Refs. 10 and 11, has shown that, first, in our range of angles 90–150° the Rayleigh background is almost constant, and second, its value is less than 1% of the resonant effect, in agreement with the data of Refs. 12–14. Therefore the usual procedure of taking the nonresonant background into account by using the counting rate at $v=\infty$ introduces no noticeable corrections in our experimental angular distributions.

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Vortex formation in small superconducting samples

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An expression is obtained for the critical magnetic field of a cylinder and of a cylindrical cavity inside a superconducting matrix. Possible states of a superconducting cylinder with radius $R\sim\xi(T)$, when the number of vortices in it is small, are investigated.

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1. INTRODUCTION

The interaction of vortices with the surface increases the critical field for nucleation[1] and produces a threshold for the penetration of the vortices into the sample. For bulky samples, a detailed investigation of this phenomenon raises considerable difficulties because of the large number of degrees of freedom. It is of interest therefore to consider small samples, in which the number of vortices is small and it is much simpler to obtain physical results. At the same time, bulky samples always contain defects whose interaction with the vortex lattice determines the dynamics of the current state. Therefore the investigation of various types of inclusions in superconducting materials is of particular interest.

We investigate below the oscillatory dependence of the critical magnetic vortex-nucleation field for a cylinder and a cylindrical pore in a superconducting matrix, when their dimension is of the order of the correlation length $\xi(T)$. We investigate the penetration of one and two vortices into a cylinder and show that a first-order transition between states of different type is realized in this case. An expression is also obtained for the critical field of nucleation on a defect in the form of a hollow small-radius sphere in a superconducting matrix.

The process of penetration of vortices into a cylindrical sample of radius R of the order of the penetration depth λ, at a large value of the Ginzburg-Landau parameter $\varkappa$, was considered both experimentally and theoretically.[2-5] In the case considered by us, that of a strong magnetic field, the surface effects are large and lead to a qualitative change of the vortex distribution in the sample.

2. CRITICAL FIELD OF FORMATION OF A SUPERCONDUCTING NUCLEUS ON A CYLINDRICAL CHANNEL

The presence of a surface increases the critical field for the formation of the superconducting nucleus.[1] At the same time, when the radius of the pore is changed, a discrete change takes place in the type of the solution that describes the superconducting nucleus. For a pore of radius on the order of $\xi(T)$ it is therefore necessary to expect an oscillatory dependence of the critical magnetic field on the radius of the pore. We confine ourselves below to temperatures close to critical:

$$\tau=1-T/T_c\ll 1. \qquad (1)$$

If the condition (1) is satisfied, we can write for the order parameter Δ the Ginzburg-Landau equation. In the approximation linear in Δ this equation takes the form

$$-\tau\Delta-\frac{\pi D}{8T}\left(\frac{\partial}{\partial\mathbf{r}}-2ie\mathbf{A}\right)^2\Delta=0 \tag{2}$$

with the boundary condition

$$\mathbf{n}(\partial/\partial\mathbf{r}-2ie\mathbf{A})\Delta=0, \tag{3}$$

where $\mathbf{n}$ is the direction of the normal to the surface, A is the vector potential, $D=vl_{tr}/3$ is the diffusion coefficient. For a superconductor with a small electron mean free path l_{tr} one can use Eq. (2) with the boundary condition (3) if the size of the region exceeds $(D/T)^{1/2}$. At an arbitrary electron mean free path, the coefficient D in (2) must be replaced by[6]

$$D\to D\eta,\quad \eta=1-\frac{8T\tau_{tr}}{\pi}\left[\psi\left(\frac{1}{2}+\frac{1}{4\pi T\tau_{tr}}\right)-\psi\left(\frac{1}{2}\right)\right], \tag{4}$$

where $\psi(x)$ is the psi-function.

We choose the vector potential in the form

$$\mathbf{A}(r)=[\mathbf{H}\mathbf{r}]/2. \tag{5}$$

We direct the z axis along the magnetic field and place the origin on the axis of the cylindrical cavity. We change over to the dimensionless variables

$$\rho=(eH)^{1/2}r,\quad \mu=8T\tau/\pi DeH. \tag{6}$$

We seek the solution of (2) in the form

$$\Delta(r)=\chi(\rho)\exp(in\varphi-\rho^2/2), \tag{7}$$

where φ is the azimuthal angle in the (x,y) plane, ρ is the distance from the cylinder axis to the observation point, and n is an integer. The function $\chi(\rho)$ satisfies the equation

$$-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)\chi+2\rho\frac{\partial\chi}{\partial\rho}+\frac{n^2}{\rho^2}\chi=(\mu-2+2n)\chi \tag{8}$$

and the boundary condition

$$\rho\chi-\partial\chi/\partial\rho=0\quad\text{at}\quad\rho=(eHR^2)^{1/2}, \tag{9}$$

where R is the radius of the cylindrical pore. Equation (8) has a solution that is regular at zero:

$$\chi_1=\rho^n\sum_0^\infty C_K\rho^{2K},\quad C_K=\frac{\Gamma(K+Q)\Gamma(n+1)}{\Gamma(Q)\Gamma(K+1)\Gamma(K+1+n)},\quad Q=\frac{2-\mu}{4}. \tag{10}$$

