=\int dxVe^{2ikx}+\int dx_1dx_2dx_3V(x_1)V(x_2)V(x_3)$$
$$\times\theta(x_1-x_2)\theta(x_2-x_3)e^{2ik(x_1-x_2+x_3)}+\int dx_1\ldots dx_5V(x_1)\ldots V(x_5)$$
$$\times\theta(x_1-x_2)\ldots\theta(x_4-x_5)e^{2ik(x_1-x_2+\ldots+x_5)},$$
$$V(x)=-\frac{1}{x}g(x-\eta_0)g(\eta_1-x). \tag{4.18}$$

In the second integral we make the change of variables $x_1-x_2+x_3=x$, $x_1-x_2=y_1$, $x_2-x_3=y_2$, and in the third integral $x_1-x_2=y_1$, $x_2-x_3=y_2$, $x_3-x_4=y_3$, $x_4-x_5=y_4$, $x_1-x_2+x_3-x_4+x_5=x$. Then (4.18) can be written as

$$e^{2ik\eta_0}\beta=\int_{\eta_0}^{\eta}dxe^{2ikx}[V+f_2+f_4]=\int_{\eta_0}^{\eta}dxe^{2ikx}\psi(x),$$

$$f_2(x)=-\int_x^\eta dx_1\int_x^{\eta_0}dx_2V(x_1)V(x_2)V(x_1+x_2-x),$$

$$f_4(x)=-\int_x^\eta dx_1\int_x^{\eta_0}dx_2\int_0^{\eta-x_1}dx_3\int_0^{x_2-\eta_0}dx_4V(x_1+x_3)$$
$$\times V(x_2-x_4)V(x_1+x_2-x)V(x_1+x_2+x_3-x)V(x_1+x_2-x_4-x).$$

The contribution to ε from the region $\lambda<k$ is

$$2\pi^2a^4\varepsilon(\lambda<k)=\int_\lambda^\infty dkk^3|\beta|^2$$
$$=\int_0^\infty dkk^3\int_{\eta_0}^{\eta}dxdy\psi(x)\psi(y)\cos 2k(x-y)$$
$$-\int_0^\lambda dkk^3\int_{\eta_0}^{\eta}dxdy\psi(x)\psi(y)\cos 2k(x-y). \tag{4.19}$$

The first integral in (4.19) is $\frac{1}{16}J_3$, where J_3 is determined by (3.6). The second integral in (4.19) can be calculated directly. A simple but lengthy calculation gives

$$\int_0^\lambda dkk^3\int_{\eta_0}^{\eta}dxdy\psi(x)\psi(y)\cos 2k(x-y)$$
$$=\frac{1}{16}J_3+2\pi^2a^4\varepsilon(k<\lambda)+o\left(\frac{1}{\lambda}\right).$$

Thus, $\varepsilon(k<\lambda)=-\varepsilon(k>\lambda)+o(1/\lambda)$, and, going to the limit $\lambda\to\infty$, we find the total energy density:

$$\varepsilon=\lim_{\lambda\to\infty}[\varepsilon(k<\lambda)+\varepsilon(\lambda<k)]=0.$$

One can show similarly that in the considered case the concentration n and the pressure P vanish. The absence of production of quanta of the massless scalar field with the Lagrangian (4.8) in Friedmann models for the equation of state $p=\varepsilon/3$ of the background matter was already noted by Parker.[1]

5. CONCLUSIONS

We conclude with some comments on the applicability of the smoothing method. The physical evolution of a space-time metric may lead to the appearance of real singularities of the time derivatives of $g_{\mu\nu}$ (for example, at singular points of the space-time). At such singular points, the smoothing procedure has no direct meaning. In this case, one can use the smoothing method to regularize divergent quantities at any point near but not coincident with the singularity. If the regularized physical quantities do not have singularities at the singular points of the space-time (as, for example, ε and P in the examples considered in Sec. 4), the results are obviously also valid at the singularities themselves. Divergence of n,ε,P, etc., as the singularity is approached indicates instability of the corresponding classical metrics with respect to the process of particle production.

