Gauge conditions for fields of higher spins in an external gravitational field

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In flat space-time, fields with higher spins have a gauge freedom. If this freedom is used to impose a complete set of gauge conditions, the number of independent potentials of the free field can be reduced to two. There is a similar gauge freedom in a curved world, i.e., in the presence of an external gravitational field. However, as is shown in the paper, this freedom cannot in general be used to satisfy a complete set of gauge conditions constituting a covariant generalization of the gauge conditions in flat space-time. Thus, because of the interaction with the external gravitational field, gauge conditions corresponding to the elimination of "longitudinal" and "scalar" particles can be introduced only in special cases. The class of external gravitational fields admitting such a possibility is found. It is shown that for the considered fields (spins 1, 3/2, 2) the restrictions on the external gravitational field are essentially the same.

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§1. INTRODUCTION

The interaction of physical fields with an external gravitational field leads to a number of interesting and important effects such as amplification of classical waves and quantum particle production. Physical fields in an external gravitational field are described by generally covariant wave equations, the generalization of the corresponding equations in flat space-time. As in a flat world, these equations have a certain gauge freedom. In other words, they admit transformations of the potentials that leave the basic equations unchanged. The presence and correct exploitation of the gauge freedom are important for both technical and fundamental reasons. At the technical level, the gauge conditions make it possible to simplify the original equations significantly and reduce the number of unknown functions. At the fundamental level, the gauge freedom makes it possible to separate the physical degrees of freedom and interpret solutions correctly.

It is well known that in flat space-time the Maxwell equations

$$\eta^{\mu\nu}A_{\alpha,\mu,\nu}-A_{,\nu,}{}^{\nu}{}_{\alpha}=0 \tag{1.1}$$

are gauge invariant under the transformation

$$\bar{A}_\mu=A_\mu+\frac{\partial\Lambda}{\partial x^\mu}, \tag{1.2}$$

where Λ is an arbitrary function of the coordinates and the time. This means that if the functions A_μ are a solution of Eqs. (1.1), then so are the functions $\bar{A}_\mu$. The gauge freedom is usually employed to make $\bar{A}_\mu$ satisfy the conditions

$$\bar{A}^\mu{}_{,\mu}=0 \tag{1.3}$$

and, in addition, to make one of the components vanish, for example,

$$\bar{A}_0=0. \tag{1.4}$$

The condition (1.4) can be expressed in the invariant form[1)]

$$\bar{A}_\mu u^\mu=0, \tag{1.5}$$

where u^μ is some vector field. It takes the value $u^\mu=(1,0,0,0)$ when the subsidiary condition is chosen in the special form (1.4). Thus, in flat space-time one can always find a vector field u^μ(which is not unique) such that any solution of Eqs. (1.1) can be made by a choice of Λ to satisfy the conditions (1.3) and (1.5) (more precisely, it can be mapped onto the class of solutions for which these conditions are satisfied) in the whole of space-time or, at least, in some region. The conditions (1.3) and (1.4) reduce the number of independent components A^μ to two, so that the physical significance of these conditions is that they eliminate the "longitudinal" and "scalar" photons (see, for example, Refs. 2, 3). Note that in the case of a plane monochromatic wave propagating, say, in the positive direction of the x axis the elimination of the two "unphysical" components can be achieved by means of (1.3) and the use of an isotropic (or even spacelike) vector u^μ in (1.5), for example, the vector $u^\mu=(1,-1,0,0)$.

In the presence of a gravitational field, i.e., in curved space-time, it is natural to take as the gauge conditions for the electromagnetic field the generally covariant generalizations of Eqs. (1.3) and (1.5) and the analogous equations for the fields of other spins. In the present paper, we consider whether the gauge invariance of the equations guarantees the possibility in every gravitational field of making an arbitrary solution of these equations satisfy a complete set of gauge conditions of the type (1.3) and (1.5). We shall see that, in contrast to flat space-time, the gauge freedom cannot, in general, be used to ensure simultaneous fulfillment of conditions of the type (1.3) and (1.5). Although these conditions can be satisfied simultaneously for any solution and in any space-time at one time, i.e., on an arbitrary initial hypersurface Σ, they cannot in general be satisfied simultaneously off Σ.[2)] As an illustration, let us imagine a space-time possessing two asymptotically flat regions and an intermediate region with

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curvature. Then, in general, an initial state defined as vacuum for the "longitudinal" and "scalar" particles in one of the asymptotically flat regions will not be such in the other. Evidently, this corresponds to the appearance of nonradiative components of the field due to the interaction with the curvature (for the occurrence of effective electric charges in curved space-time, see Ref. 4). Of course, a strong gravitational field can lead to more radical consequences such as the production of real particles (all the gauge conditions being satisfied moreover everywhere), but this effect must also evidently be taken into account in the construction of quantum field theory in an external gravitational field.

Although conditions of the type (1.3) and (1.5) cannot be satisfied simultaneously in the general case, this can happen in some spaces. The main burden of the present paper is to determine the class of external gravitational fields in which an arbitrary solution of the wave equations can be made to satisfy a complete set of gauge conditions through the use of the gauge freedom. This class of spaces is very important, since the study of physical fields in such spaces will at least make it possible to avoid the complications associated with the impossibility of reducing the number of unknown components of the field. In the literature hitherto only fields of the lowest spins (0 and $\frac{1}{2}$), for which there is no gauge freedom, have been considered, or alternatively spaces with a very special background metric in which the fulfillment of a complete set of gauge conditions is indeed possible; finally, it has sometimes been assumed incorrectly that such gauge conditions can be satisfied on any background.

As wave equations, we consider the generally covariant equations for fields of spin 1, 3/2, 2, the linear version of Einstein's equations on an arbitrary curved background being used in the last case. We find the requirements that must be imposed on the vector u^μ if it is to be used in a condition (1.5) that holds simultaneously with the condition (1.3). A remarkable unity and similarity of the considered fields is the fact that the basic equation for u^μ in all three cases is the same[3]:

$$u^{\mu;\nu}=u^\mu a^\nu+bg^{\mu\nu}, \tag{1.6}$$

where a^ν and b are a vector and a scalar. The integrability conditions of Eqs. (1.6) constitute the basic restriction on the background metric.

An arbitrary space-time does not admit a vector field u_μ satisfying Eqs. (1.6). Some characteristics of spaces in which u_μ exists [Eqs. (1.6) can be integrated] are determined and analyzed in Ref. 6; the explicit form of the metric is given in Ref. 7.

The fields of various spins are considered in turn, the main facts relating to the derivation and investigation of Eqs. (1.6) being given in §2, which is devoted to the electromagnetic field; they are not repeated subsequently. For greater generality, we consider the wave equations with sources, and then the desire to make a solution satisfy the conditions (1.3) and (1.5) leads to restrictions on the functions describing the sources, although their complete vanishing is not required. The vector field u_μ, which could be interpreted [in accordance with (1.3)–(1.5)] as the four-velocity of observers that eliminate the "longitudinal" and "scalar" particles, will not be restricted by the norm sign, and for completeness we consider also the cases of an isotropic or spacelike vector.