The function χ_1 increases exponentially at large values of ρ:

$$\chi_1(\rho\to\infty)=\frac{\Gamma(n+1)}{\Gamma(Q)\rho^{n+1}}\exp\left[\rho^2-\frac{\mu}{2}\ln\rho\right]. \tag{11}$$

The second linearly independent solution of (8) has a singularity at zero and can be represented in the form

$$\chi_2(\rho)=-\chi_1(\rho)\ln\rho-\sum_0^\infty\rho^{n+2K}C_KF(K)$$
$$+\frac{1}{2\rho^n}\sum_0^{n-1}\rho^{2K}\frac{\Gamma(1-Q)\Gamma(n+1)\Gamma(n-K)}{\Gamma(K+1)\Gamma(n-K+1-Q)}, \tag{12}$$

where

$$F(K)=\frac{1}{2}\sum_1^K\left(\frac{1}{K_1-1+Q}-\frac{1}{K_1}-\frac{1}{K_1+n}\right), \tag{13}$$

$\Gamma(n)$ is the gamma function. At large values of the argument, the function $\chi_2(\rho)$ also increases exponentially:

$$\chi_2(\rho\to\infty)={}^1/{}_2\chi_1(\rho)[\ln\gamma+\psi(Q)-\psi(n+1)], \tag{14}$$

where $\ln\gamma=C=0.577$ is the Euler constant.

It follows from (11) and (12) that the solution $\chi(\rho)$ of Eq. (8), with a power-law growth as $\rho\to\infty$, can be represented in the form

$$\chi(\rho)=\chi_1(\rho)-2\chi_2(\rho)/(\ln\gamma+\psi(Q)-\psi(n+1)). \tag{15}$$

The eigenvalue μ, and by the same token the critical nucleation field, is determined from the boundary condition (9).

For a small-radius pure $(eH_{c2}R^2\ll 1)$ the parameter $Q\ll 1$ and from (10) and (12) we get at $\rho\ll 1$

$$\chi^{(0)}(\rho)=1-2Q^{(0)}\ln\rho,\quad \chi^{(n)}(\rho)=\rho^n+\frac{Q^{(n)}\Gamma(n)}{\rho^n},\quad n\neq 0. \tag{16}$$

From (9) and (16) we obtain an expression for the critical nucleation field:

$$H_c^{(0)}/H_{c2}=1-eH_{c2}R^2,\quad H_c^{(n)}/H_{c2}=1+\frac{2}{\Gamma(n)}(eH_{c2}R^2)^n,$$
$$n\neq 0,\quad eH_{c2}=4T\tau/\pi D,\quad eH_{c2}R^2\ll 1. \tag{17}$$

It follows from these formulas that at $n=0$ the critical nucleation field is smaller than H_{c2}. At $n\neq 0$ the critical field is larger than H_{c2}. At a small pore radius the maximum increase of the field is reached at $n=1$.

With increasing radius of the pore, the number n of the solution, at which the maximum critical field is reached, increases. Since the number n assumes discrete values, the critical field is a nonmonotonic function of the pore radius. At $R\sim\xi(T)$ the critical field as a function of the radius can be obtained numerically from (9) and (15). The critical nucleation field with number n is expressed in terms of $Q^{(n)}$ by the formula

$$H_c^{(n)}=H_{c2}/(1-2Q^{(n)}). \tag{18}$$

The true critical field corresponds to the number n

TABLE I.

n	Q	ρ_0	n	Q	ρ_0	n	Q	ρ_0
1	0,05	0.224	4	0,05	0,948	7	0,05	1.5
		0.843			1,815			2.432
	0.08	0.296		0.1	1,045		0.1	1.573
		0.738			1.614			2.203
	0.1	0.351		0.13	1.096		0,15	1.626
		0.655			1,476			1.936
	0.11	0.387		0.15	1.147		0.16	1.647
		0,603			1,361			1.868
	0,1208	0.487 **		1.1606	1,236 **		0,1712	1.73 **
2	0.05	0.49	5	0.05	1.146	8	0.05	1,662
		1.25			2.041			2.606
	0.1	0.605		0.1	1.235		0,1	1.726
		1.067			1,83			2.369
	0.12	0,654		0.15	1.317		0,15	1.768
		0,98			1,575			2.094
	0.135	0.707		0,16	1,353		0,165	1.798
		0.895			1,501			1.987
	0.1433	0.792 **		0,1651	1,417 **		0,1733	1.87 **
3	0.05	0.731	6	0,05	1.329	9	0,05	1.814
		1.557			2.245			2,77
	0,1	0.837		0,1	1,41		0.1	1.871
		1,366			2,025			2.524
	0.13	0.899		0.15	1.476		0.15	1,903
		1,231			1.764			2.242
	0.14	0.928		0.16	1.503		0.165	1.927
		1.176			1,695			2.135
	0,154	1,032 **		0,1685	1.58 **		0,1751	2.002 **