The examples considered in the previous section correspond to the case of a continuous spectrum. It is obvious that the smoothing method can also be used when k form a discrete set of quantum numbers and the divergent quantities are represented by sums. At the same time, in the region of large k, using the well-known summation methods, we can always go over from sums to integrals, i.e., to the case when the regularization rules proposed in Secs. 2 and 3 apply directly. In closed spaces, the normal ordering operation (2.4) subtracts from the dynamical quantities not only the divergences associated with the zero-point fluctuations of the vacuum but also the finite terms that arise because of the difference between the topology of the closed space and Euclidean space. To calculate these terms, one can, instead of using (2.4), make a direct subtraction of the corresponding divergences.[2-4]

The values of ε and P calculated in Sec. 4 must include the contribution of the actually produced particles as well as the contribution from the vacuum polarization in the varying external field. In the case of a small coupling constant (for example, in quantum electrodynamics), one can distinguish the effects associated with vacuum polarization from the production of real particles.[9] It is therefore worth considering to what extent these contributions can be separated in the present case. Weak coupling corresponds to a low interaction intensity, $|\dot{g}_{\mu\nu}| \ll 1$, and the probability amplitude β is given by (3.4). Then, for example, the mean energy density ε is proportional to the integral J_3. Specifying the initial conditions at the point $t_0 = -\infty$ and requiring there the vanishing of $\dot{g}_{\mu\nu}$ and the higher derivatives, we obtain from (3.6) for $m = 2$

$$\varepsilon \sim \int_{-\infty}^{t} dx\, dy \ln|x-y| V(x) V^{(4)}(y)$$

$$- \int_{-\infty}^{t} dx \ln(t-x) [V(x) V'''(t) - V'''(x) V(t) + V''(x) V'(t) - V'(x) V''(t)]. \quad (5.1)$$

It is obvious that the first term in (5.1) can be interpreted as the contribution to ε from the actually produced particles, and the second, which is proportional to the derivatives of $g_{\mu\nu}$ at the considered time t, as the contribution from the vacuum polarization in the varying external field. In the case when V depends on $\mathbf{k}$, $V = V(\mathbf{k}, t)$, the problem becomes more complicated, and (5.1) gives only the high-frequency asymptotic behavior of the corresponding contributions. [It is possible that for $V = V(\mathbf{k}, t)$ a unique separation in ε of the contributions from the vacuum polarization and the real particles does not exist, as in the case of an arbitrary strong field.[2]]

The method proposed in Sec. 3 for removing the smoothing leads automatically to subtraction of all the surface terms. A more detailed examination of the neglected terms shows that in the case $n = 2m$ all the neglected terms diverge in the limit $g(x) \to \theta(x)$. For odd $n = 2m - 1$, the neglected surface terms include some that diverge and some that remain finite in the considered limit. The latter form a bilinear form composed of products of the potentials V and their time derivatives taken at the initial t_0 and final t_1 times. We do not know a sensible physical interpretation of these local surface terms.

Note added in proof (December 1, 1980). If the derivative $\dot{g}_{\mu\nu}(t_0)$ and a sufficient number of the higher time derivatives of the metric tensor $g^{(n)}_{\mu\nu}(t_0)$ vanish at the initial time t_0, the smoothing method can be reformulated in terms of renormalization of the constants in the generalized Einstein equations. For example, for a spinor ($s = \frac{1}{2}$) field in the metric (4.1), the corresponding result is

$$G_{\mu\nu} = 8\pi k_0 (T^{(reg)}_{\mu\nu} + T^{(div)}_{\mu\nu})$$

where

$$T^{(div)}_{\mu\nu} = \frac{m^2}{24\pi^2} (\gamma + \ln 2\lambda) G_{\mu\nu}, \qquad \lambda \to \infty.$$

In other words, the bare gravitational constant k_0 is renormalized:

$$k_0 = k \Big/ \left\{ 1 + \frac{km^2}{3\pi} (\gamma + \ln 2\lambda) \right\}.$$

It is possible that the requirement of renormalizability is necessary for unambiguous physical interpretation of the results of the regularization. I am grateful to A. A. Starobinskiĭ for drawing my attention to these questions.

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