In the final §5, the method we have developed is used to make the linearized Kerr solution satisfy the TT gauge.

§2. THE ELECTROMAGNETIC FIELD (SPIN 1)

In a curved space-time, the equations of electrodynamics for the four-potential

$$A_{\mu;\nu}{}^{;\nu}-A^\nu{}_{;\nu;\mu}-R_{\mu\nu}A^\nu=I_\mu \tag{2.1}$$

are invariant under the gauge transformation

$$\bar{A}_\mu=A_\mu+\Lambda_{,\mu}. \tag{2.2}$$

By a suitable choice of Λ and for any A_μ, one can make $\bar{A}_\mu$ satisfy the condition

$$\bar{A}^\mu{}_{;\mu}=0 \tag{2.3}$$

or the condition

$$\bar{A}_\mu u^\mu=0, \tag{2.4}$$

where u_μ is an arbitrary vector field. However, in general, it is not possible to achieve simultaneous fulfillment of the conditions (2.3) and (2.4).

Suppose that by the choice of Λ an arbitrary solution of Eqs. (2.1) has been made to satisfy the condition (2.3). The remaining gauge freedom is contained in functions satisfying the equation

$$\Lambda_{;\mu}{}^{;\mu}=0. \tag{2.5}$$

The general solution of this equation is determined by two functions of three variables—the initial data for Λ on some hypersurface Σ. We require that on Σ

$$\begin{gathered}\bar{A}_\mu u^\mu|_\Sigma=0=(A_\mu+\Lambda_{,\mu})u^\mu|_\Sigma,\\ (\bar{A}_\mu u^\mu)_{,\nu}n^\nu|_\Sigma=0=[(A_\mu+\Lambda_{,\mu})u^\mu]_{,\nu}n^\nu|_\Sigma,\end{gathered} \tag{2.6}$$

where n^ν is the vector of the normal to Σ. The system of equations (2.6) for the initial data $\Lambda|_\Sigma$ and $(\Lambda,_\nu n^\nu)|_\Sigma$ is always solvable. Thus, besides the condition (2.3), we can always achieve (2.6), which exhausts the gauge freedom (2.2) if we do not count the two functions of two variables that arise on the solution of (2.6). However, the fulfillment of the condition (2.4) off Σ depends on the propagation equations for $A_\mu u^\mu$. To obtain these equations, we multiply (2.1) by u^μ, use (2.3), and rewrite the obtained equation in the form (omitting the bar above the potentials)

$$(A_\mu u^\mu)_{;\nu}{}^{;\nu}-[2A_{\mu;\nu}u^{\mu;\nu}+A_\mu u^\mu{}_{;\nu}{}^{;\nu}+R_{\mu\nu}u^\mu A^\nu]=I_\mu u^\mu. \tag{2.7}$$

The null initial data (2.6) for $A_\mu u^\mu$ will guarantee fulfillment of $A_\mu u^\mu=0$ off Σ, if the second-order equation (2.7) is linear and homogeneous in $A_\mu u^\mu$. This is possible if and only if the right-hand side of (2.7), which does not contain A_μ, vanishes,

$$I_\mu u^\mu=0, \tag{2.8}$$

and the term in the square brackets, which contains the functions A_μ and their first derivatives, is a linear combination of the expressions $A^\mu_{;\mu}$, $A_\mu u^\mu$, and $(A_\mu u^\mu)_{;\nu}$. We multiply these expressions by $2b$, $-c$, and $2a^\nu$, respectively, where b and c are arbitrary scalars and a^ν is an arbitrary vector; we then add the obtained quantities and require that their sum be equal to the term in the square brackets. This gives the equation

$$2A_{\mu;\nu}(u^{\mu;\nu}-u^\mu a^\nu-bg^{\mu\nu})$$
$$+A_\mu(u^\mu{}_{;\nu}{}^{;\nu}-2u^\mu{}_{;\nu}a^\nu+R_\nu{}^\mu u^\nu+cu^\mu)=0.$$

Since, by hypothesis, A_μ is an arbitrary solution, the coefficients of A_μ and $A_{\mu,\nu}$ must vanish separately:

$$u_{\mu;\nu}-u_\mu a_\nu-bg_{\mu\nu}=0, \quad (2.9)$$

$$u_{\mu;\nu}{}^{;\nu}-2u_{\mu;\nu}a^\nu+R_{\mu\nu}u^\nu+cu_\mu=0. \quad (2.10)$$

If Eqs. (2.9) and (2.10) could be solved always, the gauge vector u_μ (perhaps, not unique) would exist in an arbitrary space-time, and any solution A_μ could be made to satisfy the conditions (2.3) and (2.4) simultaneously. However, Eqs. (2.9) and (2.10) are not in general integrable. The conditions of their integrability lead to restrictions on the space-time metric, and we now analyze these restrictions.

We note first that the condition (2.4), which we achieve, indicates that the vector u_μ is defined up to multiplication by an arbitrary scalar. For this reason, Eqs. (2.9) and (2.10) preserve their form under the substitution $\tilde{u}_\mu=\alpha u_\mu$ and an appropriate renotation of a_ν, b, c. Second, since the right-hand side of (2.9) after multiplication by u^ν is proportional to u_μ, the vector field u_μ defines a geodesic congruence. Third, since the tensor $u_{[\alpha}u_{\mu;\nu]}$ vanishes identically as a consequence of (2.9), the vector u_μ differs from a gradient vector only by a scalar factor.[8]

Using this last property, we introduce $l_\mu=e^{-\alpha}u_\mu$, where

$$l_{\mu;\nu}-l_{\nu;\mu}=0. \quad (2.11)$$

Substituting $u_\mu=e^\alpha l_\mu$ in (2.9), alternating with respect to the indices μ and ν, and using (2.11), we find

$$a_\mu=du_\mu+\alpha_{,\mu},$$

where d is an arbitrary scalar, and Eq. (2.9) takes the form

$$l_{\mu;\nu}=ml_\mu l_\nu+ng_{\mu\nu}, \quad (2.12)$$

where $m=de^\alpha$ and $n=be^{-\alpha}$ are arbitrary scalar functions. We introduce further the norm of the vector u_μ: $u_\mu u^\mu=\pm\rho^2$. Differentiating this equation and using (2.9), we obtain

$$(\rho^2)_{,\nu}=2\rho^2a_\nu\pm2bu_\nu,$$

from which it follows that if u_μ is isotropic ($\rho=0$), then $b=0$ and $n=0$; if u_μ is not isotropic, then

$$a_\nu=(\ln\rho)_{,\nu}\mp bu_\nu.$$

We reduce the nonisotropic vector field to unit norm. Suppose $v_\mu=\rho^{-1}u_\mu$ ($v_\mu v^\mu=\pm1$); then

$$v_{\mu;\nu}=p(g_{\mu\nu}\mp v_\mu v_\nu). \quad (2.13)$$

In what follows, it is convenient to consider the cases of isotropic and nonisotropic u^μ separately.