at which $Q^{(n)}$ is a maximum. Table I lists the results of a numerical calculation of $A^{(n)}$ as a function of the dimensionless quantity ρ_0 (the value of ρ_0 at which $Q^{(n)}$ reaches the maximum value are maked in the table by a double asterisk):

$$\rho_0=(eH_{c2}R^2)^{1/2}. \tag{19}$$

We find now the behavior of the function $Q^{(n)}(R)$ at large values of n and R. We assume

$$\rho=n^{1/2}+t. \tag{20}$$

The value of interest to us is $t\sim 1$, inasmuch as the function $Q^{(n)}(R)$ decreases rapidly with increasing $|t|$. Equation (9) at large values of n reduces, when account is taken of (20), to the form

$$I(Q,t)+\frac{2^{1/2}t}{(2n)^{Q/2}}\Phi(Q)+\frac{1}{n^{1/2}}[I_1(Q,t)+I_2(Q,t)]=0, \tag{21}$$

where

$$I(Q,t)=\int_0^\infty dz\, z^{(Q-1)/2}\left(1+\frac{t}{(2z)^{1/2}}\right)\exp[-z-2^{1/2}tz^{1/2}],$$

$$I_1(Q,t)=\frac{1}{2^{1/2}}\int_0^\infty dz\, z^{Q/2-1}\left[\frac{t^2}{2}+2z^{1/2}\left(z^{1/2}+\frac{t}{2^{1/2}}\right)\right.$$

$$\left.\times\left(t^2+\frac{1}{2}-\frac{z}{3}\right)\right]\exp(-z-2^{1/2}tz^{1/2}),$$

$$I_2(Q,t)=\frac{\sin\pi Q}{2^{1/2}\pi}\int_0^\infty dz\, z^{Q/2-1}[Q+2^{1/2}tz^{1/2}-2z]\exp(-z+2^{1/2}tz^{1/2}), \tag{22}$$

$$\Phi(Q)=\sum_1^\infty\left[\frac{\Gamma(K)}{\Gamma(K+1-Q)}-\frac{1}{Q}((K+1/2)^Q-(K-1/2)^Q)\right]$$

$$-\frac{1}{2^QQ}+\frac{\sin\pi Q}{\pi}\sum_0^\infty\frac{\Gamma(K+Q)}{\Gamma(K+1)}[\psi(K+1)-\psi(K+Q)].$$

It follows from (21) that as $n\to\infty$ the quantity Q, regarded as a function of the dimensionless parameter, has a maximum. The maximum value $A=Q_0$ is reached at the point $t=-t_0$:

$$t_0=0.5432,\quad Q_0=0.20494. \tag{23}$$

Near this extremal point we can expand (21) in terms of the parameters $Q-Q_0$ and $t-t_0$ after which it reduces to

$$10.47(t+t_0)^2+17.38(Q-Q_0)+1.112n^{-1/2}=0. \tag{24}$$

From (6), (19), (30), and (24) we obtain an expression for $Q^{(n)}(R)$:

$$Q^{(n)}_{(R)}=Q_0-0.602[0{,}543-n^{1/2}+1.3(eH_{c2}R^2)^{1/2}]^2-0.064/n^{1/2}. \tag{25}$$

We note that the extremal point (Q_0, t_0) is obtained from the condition

$$I(Q_0,t_0)=0,\quad \left.\frac{\partial I}{\partial t}\right|_{(Q_0,t_0)}=0. \tag{26}$$

At $Q=Q_0$ the quantity $\Phi(Q)$ vanishes:

$$\Phi(Q_0)=0. \tag{27}$$

It follows from (25) that when the radius of the pore is increased the critical field increases in oscillatory manner and approaches the limiting value

$$H_c^{(\infty)}=H_{c2}/(1-2Q_0)=1.695\,H_{c2}, \tag{28}$$

which coincides with the third critical field for a half-space.[1] The amplitude of the oscillations decreases like R^{-2} with increasing R.

3. CRITICAL NUCLEATION FIELD ON A HOLLOW SPHERE OF SMALL RADIUS ($R\ll\xi(T)$)

We choose as before a coordinate frame with the z axis along the magnetic field. For the vector potential defined by (5), we seek the order parameter $\Delta(r)$ in the form

$$\Delta(\mathbf{r})=\Delta(\rho,z)e^{in\varphi}. \tag{29}$$

The function $\Delta(\rho,z)$ satisfies the equation

$$\left\{-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\left(\frac{n}{\rho}-\rho\right)^2-\frac{\partial}{\partial z^2}\right\}\Delta(\rho,z)=\mu\Delta(\rho,z), \tag{30}$$

where the dimensionless quantities μ and ρ are defined in (6). The boundary condition (3) for the function $\Delta(\rho,z)$ takes the form

$$\left(\rho\frac{\partial}{\partial\rho}+z\frac{\partial}{\partial z}\right)\Delta(\rho,z)=0,\quad \rho^2+z^2=\rho_0^2=eHR^2, \tag{31}$$

where R is the radius of the sphere.

The order parameter $\Delta(\rho,z)$ at $\rho>\rho_0$ can be represented in the form

$$\Delta(\rho,z)=\int\frac{dK}{2\pi}\Delta(\rho,K)e^{iKz}, \tag{32}$$

where the function $\Delta(\rho,K)$ satisfies the equation

$$\left\{-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\left(\frac{n}{\rho}-\rho\right)^2\right\}\Delta(K,\rho)=(\mu-K^2)\Delta(K,\rho). \tag{33}$$

Equation (33) has a solution that decreases at infinity, in the form

$$\Delta(K,\rho)=B(K)\chi(\rho,Q)e^{-\rho^2/2},\quad Q=(2-\mu+K^2)/4, \tag{34}$$

where the function $\Delta(\rho,Q)$ is defined by formulas (10), (12), (15), and $B(K)$ depends only K.

At $\rho_0\ll 1$, as well as in the case of a cylindrical pore, the maximum of the field is reached on a nucleus with $n=1$. The order parameter $\Delta(\rho,z)$ can then be represented in the form

$$\Delta(\rho,z)=\left[\frac{\rho}{(\rho^2+z^2)^{1/2}}\Phi_1(\rho,z)+\rho\Phi_2(\rho,z)\right]e^{-\rho^2/2}, \tag{35}$$

where the functions Φ_1 and Φ_2 are expanded in powers of ρ and z. Using the explicit form of $\chi(\rho,Q)$ and formula (35), we obtain for the order parameter $\Delta(\rho,z)$

$$\Delta(\rho,z)=Ce^{-\rho^2/2}\left[\frac{\rho}{(\rho^2+z^2)^{1/2}}+4\rho\left(2-\mu-\frac{\partial^2}{\partial z^2}\right)^{-1}\frac{\rho^2}{(\rho^2+z^2)^{3/2}}\right],\quad \rho>\rho_0. \tag{36}$$

For small values of ρ and z we have from (36)

$$\Delta(\rho,z)=C\left[\frac{\rho}{(\rho^2+z^2)^{1/2}}+\frac{4\rho}{(2-\mu)^{1/2}}\right]. \tag{37}$$

Substituting this value of Δ in the boundary condition (31), we obtain an expression for the critical field H_c:

$$2-\mu=4\rho_0^6,\quad H_c=H_{c2}[1+2(eH_{c2}R^2)^3]. \tag{38}$$

We note that the correction to the critical field is proportional to the sixth power of the radius. This apparently explains why no shift of the critical field is observed in experiment when small cavities are made in a superconducting matrix.

4. CYLINDRICAL SAMPLE IN A LONGITUDINAL MAGNETIC FIELD

We obtain first the dependence of the critical magnetic field on the radius of the cylinder. The superconducting nucleus satisfies Eq. (8) with boundary condition (9). The difference from the case of a cylindrical pore is that the function χ for a cylinder should be regular at zero. Therefore

$$\chi=\chi_1, \tag{39}$$

where the function χ_1 is defined in (10). The dependence of the critical field on the radius is obtained from the boundary condition (9). We consider first a cylinder with a small radius $R \ll \xi(T)$. In this case the most convenient is the formation of a nucleus with $n=0$. From (10) we obtain

$$\chi_1=1+Q\rho^2+Q(1+Q)\rho^4/4. \tag{40}$$

Substituting this expression in the boundary condition (9) we obtain the critical magnetic field H_c:

$$2Q=1-Q^2\rho^2, \quad eH_c=\frac{4}{R}\left(\frac{T\tau}{\pi D}\right)^{1/2}. \tag{41}$$

Just as in the case of a pore, when the cylinder radius increases an increase takes place in the number n of the solution at which the maximum critical field is obtained. The expression of the critical nucleation field numbered n in terms of $Q^{(n)}$ is given in (18). The results of the numerical calculation of $Q^{(n)}$ as a function of the dimensionless quantity ρ_0 defined in (19) are listed in Table II.