Isotropic gauge vector

An isotropic gauge vector satisfies the equation

$$l_{\mu;\nu}=ml_\mu l_\nu. \quad (2.14)$$

Equation (2.14) described an geodesic null congruence with the following kinematic characteristics[9]: zero rotation, zero expansion, and nonzero shear.

Substitution of (2.14) in (2.10) leads only to the concrete expression for c without in any way restricting the vector l_μ. Thus, the restrictions on the metric are exhausted by Eqs. (2.14).

The conditions of integrability of these equations are

$$l^\mu R_{\mu\nu\alpha\beta}=l_\nu(m_{,\alpha}l_\beta-m_{,\beta}l_\alpha) \quad (2.15)$$

and serve as a basis for the Petrov classification of the required spaces (for more detail, see Ref. 6). It follows from this investigation that in nonflat spaces the vector l_μ must be a multiple of a null principal direction of the Weyl tensor, i.e., it is unique except perhaps for metrics of type D, in which there are two such directions. In flat space-time, there exists an uncountable set of fields of gauge vectors l_μ. One can find the form of the Ricci tensor of spaces admitting the vector field (2.14) (for more details, see Ref. 6):

$$R_{\mu\nu}=B_\mu l_\nu+B_\nu l_\mu+(m_{,\alpha}-B_\alpha)l^\alpha g_{\mu\nu}, \quad (2.16)$$

where B_μ is an arbitrary vector.

Nonisotropic gauge vector

We take Eq. (2.9) in the form (2.13). It describes a geodesic congruence possessing zero rotation, zero shear, and nonzero expansion. The Ricci tensor of spaces admitting the vector field (2.13) has the form[6]

$$R_{\mu\nu}=P_{\mu\nu}\mp2(p_{,\mu}v_\nu+p_{,\nu}v_\mu)$$
$$+2v_\mu v_\nu p_{,\sigma}v^\sigma\mp(p_{,\sigma}v^\sigma+3p^2)g_{\mu\nu}, \quad (2.17)$$

where $P_{\mu\nu}$ is the Ricci tensor of the three-dimensional space orthogonal to the vector v^μ: $P_{\mu\nu}v^\mu=0$.

Substitution of (2.13) and its consequences in (2.10) leads to the concrete expression for the factor c and, in addition, to the relation

$$p_{,\mu}=\pm v_\mu(p_{,\nu}v^\nu). \quad (2.18)$$

This relation makes it possible to introduce the vector $w_\mu=e^\beta v_\mu$ satisfying the equation

$$w_{\mu;\nu}=qg_{\mu\nu}, \quad (2.19)$$

where $q=pe^\beta$ is a scalar function restricted by the condition

$$(w_\alpha w^\alpha)q_{,\mu}=\pm w_\mu(w^\nu q_{,\nu}). \quad (2.20)$$

Indeed, if the reduction of Eq. (2.13) to the form (2.19) is to be possible, β must satisfy the equation $\beta_{,\nu}=\pm pv_\nu$. The conditions of integrability of this equation are identical to (2.18), and, therefore, such a β can always be found. Thus, the restrictions on the metric are exhausted by Eqs. (2.19) and (2.20).

Spaces that admit the vector field (2.19), (2.20) are said to be equidistant spaces. (See Ref. 10 for the reason for this designation and some properties of these spaces.)

The Petrov classification of equidistant spaces and the question of the uniqueness of the gauge vector w_μ are considered in Ref. 6. We shall merely say here that the admissible background spaces include some metrics of type I, and the vector w_μ in the general case ($q\neq0$) is unique.

§3. FERMION FIELD (SPIN 3/2)

In flat space-time, the equations of a field of 3/2 spin (we omit the spinor index)

$$\gamma^\mu(\psi_{\nu,\mu}-\psi_{\mu,\nu})=0 \tag{3.1}$$

are invariant under the gauge transformation $\bar{\psi}_\mu=\psi_\mu+\varepsilon_{,\mu}$ with arbitrary four-spinor ε. Using this transformation in the case of arbitrary ψ_μ one can make $\bar{\psi}_\mu$ satisfy the condition

$$\gamma^\mu\bar{\psi}_\mu=0. \tag{3.2}$$

Then contraction of (3.1) with γ^ν automatically leads to

$$\bar{\psi}^\mu{}_{,\mu}=0. \tag{3.3}$$

The remaining gauge freedom is determined by four functions of three variables—the initial data $\varepsilon|_\Sigma$ for the equations $\gamma^\mu\varepsilon_{,\mu}=0$. These functions can be chosen such that on Σ

$$\bar{\psi}_\mu u^\mu|_\Sigma=0=(\psi_\mu+\varepsilon_{;\mu})u^\mu|_\Sigma.$$

In a flat space one can always find a vector u^μ (for example, with constant components in Lorentz coordinates) such that off Σ as well[11]

$$\bar{\psi}_\mu u^\mu=0. \tag{3.4}$$

The generally covariant generalization of Eqs. (3.1) is[4)]

$$\gamma^\mu(\psi_{\nu;\mu}-\psi_{\mu;\nu})=0. \tag{3.5}$$

These equations remain unchanged under the gauge transformation

$$\bar{\psi}_\mu=\psi_\mu+\varepsilon_{;\mu}, \tag{3.6}$$

if $\gamma^\mu(\varepsilon_{;\nu;\mu}-\varepsilon_{;\mu;\nu})\equiv\frac{1}{2}R_{\mu\nu}\gamma^\nu\varepsilon=0$ (i.e., in vacuum space-times, $R_{\mu\nu}=0$). In a nonvacuum space-time, Eqs. (3.5) do not admit (3.6) and, in addition, they are, in general, incompatible,[12] since their integrability conditions have the form

$$(R_{\mu\nu}-{}^1/{}_2g_{\mu\nu}R)\gamma^\mu\psi^\nu=0. \tag{3.7}$$

However, Eqs. (3.5) can admit a restricted class of solutions on a nonvacuum background, i.e., the class of solutions satisfying (3.7), and we include these solutions in our study.

We now establish the conditions under which the gauge freedom (3.6) (on a vacuum background) can be used to ensure that an arbitrary solution of Eqs. (3.5) can be made to satisfy simultaneously the conditions (3.2) and (3.4) and

$$\bar{\psi}^\mu{}_{;\mu}=0. \tag{3.8}$$

For a nonvacuum background, we investigate the possible existence of solutions satisfying (3.2), (3.4), and (3.8) without relating this to gauge freedom.

It is more convenient to work with equations of second order for ψ_μ, so as to be able to follow more readily the analogy between the conclusions in the case of this field and the fields with spins $s=1$ and $s=2$. The first-order equations (3.5) will be satisfied if they hold on the initial hypersurface. Differentiating (3.5) covariantly with respect to σ and multiplying from the left by γ^σ, we obtain the second-order equations

$$\psi_{\mu;\nu}{}^{;\nu}-\psi^\nu{}_{;\nu;\mu}+R_{\mu\nu\alpha\beta}\sigma^{\alpha\beta}\psi^\nu-R_{\mu\nu}\gamma^\nu\gamma^\sigma\psi_\sigma-{}^1/{}_4R\psi_\mu=0, \tag{3.9}$$

where $\sigma^{\alpha\beta}=\frac{1}{4}(\gamma^\alpha\gamma^\beta-\gamma^\beta\gamma^\alpha)$.