The plot of $Q^{(n)}$ vs. R has a characteristic S shape. The quantity $Q^{(n)}$ vanishes at $eH_{c2}R^2=n$ $(n\neq 0)$. At this point, the derivative $\partial Q^{(n)}/\partial R<0$ and $Q^{(n)}$ increases with decreasing radius of the cylinder to the turning point marked in Table II by a single asterisk. The cylinder radius at which $Q^{(n)}$ is a maximum is marked in Table II by two asterisks.

TABLE II.

n	Q	ρ_0	n	Q	ρ_0	n	Q	ρ_0	n	Q	ρ_0
0	0.45	0.2	3	0.05	1.69			2.413		0.2383	2.319 **
	0.4	0.401			3.068		0.2284	1.943 *	8	0.05	2.736
	0.3	0.812		0.1	1.644			2.196			4.027
	0.2	1.246			2.696		0.235	1.946		0.1	2.638
	0.15	1.483		0.15	1.598			2.136			3.605
	0.1	1.747			2.375		0.24	1.956		0.15	2.536
	0.05	2.077		0.2	1.554			2.083			3.227
1	0.05	0.985			2.061		0.2443	2.001 **		0.2	2.439
		2.497		0.2319	1.539 *	6	0.05	2.374			2.842
	0.1	0.97			1.846			3.688		0.2252	2.411 *
		2.151		0.24	1.541		0.1	2.296			2.622
	0.15	0.954			1.784			3.285		0.23	2.414
		1.863		0.25	1.557		0.15	2.214			2.571
	0.2	0.939			1.695			2.926		0.2362	2.461 **
		1.593		0.2551	1.607 **		0.2	2.136	9	0.05	2.899
	0.2392	0.933 *	4	0.05	1.946			2.566			4.181
		1.379			3.295		0.2271	2.112 *		0.1	2.793
	0.25	0.934		0.1	1.888			2.348			3.752
		1.317			2.912		0.235	2.118		0.15	2.682
	0.28	0.96		0.15	1.829			2.27			3.364
		1.12			2.577		0.2409	2.167 **		0.2	2.576
	0.2876	1.018 **		0.2	1.773	7	0.05	2.562			2.968
2	0.05	1.385			2.246			3.863		0.22	2.548
		2.808		0.2299	1.755 *		0.1	2.473			2.791
	0.1	1.353			2.03			3.45		0.2244	2.546 *
		2.449		0.24	1.761		0.15	2.381			2.747
	0.15	1.321			1.944			3.081		0.23	2.551
		2.143		0.2487	1.817 **		0.2	2.294			2.683
	0.2	1.291	5	0.05	2.171			2.709		0.2345	2.594 **
		1.848			3.5		0.22	2.27			
	0.2347	1.28 *		0.1	2.102			2.545			
		1.634			3.106		0.2261	2.267 *			
	0.25	1.285		0.15	2.031			2.489			
		1.528			2.759		0.235	2.279			
	0.2656	1.355 **		0.2	1.964			2.392			

We find now the behavior of the function $Q^{(n)}(R)$ at large values of n and R. We put

$$\rho=n^{1/2}+t, \tag{42}$$

where, as before, the significant values are $t\sim 1$. Equation (9) reduces at large n to

$$I(Q,-t)-\frac{t\,2^{1/2}}{(2n)^{Q/2}}\Phi_1(Q)-\frac{I_1(Q,-t)}{n^{1/2}}=0, \tag{43}$$

where

$$\Phi_1(Q)=\Gamma(Q)-\frac{1}{Q\,2^Q}+\sum_1^\infty\left[\frac{\Gamma(K+Q)}{\Gamma(K+1)}-\frac{1}{Q}\left(\left(K+\frac{1}{2}\right)^Q-\left(K-\frac{1}{2}\right)^Q\right)\right], \tag{44}$$

and the integrals I and I_1 are given by (21) and (22).

With increasing cylinder radius, Q reaches the extremal value Q_0 at $t=t_0$, where Q_0 and t_0 are given by (23). At the extremal point we have

$$\Phi_1(Q_0)=0. \tag{45}$$

Since the integral I_2 of (22) vanishes at the extremal point (Q_0, t_0), the equations for $Q^{(n)}$ in the cases of a cavity and a cylinder go over into each other following the substitutions $R\to -R$, $n^{1/2}\to -n^{1/2}$:

$$Q^{(n)}(R)=Q_0-0.602[0.543+n^{1/2}-1.3(eH_{c2}R^2)^{1/2}]^2+0.064/n^{1/2}. \tag{46}$$

Given the cylinder radius, the critical field is determined by the nucleus numbered n for which $Q^{(n)}$ is maximal. With increasing cylinder radius, the critical field decreases in oscillatory fashion. The amplitude of the oscillations is proportional to R^{-2}.

We consider now the penetration of vortices in a cylinder. We deal with a situation when the cylinder can contain 0, 1, or 2 vortices. To this end we consider a cylinder with radius R such that

$$|R-R_0|\ll\xi(T), \tag{47}$$

where R_0 is the solution of the equation

$$Q^{(n)}(R_0)=Q^{(n+1)}(R_0), \quad n=0,\ 1. \tag{48}$$

At $n=0$ we have

$$eH_{c2}R_0^2=0.8886, \quad Q^{(0)}(R_0)=0.269. \tag{49}$$

At $n=1$ we have

$$eH_{c2}R_0^2=1.665; \quad Q^{(1)}(R_0)=0.2545. \tag{50}$$

We consider now magnetic fields close to the critical one determined by the quantity $Q^{(n)}(R_0)$. We express the free energy in the form

$$F=\nu\int d^3\mathbf{r}\left\{-\tau|\Delta|^2+\frac{7\zeta(3)}{16\pi^2T^2}|\Delta|^4+\frac{\pi D}{8T}|\partial_-\Delta|^2\right\}+\frac{1}{8\pi}\int(H^2-2H_0H)\,d^3\mathbf{r}, \tag{51}$$

where H_0 is the external magnetic field, $\nu=mp/2\pi^2$ is the state density on the Fermi surface, and $\partial_-=\partial/\partial\mathbf{r}-2ie\mathbf{A}$. In the vicinity of the critical point (48) the order parameter Δ can be represented in the form

$$\Delta=\Delta_n(\rho)e^{in\varphi}+\Delta_{n+1}(\rho)e^{i(n+1)\varphi}. \tag{52}$$

Substituting (52) in (51), we obtain the free energy per unit cylinder length:

$$\frac{F}{\nu}=\int d^2r\left[\frac{7\zeta(3)}{16\pi^2T^2}\left[|\Delta_n|^4+|\Delta_{n+1}|^4+4|\Delta_n\Delta_{n+1}|^2\right.\right.$$
$$-\frac{2}{\varkappa^2}\left(\int_r^R dr\left[\left(\frac{n}{r}-eHr\right)|\Delta_n|^2+\left(\frac{n+1}{r}-eHr\right)|\Delta_{n+1}|^2\right]\right)^2$$
$$\left.-\frac{r^2}{\varkappa^2}\left(|\Delta_{n+1}|\frac{\partial|\Delta_n|}{\partial r}-|\Delta_n|\frac{\partial|\Delta_{n+1}|}{\partial r}\right)^2\right]+2\tau\left(\frac{1}{1-2Q^{(n)}}-\frac{H_0}{H_{c2}}\right)$$
$$\int_r^R dr\left(\frac{n}{r}-eHr\right)|\Delta_n|^2+2\tau\left(\frac{1}{1-2Q^{(n+1)}}-\frac{H_0}{H_{c2}}\right)\int_r^R dr\left(\frac{n+1}{r}-eHr\right)|\Delta_{n+1}|^2\Bigg\}. \qquad (53)$$

At $n=0$ we obtain from (53)

$$\frac{F}{\nu/eH}=\frac{16\pi^2T^2\tau^2}{7\zeta(3)}\left(2.165-\frac{H_0}{H_{c2}}\right)^2\mathcal{F}, \qquad (54)$$

where

$$\mathcal{F}=3.41\left(1-\frac{0.53}{\varkappa^2}\right)|C_0|^4+1.254\left(1-\frac{0.03}{\varkappa^2}\right)|C_1|^4$$
$$+7.293\left(1-\frac{0.102}{\varkappa^2}\right)|C_0C_1|^2-W[3.97(1-2.195z)|C_0|^2$$
$$+0.375(1+10.14z)|C_1|^2], \qquad (55)$$
$$W=\mathrm{sign}\left(2.165-\frac{H_0}{H_{c2}}\right),\qquad z=\frac{(R-R_0)(eH_{c2})^{1/2}}{2.165-H_0/H_{c2}}.$$

In (55), $\varkappa^2$ is the Ginzburg-Landau parameter:

$\varkappa^2=63\zeta(3)/2\pi^3e^2p^2v^3\tau_{tr}{}^2\eta^2$.