Proceeding as in §2, we use (3.6) to achieve (3.8). We use the remaining gauge freedom contained in the equation $\varepsilon_{;\nu}{}^{;\nu}=0$ to ensure that on Σ

$$\begin{gathered}\bar{\psi}_\mu u^\mu|_\Sigma=0=(\psi_\mu+\varepsilon_{;\mu})u^\mu,\\ (\bar{\psi}_\mu u^\mu)_{;\nu}n^\nu|_\Sigma=0=[(\psi_\mu+\varepsilon_{,\mu})u^\mu]_{;\nu}n^\nu|_\Sigma.\end{gathered} \tag{3.10}$$

Equations (3.10) can be solved for the functions $\varepsilon|_\Sigma$ and $\varepsilon_{,\nu}n^\nu|_\Sigma$. Moreover, they can be determined from Eqs. (3.10) up to eight arbitrary functions of two variables, which express the gauge freedom that still remains. We shall use these functions in what follows.

For nonvacuum spaces, for which the gauge freedom (3.6) does not occur, we consider the functions for which (3.8) and (3.10) are satisfied *a priori*.

The initial conditions (3.10) guarantee fulfillment of (3.4) off Σ if the propagation equations for $\bar{\psi}_\mu u^\mu$ are linear and homogeneous. Contracting (3.9) with u^μ and using (3.8), we find (omitting the bar over the potentials)

$$\begin{aligned}&(\psi_\mu u^\mu)_{;\nu}{}^{;\nu}-{}^1/{}_4R\psi_\mu u^\mu-[2\psi_{\mu;\nu}u^{\mu;\nu}+\psi_\mu u^\mu{}_{;\nu}{}^{\nu}\\ &\quad+R_{\alpha\beta\mu\nu}u^\nu\sigma^{\alpha\beta}\psi^\mu+R_{\mu\nu}u^\mu\gamma^\nu\gamma^\alpha\psi_\alpha]=0.\end{aligned} \tag{3.11}$$

Therefore, the term in the square brackets in (3.11) must be a linear combination of the expressions $\psi^\mu{}_{;\mu}$, $\psi_\mu u^\mu$, and $(\psi_\mu u^\mu)_{;\nu}$. Multiplying these expressions from the left by the matrices $2B$, C, and $2A^\nu$, respectively, adding them, and comparing the result with term in the square brackets, we obtain the equation

$$\begin{aligned}&2(u^{\mu;\nu}-A^\nu u^\mu-Bg^{\mu\nu})\psi_{\mu;\nu}+(u^\mu{}_{;\nu}{}^{;\nu}-2u^\mu{}_{;\nu}A^\nu\\ &\quad+R_{\alpha\beta}{}^\mu{}_\nu u^\nu\sigma^{\alpha\beta}+R_{\alpha\beta}u^\alpha\gamma^\beta\gamma^\mu-Cu^\mu)\psi_\mu=0.\end{aligned} \tag{3.12}$$

The matrix coefficients of $\psi_{\mu;\nu}$ and ψ_μ must vanish separately. Taking their trace and introducing the notation

$$a^\nu={}^1/{}_4\,\mathrm{Sp}\,A^\nu,\quad b={}^1/{}_4\,\mathrm{Sp}\,B,\quad c=-{}^1/{}_4\,\mathrm{Sp}\,C,$$

we obtain equations that are identical to (2.9) and (2.10). The traceless parts of the matrices B, C, and A^ν do not lead to any new restrictions over and above the conditions (2.9) and (2.10). Thus, solutions of Eqs. (3.9) satisfying the conditions (3.8) and (3.4) exist is space-times that admit a vector field u^μ satisfying Eqs. (2.9) and (2.10).

We now consider the possibility of satisfying the condition (3.2). We recall that we are interested in solutions of the first-order equations (3.5) for which the solutions of Eqs. (3.9) must satisfy (3.5) on Σ. We multiply (3.5) by γ^ν and use (2.9). We obtain the equation

$$u^\nu(\gamma^\mu\psi_\mu)_{;\nu}=0. \tag{3.13}$$

We now multiply (3.5) by γ^ν and use (3.8). We obtain the equation

$$\gamma^\nu(\gamma^\mu\psi_\mu)_{;\nu}=0. \tag{3.14}$$

It follows from (3.13) and (3.14) that if on a two-dimensional surface Σ' belonging to Σ

$$(\gamma^\mu\psi_\mu)|_{\Sigma'}=0, \tag{3.15}$$

then on the complete hypersurface Σ we shall have $(\gamma^\mu\psi_\mu)|_\Sigma=0$, $(\gamma^\mu\psi_\mu)_{;\nu}n^\nu|_\Sigma=0$. Null initial data for $\gamma^\mu\psi_\mu$ on Σ guarantee fulfillment of the condition (3.2) off Σ since the propagation equation for $\gamma^\mu\psi_\mu$ that follows from (3.9) when (3.8) is used has the form

$$(\gamma^\mu\psi_\mu)_{;\nu}{}^{;\nu}-{}^3/_4R\gamma^\mu\psi_\mu=0.$$

With regard to the condition (3.15), it can always be achieved on a vacuum background by using the remaining gauge freedom, i.e., by a suitable specification of four of the eight functions of two variables. [On a nonvacuum background, we assume that the condition (3.15) is satisfied *a priori*.]

Thus, solutions of Eqs. (3.5) satisfying the conditions (3.2), (3.4), and (3.8) exist in space-times admitting the vector field (2.9), (2.10), and on a vacuum background admitting such a vector field an arbitrary solution of Eqs. (3.5) can always be made to satisfy the conditions (3.2), (3.4), and (3.8) by using the gauge freedom.

On a nonvacuum background, the solutions of Eqs. (3.5) must, besides everything else, satisfy the conditions (3.7). Since we regard (3.2), (3.4), and (3.8) as constraints that distinguish the physical degrees of freedom, (3.7) must be satisfied without leading to additional restrictions on ψ_μ. The form of the Ricci tensor of spaces admitting an isotropic or nonisotropic gauge vector u^μ is known [see (2.16) and (2.17), respectively]. Substituting this form in (3.7), we find that in the case of isotropic u^μ the vector B^μ must be proportional to u^μ, which restricts the class of admissible metrics, but still retains among them representatives of spaces in the Petrov type O, N, and III. In the case of nonisotropic u^μ, the Ricci tensor must have the form $R_{\mu\nu}=\alpha u_\mu u_\nu+\beta g_{\mu\nu}$ with arbitrary α and β, which restricts the admissible metrics to type O, i.e., to conformally flat spaces [the solution of Eqs. (3.5) on such a background was used in Ref. 11].