The coefficients C_0 and C_1 are obtained from the condition that the function $\mathcal{F}$ be a minimum. The extremum condition are satisfied by three solutions:

$$\text{a) } C_1=0,\quad |C_0|^2=W\cdot 0.583(1-2.195z)/(1-0.529/\varkappa^2),$$
$$\mathcal{F}^a=-3.41|C_0|^4\left(1-\frac{0.529}{\varkappa^2}\right). \qquad (56)$$

A solution of type a) exists in the region $\varkappa^2>0.529$:

$$z<0.456 \text{ at } W>0;\quad z>0.456 \text{ at } W<0. \qquad (57)$$

At $\varkappa^2<0.529$, a first-order transition takes place.

$$\text{b) } C_0=0,\quad |C_1|^2=W\cdot 0.15\frac{1+10.14z}{1-0.031/\varkappa^2}$$
$$\mathcal{F}^b=-1.254|C_1|^4(1-0.031/\varkappa^2). \qquad (58)$$

A solution of type b) exists in the region $\varkappa^2>0.031$:

$$z>-0.1 \text{ at } W>0,\quad z<-0.1 \text{ at } W<0; \qquad (59)$$

$$\text{c) } |C_0|^2=d^{-1}W49.61(1-0.0707/\varkappa^2)(z-z_0),$$
$$|C_1|^2=-d^{-1}W89.48(1-0.226/\varkappa^2)(z-z_1),$$
$$\mathcal{F}^c=-1.254\left(1-\frac{0.031}{\varkappa^2}\right)|C_1|^4-3.41\left(1-\frac{0.529}{\varkappa^2}\right)|C_0|^4$$
$$-7.293\left(1-\frac{0.102}{\varkappa^2}\right)|C_0C_1|^2, \qquad (60)$$

where

$$d=53.2(1-0.102/\varkappa^2)^2-17.1(1-0.529/\varkappa^2)(1-0.031/\varkappa^2),$$
$$z_0=0.146\frac{1-0.004/\varkappa^2}{1-0.0707/\varkappa^2},\qquad z_1=0.295\frac{1-0.0607/\varkappa^2}{1-0.226/\varkappa^2}. \qquad (61)$$

A solution of type c) exists only at $W>0$ in the field region defined by the condition

$$z_0<z<z_1. \qquad (62)$$

In the state of type c), the free energy has a maximum. The difference between the free energies, $\mathcal{F}^c-\mathcal{F}^a$ or $\mathcal{F}^c-\mathcal{F}^b$ is the height of the energy barrier for the transition from a metastable state ($a\rightleftharpoons b$) into another energywise more favored state. If z lies outside the region (62), then the metastable state is absolutely unstable and a barrierless transition takes place. At $W>0$ a solution of type a) gives an absolute minimum of the free energy in the region

$$z<0.456\left(\frac{1}{(1-0.529/\varkappa^2)^{1/2}}-\frac{0.156}{(1-0.031/\varkappa^2)^{1/2}}\right)$$
$$\times\left(\frac{1}{(1-0.529/\varkappa^2)^{1/2}}+\frac{0{,}721}{(1-0.031/\varkappa^2)^{1/2}}\right)^{-1}. \qquad (63)$$

A solution corresponding to a conditional extremum at a given position of the vortex relative to the cylinder axis has a free energy

$$\mathcal{F}=-\frac{[3.97(1-2.19z)+0.375t(1+10.14z)]^2}{4[3.41(1-0.519/\varkappa^2)+t^2\,1.254(1-0.031/\varkappa^2)+7.29t\left(1-\frac{0.102}{\varkappa^2}\right)]}, \qquad (64)$$

where $t=|C_1/C_0|^2$. This free energy has an extremum with respect to t at

$$t=-1.8\frac{1-0.226/\varkappa^2}{1-0.0707/\varkappa^2}\,\frac{z-z_1}{z-z_0}. \qquad (65)$$

The value of the free energy (64) at this point coincides with the free energy in the state c) and determines the value of the barrier in the transition ($a\rightleftharpoons b$). The region of the values of z at which $0<t<\infty$ is the region (62) of the existence of metastable states.

We can treat similarly the case $n=1$, when states with one or two vortices in the cylinder are possible. The free energy per unit length of the cylinder is in this case

$$\frac{F}{\nu/eH}=\frac{16\pi^2T^2\tau^2(2.037-H_0/H_{c2})^2}{7\zeta(3)}\mathcal{F}, \qquad (66)$$

where

$$\mathcal{F}=2.494\left(1-\frac{0.433}{\varkappa^2}\right)|C_1|^4+5.385\left(1-\frac{0.05}{\varkappa^2}\right)|C_2|^2$$
$$+14.04\left(1-\frac{0.117}{\varkappa^2}\right)|C_1C_2|^2$$
$$-W[4.384(1-1.396z)|C_1|^2+1.2(1+5.539z)|C_2|^2], \qquad (67)$$
$$W=\mathrm{sign}(2.037-H_0/H_{c2}),\quad z=(R-R_0)(eH_{c2})^{1/2}/(2.037-H_0/H_{c2}). \qquad (68)$$

The quantity R_0 was defined by us earlier [formula (50)].