§4. WEAK GRAVITATIONAL FIELD (SPIN 2)

Small corrections $h_{\mu\nu}$ to the Minkowski metric $\eta_{\mu\nu}$ satisfy the linearized Einstein equations

$$-\psi_{\mu\nu,\alpha}{}^{,\alpha}+\psi_\mu{}^\alpha{}_{,\alpha,\nu}+\psi_\nu{}^\alpha{}_{,\alpha,\mu}-\eta_{\mu\nu}\psi^{\alpha\beta}{}_{,\alpha,\beta}=2T_{\mu\nu}, \tag{4.1}$$

where

$$\psi_{\mu\nu}\equiv h_{\mu\nu}-{}^1/_2\eta_{\mu\nu}h,\quad h=h_{\mu\nu}\eta^{\mu\nu}.$$

Equations (4.1) are invariant under the gauge transformation

$$\tilde\psi_{\mu\nu}=\psi_{\mu\nu}+\xi_{\mu,\nu}+\xi_{\nu,\mu}-\xi^\alpha{}_{,\alpha}\eta_{\mu\nu} \tag{4.2}$$

with arbitrary vector ξ_α. It is well known[1,13] that in a space without sources, i.e., for $T_{\mu\nu}=0$, the plane-wave solutions of Eqs. (4.1) can be made to satisfy the following conditions by virtue of the gauge freedom (4.2):

$$\psi_\mu{}^\nu{}_{,\nu}=0,\quad \psi=\psi_{\mu\nu}\eta^{\mu\nu}=0,\quad \psi_{\mu\nu}u^\nu=0,$$

where the vector u^ν is usually chosen in the form $u^\nu=(1,0,0,0)$. Eight of these conditions are independent. Thus, there remain only two independent ("physical") components of the field $\psi_{\mu\nu}$. In §5, we shall return to the question of the gauge conditions for arbitrary (not necessarily plane-wave) solutions of Eqs. (4.1); here we shall consider the generalization of Eqs. (4.1) to the case of a curved background space-time.

Replacing the metric $g_{\mu\nu}$ by $g_{\mu\nu}+h_{\mu\nu}$ and the energy-momentum tensor $T_{\mu\nu}$ by $T_{\mu\nu}+\delta T_{\mu\nu}$, we write the linear version of the Einstein equations $R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=T_{\mu\nu}$ in the form

$$-\psi_{\mu\nu;\alpha}{}^{;\alpha}+\psi_{\mu\,;\alpha;\nu}+\psi_{\nu\,;\alpha;\mu}-g_{\mu\nu}\psi^{\alpha\beta}{}_{;\alpha;\beta}$$
$$+2R_{\mu\alpha\beta\nu}\psi^{\alpha\beta}+R_\mu^\alpha\psi_{\alpha\nu}+R_\nu^\alpha\psi_{\alpha\mu}+g_{\mu\nu}R_{\alpha\beta}\psi^{\alpha\beta}-R\psi_{\mu\nu}=2\delta T_{\mu\nu}. \tag{4.3}$$

A consequence of (4.3) is the equation

$$-\psi_{;\alpha;}{}^\alpha-2\psi^{\alpha\beta}{}_{;\beta;\alpha}+4R_{\alpha\beta}\psi^{\alpha\beta}-R\psi=2g^{\mu\nu}\delta T_{\mu\nu}. \tag{4.4}$$

Indices are raised and lowered and covariant differentiation performed by means of the background metric $g_{\mu\nu}$.

Equations (4.3) are invariant under the following gauge transformation with arbitrary vector ξ^α:

$$\tilde\psi_{\mu\nu}=\psi_{\mu\nu}+\xi_{\mu;\nu}+\xi_{\nu;\mu}-\xi^\alpha{}_{;\alpha}g_{\mu\nu},$$
$$\delta\tilde T_{\mu\nu}=\delta T_{\mu\nu}+T_{\mu\nu;\alpha}\xi^\alpha+T_{\mu\alpha}\xi^\alpha{}_{;\nu}+T_{\nu\alpha}\xi^\alpha{}_{;\mu}. \tag{4.5}$$

Let us consider the conditions under which an arbitrary solution of Eqs. (4.3) can be made to satisfy the following conditions simultaneously by virtue of the transformations (4.5):

$$\psi_\mu{}^\nu{}_{;\nu}=0, \tag{4.6}$$
$$\psi_{\mu\nu}u^\nu=0, \tag{4.7}$$
$$\psi=0. \tag{4.8}$$

An important difference between Eqs. (4.3) and the equations for other fields is that the right-hand side of (4.3) depends, in general, on the gravitational variables $\psi_{\mu\nu}$. This occurs because the energy-momentum tensor contains not only variables such as the density, pressure, velocity components, etc., that describe the matter but also the metric and, perhaps, its derivatives as well. For this reason, $\psi_{\mu\nu}$ arises on the variation of $T_{\mu\nu}$. For each definite $T_{\mu\nu}$, once it has been decided which variables are material variables, the dependence of $\delta T_{\mu\nu}$ on $\psi_{\mu\nu}$ can be written down explicitly. We prefer to work with Eqs. (4.3) in general form, assuming only that the energy-momentum tensor depends on the metric and not on its derivatives. (Below, we shall also consider definite examples of $T_{\mu\nu}$ corresponding to pure radiation and an ideal fluid.) Then $\delta T_{\mu\nu}$ can be represented in the form

$$\delta T_{\mu\nu}=t_{\mu\nu}+\tau_{\mu\nu},\quad \tau_{\mu\nu}=A_{\mu\nu}{}^{\alpha\beta}\psi_{\alpha\beta}, \tag{4.9}$$

where $t_{\mu\nu}$ contains the background metric and the perturbations of the material variables, and $\tau_{\mu\nu}$ contains a linear combination of the $\psi_{\mu\nu}$ with certain coefficients, which are as yet undetermined and depend on the background values of the material variables and the background metric.

As before, we first use the gauge freedom (4.5) to achieve (4.6). The remaining gauge freedom is described by the equations

$$\xi_{\mu;\alpha}{}^{;\alpha}+\xi_\alpha R_\mu{}^\alpha=0$$

and consists of eight functions of three variables—the initial data $\xi_\mu|_\Sigma$ and $(\xi_{\mu;\alpha}n^\alpha)|_\Sigma$. These gauge functions can be chosen in such a way that for arbitrary $\psi_{\mu\nu}$ and u^α the following conditions are satisfied on Σ:

$$\tilde\psi_{\mu\nu}u^\nu|_\Sigma=0=(\psi_{\mu\nu}+\xi_{\mu;\nu}+\xi_{\nu;\mu}-g_{\mu\nu}\xi^\alpha{}_{;\alpha})u^\nu|_\Sigma,$$
$$(\tilde\psi_{\mu\nu}u^\nu)_{;\alpha}n^\alpha|_\Sigma=0=[(\psi_{\mu\nu}+\xi_{\mu;\nu}+\xi_{\nu;\mu}-g_{\mu\nu}\xi^\alpha{}_{;\alpha})u^\nu]_{;\beta}n^\beta|_\Sigma. \tag{4.10}$$

Moreover, the initial data $\xi_\mu|_\Sigma$ and $(\xi_{\mu;\alpha}n^\alpha)|_\Sigma$ are determined from (4.10) up to eight arbitrary functions of two variables.

The null initial data (4.10) for $\bar{\psi}_{\mu\nu}u^{\nu}$ guarantee fulfillment of (4.7) off Σ provided linear homogeneous propagation equations for $\psi_{\mu\nu}u^{\nu}$ follow from (4.3). Multiplying (4.3) by u^{ν} and using (4.6), we obtain the equation (omitting the bars above the potentials $\psi_{\mu\nu}$)

$$-(\psi_{\mu\nu}u^{\nu})_{;\alpha}{}^{;\alpha}+R_{\mu}{}^{\alpha}\psi_{\alpha\nu}u^{\nu}-R\psi_{\mu\nu}u^{\nu}+[2\psi_{\mu\alpha;\beta}u^{\alpha;\beta}+\psi_{\mu\alpha}u^{\alpha}{}_{;\beta}{}^{;\beta}+R_{\nu}{}^{\alpha}u^{\nu}\psi_{\alpha\mu}+2R_{\mu\alpha\beta\nu}u^{\nu}\psi^{\alpha\beta}+u_{\mu}R_{\alpha\beta}\psi^{\alpha\beta}-2A_{\mu\nu}{}^{\alpha\beta}u^{\nu}\psi_{\alpha\beta}]=2t_{\mu\nu}u^{\nu}. \qquad (4.11)$$

The right-hand side of Eqs. (4.11), which does not contain $\psi_{\mu\nu}$ at all, must vanish:

$$t_{\mu\nu}u^{\nu}=0, \qquad (4.12)$$

and the term in the square brackets must be represented by a linear combination of the expressions $(\psi_{\alpha\nu}u^{\nu})_{;\beta}, \psi_{\alpha\nu;}{}^{\nu}, \psi_{\alpha\nu}u^{\nu}$. Multiplying them by $2a_{\mu}{}^{\alpha\beta}, 2b_{\mu}{}^{\alpha}$, $c_{\mu}{}^{\alpha}$, respectively, adding, and comparing the result with the term in the square brackets, and setting the coefficients of $\psi_{\alpha\beta;\nu}$ and $\psi_{\alpha\beta}$ equal to zero separately, we obtain the equations (which are symmetrized with respect to α and β because of the symmetry of $\psi_{\alpha\beta}$, which is indicated by placing the indices α and β within brackets)

$$\delta_{\mu}^{(\alpha}u^{\beta)}{}_{;\nu}-u^{(\alpha}a_{\mu}{}^{\beta)}{}_{\nu}-b_{\mu}{}^{(\alpha}\delta_{\nu}{}^{\beta)}=0, \qquad (4.13)$$

$$\delta_{\mu}^{(\alpha}u^{\beta)}{}_{;\sigma}{}^{;\sigma}+\delta_{\mu}^{(\alpha}R_{\sigma}^{\beta)}u^{\sigma}+2R_{\mu}{}^{(\alpha\beta)}{}_{\sigma}u^{\sigma}+2u_{\mu}R^{\alpha\beta}-4A_{\mu\sigma}{}^{\alpha\beta}u^{\sigma}-2u^{(\alpha}{}_{;\sigma}a_{\mu}{}^{\beta)\sigma}-c_{\mu}{}^{(\alpha}u^{\beta)}=0. \qquad (4.14)$$

Equations (4.13) are equivalent to the system of equations

$$u_{\mu;\nu}=u_{\mu}a_{\nu}+bg_{\mu\nu}, \qquad (4.15)$$

$$a_{\mu}{}^{\alpha\beta}=\delta_{\mu}^{\alpha}a^{\beta}+m_{\mu}g^{\alpha\beta}, \qquad (4.16)$$

$$b_{\mu}{}^{\alpha}=\delta_{\mu}^{\alpha}b-m_{\mu}u^{\alpha}, \qquad (4.17)$$

where a_{ν} and m_{μ} are arbitrary vectors and b is an arbitrary scalar. Note that Eq. (4.15), which is a restriction on the background metric, is identical to (2.9). It is more convenient to analyze (4.14) and the possibility of satisfying the condition (4.8) separately for isotropic and nonisotropic u^{μ}.

Isotropic gauge vector

We represent Eq. (4.15) in the form (2.14). Substituting (2.14) and its consequences, and also (2.16) and (4.16), (4.17) (with appropriate new notation) in (4.14), we can find the actual expressions for the coefficients c_{μ}^{α} (which is not important for us) and, moreover, derive a connection between $A_{\mu\sigma}^{\alpha\beta}l^{\sigma}$ and the remaining quantities:

$$A_{\mu\sigma}{}^{\alpha\beta}l^{\sigma}={}^{1}/{}_{2}l_{\mu}(m_{,\sigma}-B_{\sigma})l^{\sigma}g^{\alpha\beta}+\rho_{\mu}{}^{\alpha}l^{\beta}+\rho_{\mu}{}^{\beta}l^{\alpha}, \qquad (4.18)$$

where ρ_{μ}^{α} is an arbitrary tensor. Thus, if the structure of quantities $\tau_{\mu\nu}$ in (4.9) for the given energy–momentum tensor satisfies the relation (4.18), Eqs. (4.14) do not lead to any restrictions on the background metric additional to (4.15).

We now consider the possibility of satisfying the condition (4.8). Decisive here is the following circumstance. Although for arbitrary vector field u_{μ} all the eight equations (4.10) are independent and require the use of all eight gauge functions of three variables if they are to be satisfied, for vector field l_{μ} satisfying Eq. (2.14) only six of Eqs. (4.10) are important. The two remaining equations are satisfied as a consequence of them provided one gauge function of two variables is suitably chosen. Thus, two gauge functions of three variables remain in reserve and can be used to specify on Σ null initial data for ψ (for the details, see Ref. 7):

$$\psi|_{\Sigma}=0, \quad \psi_{,\alpha}n^{\alpha}|_{\Sigma}=0. \qquad (4.19)$$

Thus, in spaces admitting the vector (2.14) the gauge freedom can be used to satisfy not only (4.6) but also the conditions (4.10) and (4.19). For this reason, one can weaken the requirements on $A_{\mu\sigma}^{\alpha\beta}l^{\sigma}$ by permitting the term in the square brackets in (4.11) to contain as well $d_{\mu}\psi$, where d_{μ} is an arbitrary vector. Then instead of (4.18) we obtain

$$A_{\mu\sigma}{}^{\alpha\beta}l^{\sigma}=s_{\mu}g^{\alpha\beta}+\rho_{\mu}{}^{\alpha}l^{\beta}+\rho_{\mu}{}^{\beta}l^{\alpha}, \qquad (4.20)$$

where s_{μ} and ρ_{μ}^{α} are, respectively, an arbitrary vector and an arbitrary tensor. On the other hand, the propagation equation (4.4) for ψ contains by virture of (2.16) a combination of the expressions $\psi_{\mu\nu}u^{\nu}$ and ψ, and also the terms $2g^{\mu\nu}t_{\mu\nu}+2A_{\mu}^{\mu\alpha\beta}\psi_{\alpha\beta}$ on the right-hand side.