The extremum conditions are satisfied by three solutions:

$$\text{a) } C_2=0,\quad |C_1|^2=W\cdot 0.879\frac{1-1.396z}{1-0.433/\varkappa^2},$$
$$\mathcal{F}^a=-2.494|C_1|^4(1-0.433/\varkappa^2), \qquad (69)$$

a solution of type a) exists in the region $\varkappa^2>0.433$:

$$z<0.716 \text{ at } W>0,\quad z>0.716 \text{ at } W<0; \qquad (70)$$

$$\text{b) } C_1=0,\quad |C_2|^2=W\cdot 0.111(1+5.539z)/(1-0.05/\varkappa^2),$$
$$\mathcal{F}^b=-5.385|C_2|^2(1-0.05/\varkappa^2). \qquad (71)$$

This solution exists in the region $\varkappa^2>0.05$:

$$z>-0.181 \text{ at } W>0,\quad z<-0.181 \text{ at } W<0. \qquad (72)$$

The solution of type a) is energywise more favored than the solution b) in the region

$$
\begin{gathered}
z<0.716[(1-0.433/\varkappa^2)^{-1/2}-0.185(1-0.05/\varkappa^2)^{-1/2}] \\
\times[(1-0.433/\varkappa^2)^{-1/2}+0.736(1-0.05/\varkappa^2)^{-1/2}]^{-1}; \\
\text{c)}\ |C_1|^2=d^{-1}W159.24(1-0.089/\varkappa^2)(z-z_3), \\
|C_2|^2=-d^{-1}W119.1(1-0.205/\varkappa^2)(z-z_4), \\
\mathcal{F}^c=-2.49|C_1|^4(1-0.433/\varkappa^2)-5.385|C_2|^4(1-0.05/\varkappa^2) \\
-14.04(1-0.117/\varkappa^2)|C_1C_2|^2,
\end{gathered}
\tag{73}
$$

where

$$
\begin{gathered}
d=197.15(1-0.117/\varkappa^2)^2-53.72(1-0.05/\varkappa^2)(1-0.433/\varkappa^2), \\
z_3=0.191\frac{1-0.013/\varkappa^2}{1-0.089/\varkappa^2}, \quad z_4=0.467\frac{1-0.083/\varkappa^2}{1-0.205/\varkappa^2}.
\end{gathered}
\tag{74}
$$

Just as before, a solution of type c) gives the maximum of the free energy and the difference $\mathcal{F}^c-\mathcal{F}^a$ or $\mathcal{F}^b-\mathcal{F}^c$ determines the value of the barrier on going from the metastable to the ground state. A solution of type c) exists only at $W>0$ in the region of fields defined by the inequality

$$
z_3<z<z_4. \tag{75}
$$

The free energy corresponding to the conditional extremum at a given vortex position is

$$
\mathcal{F}=-\frac{[4.384(1-1.396z)+1.2t(1+5.539z)]^2}{4[2.494(1-0.433/\varkappa^2)+5.385t^2(1-0.05/\varkappa^2)+14.04t(1-0.117/\varkappa^2)]}, \tag{76}
$$

where $t=|C_2/C_1|^2$. It has an extremum with respect to t at

$$
t=-0.748\frac{1-0.205/\varkappa^2}{1-0.089/\varkappa^2}\frac{z-z_4}{z-z_3}, \tag{77}
$$

which forms a solution of type c).

When z varies in the interval (75), the parameter t ranges from infinity to zero. In the region $W>0$ one of the states of type a) with one vortex or of type b) with two vortices is metastable.

We note that a state of type b) corresponds to two congruent vortices. In the investigated region, the spatial separation of the vortices is energywise unfavored.

CONCLUSION

The critical magnetic field of a cylinder or of a cylindrical cavity has a nonmonotonic dependence on the radius. This nonmonotonicity is due to the phase quantization that leads to a discrete set of possible types of superconducting nuclei. The oscillations of the critical magnetic field decrease rather weakly with increasing radius (in proportion to R^{-2}). Since the critical magnetic field is a function of the dimensionless parameter $(eH_{c2}R^2)^{1/2}$, and the critical field H_{c2} itself is proportional to $1-T/T_c$, a nonmonotonic temperature dependence of the critical magnetic field of the cylinder should be observed. The corresponding dependence can be easily determined on the basis of Tables I and II.

With increasing cylinder radius, the number n that sets the increment of the phase of the order parameter after one closed circuit around the origin increases. This number can be regarded as the number of vortices inside the cylinder. At the critical point itself, all n vortices are congruent and can be regarded as one vortex with a large phase increment. Away from the transition point, one or more vortices can split off and become spatially separated. In the case considered, when only states with 0, 1, or 2 vortices could be realized, no such separation took place. We note that on going from one type of solution to another when, say, the magnetic field is changed, metastable states can be produced. With further change of field the threshold is lowered and the metastable state becomes absolutely unstable with respect to the entry or emergence of vortices.

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