The null initial data (4.10) and (4.19) guarantee fulfillment of (4.7) and (4.8) off Σ and do not require any additional restrictions on the background space apart from the fulfillment of Eqs. (2.14) if the variation of the energy–momentum tensor satisfies Eqs. (4.12) and (4.20), and also

$$g^{\mu\nu}t_{\mu\nu}=0 \qquad (4.21)$$

and

$$A_{\mu}{}^{\mu\alpha\beta}=f^{\alpha}l^{\beta}+f^{\beta}l^{\alpha}+sg^{\alpha\beta} \qquad (4.22)$$

with arbitrary vector and scalar functions f^{α} and s.

An example of an energy–momentum tensor for which (4.20) and (4.22) are satisfied automatically is the pure-radiation tensor

$$T_{\mu\nu}=\rho k_{\mu}k_{\nu}, \quad k_{\mu}k^{\mu}=0,$$

where it is assumed that the unperturbed value of k_{μ} is equal to the gauge vector $l_{\mu}, k_{\mu}=l_{\mu}+\delta k_{\mu}$. Taking ρ and k^{μ} to be material variables, we obtain a decomposition of (4.9) in the form

$$t_{\mu\nu}=\delta\rho l_{\mu}l_{\nu}+\rho\delta k^{\sigma}(g_{\mu\sigma}l_{\nu}+g_{\nu\sigma}l_{\mu}),$$

$$\tau_{\mu\nu}=\rho(\delta_{\nu}^{\alpha}l_{\mu}l^{\beta}+\delta_{\mu}^{\beta}l_{\nu}l^{\alpha}-l_{\mu}l_{\nu}g^{\alpha\beta})\psi_{\alpha\beta},$$

from which (4.20) and (4.22) follow directly.

On a vacuum background ($T_{\mu\nu}=0$), the relations (4.12), (4.20), (4.21), and (4.22) are satisfied trivially.

Nonisotropic gauge vector

We represent Eq. (4.15) in the form (2.13). Substituting (2.13), (2.19), (4.16), and (4.17) in (4.14), we find the concrete expressions for the coefficients c_{μ}^{α} (which we do not need) and, in addition, the relation

$$A_{\mu\sigma}{}^{\alpha\beta}v^{\sigma}={}^{1}/{}_{2}v_{\mu}P^{\alpha\beta}+g^{\alpha\beta}(p_{,\mu}-m_{\mu}\pm v_{\mu}p^{2})-{}^{3}/{}_{4}\delta_{\mu}^{\alpha}p_{,}{}^{\beta}-{}^{3}/{}_{4}\delta_{\mu}^{\beta}p_{,}{}^{\alpha}+\rho_{\mu}{}^{\alpha}v^{\beta}+\rho_{\mu}{}^{\beta}v^{\alpha}, \qquad (4.23)$$

where ρ_{μ}^{α} is an arbitrary tensor.

We now consider the condition (4.8). Multiplying (4.6) by v^μ and using (2.13), we obtain

$$(\psi_\mu{}^\nu v^\mu)_{;\nu} \pm p\psi_{\mu\nu}v^\mu v^\nu - p\psi = 0, \qquad (4.24)$$

from which it follows that when $p \neq 0$ the condition (4.8) is satisfied by virtue of (4.6) and (4.7). But if $p = 0$, then as in the case of an isotropic gauge vector it is possible to satisfy Eqs. (4.10) by using six gauge functions of three variables, which leaves two to satisfy (4.19). Then the condition (4.23) is weakened and takes the form

$$A_{\mu\sigma}{}^{\alpha\beta}v^\sigma = {}^1/_2\, v_\mu P^{\alpha\beta} - {}^3/_4\, \delta_\mu^\alpha p_,{}^\beta - {}^3/_4\, \delta_\mu^\beta p_,{}^\alpha + s_\mu g^{\alpha\beta} + \rho_\mu{}^\alpha v^\beta + \rho_\mu{}^\beta v^\alpha. \qquad (4.25)$$

From Eq. (4.4) we also find that we must have

$$t_{\mu\nu}g^{\mu\nu} = 0, \qquad (4.26)$$

$$A_\mu{}^{\mu\alpha\beta} = 2P^{\alpha\beta} + f^\alpha v^\beta + f^\beta v^\alpha + sg^{\alpha\beta} \qquad (4.27)$$

with arbitrary f^α and s. Thus, in spaces that admit the vector field (2.13) we can satisfy the conditions (4.6)–(4.8) by using the gauge freedom, and Eq. (2.13) exhausts all the restrictions on the background metric if the variation of the energy–momentum tensor satisfies (4.25) and (4.27).

We consider the energy–momentum tensor of an ideal medium:

$$T_{\mu\nu} = (\varepsilon + P)k_\mu k_\nu - Pg_{\mu\nu}, \quad k_\nu k^\nu = 1,$$

where $k_\mu = v_\mu + \delta k_\mu$, $v_\mu v^\mu = 1$. Regarding ε, P, and k^μ as material variables, we obtain (4.9) in the form

$$t_{\mu\nu} = \delta(\varepsilon+P)v_\mu v_\nu + (\varepsilon+P)\delta k^\sigma(g_{\mu\sigma}v_\nu + g_{\nu\sigma}v_\mu) - \delta P g_{\mu\nu},$$
$$\tau_{\mu\nu} = (\varepsilon+P)v^\sigma(\psi_{\mu\sigma}v_\nu + \psi_{\nu\sigma}v_\mu) - (\varepsilon+P)v_\mu v_\nu\psi - P(\psi_{\mu\nu} - {}^1/_2 g_{\mu\nu}\psi). \qquad (4.28)$$

Comparing (4.28) with (4.25) and (4.27), we find that in this case there are additional restrictions on the background metric, namely, $p_,{}^\alpha$ must be proportional to v^α, which gives a condition identical to (2.20). In addition, $P^{\alpha\beta}$ must be proportional to v^α, v^β, and $g^{\alpha\beta}$. The same restrictions arise when the background space is a vacuum space. Equations (2.13) and (2.20) return us to the equidistant spaces considered in §2.

§5. WEAK GRAVITATIONAL FIELD ON A FLAT BACKGROUND

An important special case of the equations for a weak gravitational field is provided by the equations (4.1) on a flat background. As follows from the results of §4, in this case the gauge vector u^μ always exists, and, therefore, any solution of these equations without sources can be made to satisfy the conditions (4.6)–(4.8) in some space-time region at least. The conditions (4.6)–(4.8) are called the *TT* (transverse traceless) gauge. It is easy to show that any plane-wave solution or sum of such solutions can be reduced to the *TT* gauge.[1,13] With regard to other solutions, it has been asserted that they do not satisfy the *TT* gauge. The linearized Kerr solution is given as an example.[1] We shall show that this solution can nevertheless be made to satisfy the *TT* gauge in complete agreement with the results obtained in §4.

In Lorentz coordinates, this solution has the form[1]

$$h_{00} = \frac{2M}{r}, \quad h_{jk} = \frac{2M}{r}\delta_{jk}, \quad h_{0k} = -2e_{klm}\frac{s^l x^m}{r^3}, \qquad (5.1)$$

where $r = (x^2+y^2+z^2)^{1/2}$. Using (5.1) to find the components $\psi_{\mu\nu}$, we obtain

$$\psi_{00} = \frac{4M}{r}, \quad \psi_{0i} = -2e_{ikl}\frac{s^k x^l}{r^3}, \quad \psi_{ij} = 0. \qquad (5.2)$$

The condition $\psi^\nu_{\mu;\nu} = 0$ is satisfied for the solution (5.2) automatically. The gauge transformations that do not violate this condition are described by the equations

$$\Box\xi_\nu = 0. \qquad (5.3)$$

The solutions of these equations are determined by the initial data on the hypersurface $\Sigma(t=0)$:

$$\xi_\mu|_\Sigma = f_\mu(x, y, z), \quad \xi_{\mu,0}|_\Sigma = \varphi_\mu(x, y, z).$$

As gauge vector, we choose $u^\mu = (1, 0, 0, 0)$. The gauge functions f_μ and φ_μ can be specified in such a way that on $t = 0$

$$\bar\psi_{0\mu}|_\Sigma = 0, \quad \bar\psi_{0\mu,0}|_\Sigma = 0, \quad \bar\psi|_\Sigma = 0, \quad \bar\psi_0|_\Sigma = 0. \qquad (5.4)$$

To see this, we substitute in (5.4) the actual values of (5.2), and also $\xi_{\mu,00}|_\Sigma = \Delta f_\mu$, obtaining the following system of equations: $\varphi_0 = -M/r$,

$$-2e_{ikl}\frac{s^k x^l}{r^3} + f_{0,i} + \varphi_i = 0, \quad \Delta f_0 = 0, \quad \operatorname{div}\varphi = 0, \qquad (5.5)$$

$$\Delta f_i - (\operatorname{div} f)_{,i} - (4M/r)_{,i} = 0, \quad \operatorname{div} f = -3M/r. \qquad (5.6)$$

As one would expect, not all of the equations (5.5) and (5.6) are independent. A special solution of the system (5.5) is

$$f_0 = 0, \quad \varphi_i = 2e_{ikl}\frac{s^k x^l}{r^3}.$$

To solve the system (5.6), we represent the three-dimensional vector f_i in the form $f_i = f_i^{(1)} + f_i^{(2)}$, where $\operatorname{curl} f^{(1)} = 0$ and $\operatorname{div} f^{(2)} = 0$. We then obtain a special solution for $f_i^{(1)}$:

$$f_i^{(1)} = {}^3/_2 Mr_{,i},$$

and an equation for $f_i^{(2)}$:

$$\Delta f_i^{(2)} - 4M(1/r)_{,i} = 0. \qquad (5.7)$$

We choose the boundary conditions for Eq. (5.7) in the form

$$\operatorname{div} f^{(2)}|_{r=r_0} = 0. \qquad (5.8)$$

Since it follows from Eq. (5.7) that $\Delta \operatorname{div} f^{(2)} = 0$, the boundary conditions (5.8) ensure $\operatorname{div} f^{(2)} = 0$ everywhere. Equation (5.7) with the boundary conditions (5.8) has a unique solution for $r < r_0$ and $r > r_0$. Thus, the conditions (5.4) are satisfied. The propagation equations $\Box\bar\psi_{0\mu} = 0$ and $\Box\bar\psi = 0$ ensure fulfillment of the conditions $\psi_{0\mu} = 0$ and $\psi = 0$ off Σ.

We now carry through to the end the calculations proposed in Ref. 1. The components of the curvature tensor for the solution (5.1) are equal to

$$R_{0k0m} = -{}^1/_2 h_{00,mk}, \quad R_{0klm} = {}^1/_2(h_{0m,lk} - h_{0l,mk}). \qquad (5.9)$$

In the *TT* gauge they can be expressed in terms of h_{ik}^{TT} as follows.

$$R_{0k0m}^{TT} = -{}^1/_2 h_{km,00}^{TT}, \quad R_{0klm}^{TT} = {}^1/_2(h_{lk,0m}^{TT} - h_{km,0l}^{TT}). \qquad (5.10)$$

By virtue of the gauge invariance of the curvature tensor on the flat background, (5.9) must be identical to (5.10). From the first equation, we find

$$h_{km}{}^{TT} = {}^1/_2 h_{00,mk}x^{02} + \varphi_{mk}x^0 + f_{mk},$$

where φ_{mk} and f_{mk} are certain functions of x, y, z on which only the constraints which ensure $\Box k_{km}{}^{TT}=0$ are imposed, namely,

$$\Delta f_{mk}+h_{00,\,mk}=0,\ \Delta\varphi_{mk}=0. \tag{5.11}$$

From the components h_{km}^{TT} we find

$$R_{0klm}^{TT}={}^1/_2(\varphi_{lk,m}-\varphi_{mk,l}).$$

Comparing this expression with (5.9), we obtain the equation

$$\varphi_{lk,\,m}-\varphi_{mk,\,l}=h_{0m,\,lk}-h_{0l,\,mk},$$

which is satisfied by the choice $\varphi_{lk}=-h_{0l,k}$, and, in addition, (5.11) is also satisfied, since $\Delta h_{01}=0$. Thus, no contradictions arise in the values of R_{0klm} and R_{0klm}^{TT}.

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1) The gauge condition is derived in such a form in the Ref. 1 in connection with the equations for weak gravitational waves.

2) This means that although the count of the number of "physical" components of free fields gives the number 2, it is in the general case impossible to localize these degrees of freedom explicitly.

3) Equation (1.6) as the condition of simultaneous fulfillment of a complete set of gauge conditions for weak gravitational waves was obtained for the first time in Ref. 5.

4) The introduction of spinors in a Riemannian space requires the construction of a tetrad field. The covariant derivative here includes the spinor indices.[11]

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Regularization of the energy–momentum tensor and particle production in a strong varying gravitational field

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A generally covariant method is proposed for regularizing the vacuum expectation values of a quantized field interacting with a strong varying classical field (smoothing method). The main types of divergence are found and a simple algorithm given for calculating finite quantities for the case when an explicit expansion of the field operator with respect to quantum modes is given. The smoothing method is used to calculate the energy density and pressure of produced particles for a fermion field and a massless scalar field with minimal coupling in a Friedmann space.

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1. INTRODUCTION

Quantum field theory in a classical curved space-time is a natural first approximation to the construction of a complete quantum theory in which gravitation is also quantized. In such a quasiclassical approach, one also encounters problems which are of independent interest such as the production of particles from an initial vacuum state or vacuum polarization in spaces with non-Euclidean topology,[1-4] these effects leading to nonvanishing vacuum expectation values $\langle 0|T^{\mu\nu}|0\rangle$ of the energy–momentum tensor of the quantized field. The most important applications of these effects are to cosmology[5,6] and to black holes,[7] where one encounters strong gravitational fields that can be treated naturally as classical.

