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Energy balance of charged particles multiply scattered in a randomly inhomogeneous magnetic field

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A consistent theory of energy exchange between high-energy charged particles and random magnetic-field inhomogeneities frozen in a moving plasma is developed. It is shown that the character of the change of the particle energy, given the plasma-velocity variation in space, is determined by the concrete form of the particle distribution function. An equation is obtained for the particle energy density, and the question of formation of the energy spectrum of the charged particles in the course of multiple scattering by random magnetic-field inhomogeneities is considered.

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1. INTRODUCTION

One of the vital problems of plasma physics, cosmic-ray physics, and plasma astrophysics is that of the motion of charged particles in a random magnetic field.[1-5] The first consistent kinetic approach to the problem of the motion of charged particles in a magnetic field with random inhomogeneities was developed by Dolginov and Toptygin.[1] They obtained a kinetic equation that describes the multiple scattering of charged particles by moving magnetic-field inhomogeneities, and established the correct form of the diffusion-approximation equations, namely the equation for the density of particles with a given momentum and the expression for the particle flux-density vector in space.

On the other hand, a phenomenological theory of propagation of charged particles in a random magnetic field was developed[6-8] in connection with problems of cosmic-ray physics. In this theory it became necessary to postulate an expression for the particle flux density in the space of the absolute values of the momentum due to the exchange of energy between the charged particles and the moving magnetic-field inhomogeneities. The dominant concept in the consideration of the process of energy dissipation in a system consisting of charged particles and magnetic inhomogeneities was the conviction that the energy-exchange mechanism is limited exclusively by the spatial character of the change of the velocity of the medium in which the random magnetic-field inhomogeneities are frozen-in. This point of view was formulated most clearly for the question of energy exchange between charged particles and magnetic inhomogeneities by Parker[8] and by Jokipii and Parker,[9] who used the hypothesis of adiabatic slowing down of the charged particles. The gist of this hypothesis is that high-energy charged particles scattered by radially moving magnetic field inhomogeneities lose energy systematically. We shall show that this is a restricted concept, since it takes no account of the character of the distribution of the charged particles, so that it is necessary to review the notions concerning energy dissipation in multiple scattering of charged particles by moving magnetic-field inhomogeneities. Actually, given the law that governs the variation of the velocity of the medium in space, the character of the change of the energy of the charged particles depends essentially on the form of the particle distribution function. If the system dimensions are large enough and the particles have time to become strongly scattered, so that their spatial distribution becomes close to isotropic, then the energy-exchange process is determined by the sign of the scalar product $(\mathbf{u}\cdot\nabla)N$ [$\mathbf{u}(\mathbf{r})$ is the velocity of the medium, and $N(\mathbf{r},p,t)$ is the density of the particles with given value of the momentum p][10] and at $(\mathbf{u}\cdot\nabla)N>0$ continuous energy transfer takes place from the moving magnetic-field inhomogeneities to the charged particles. In the

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opposite case, when $(\mathbf{u}\cdot\nabla)N<0$, the particles lose energy in the course of their multiple scattering by the random inhomogeneities of the magnetic field.

The purpose of the present paper is a consistent treatment of the problem of energy exchange between high-energy charged particles and moving magnetic-field inhomogeneities. The transport equation is used to obtain relations for the energy density of the charged particles and expressions for the energy flux density and the particle-energy source density. A law of conservation of the number of particles with specified value of the momentum is formulated, compatible with the requirements that follow from the equation for the particle-energy density. The kinetic equation is used to derive an expression for the change of the energy of an individual particle. This expression is used for an independent calculation of the energy source in the equation for the particle-charge density. Boundary conditions that illustrate the character of the change of the particle energy for concrete examples are considered for the transport equation.

2. EQUATION FOR THE ENERGY DENSITY OF CHARGED PARTICLES; PARTICLE FLUX IN THE SPACE ABSOLUTE VALUES OF THE MOMENTUM.

The consistent theory of multiple scattering of particles in a magnetic field with random inhomogeneities is based on the use of the kinetic equation for the distribution function $F(\mathbf{r},\mathbf{p},t)$ (Ref. 1)

$$\frac{\partial F}{\partial t}+(\mathbf{v}\nabla)F=\frac{\partial}{\partial p_\alpha}D_{\alpha\lambda}\frac{\partial F}{\partial p_\lambda}=\mathrm{St}\,F(\mathbf{r},\mathbf{p},t), \tag{1}$$

where $\mathbf{v}=c^2\mathbf{p}/\varepsilon$ is the velocity of the particles with momentum p, total energy ε, and charge e; the diffusion-coefficient tensor in momentum space, $D_{\alpha\lambda}(\mathbf{r},\mathbf{p})$, is defined as

$$D_{\alpha\lambda}=\frac{p^2}{2\Lambda}w\left(\delta_{\alpha\lambda}-\frac{w_\alpha w_\lambda}{w^2}\right), \tag{2}$$

where $\Lambda(\mathbf{r},\mathbf{p})$ is the transport mean free path of the particle, and its concrete expression is determined by the correlation function of the random magnetic field[11]; $\mathbf{w}=\mathbf{v}-\mathbf{u}$, where $\mathbf{u}(\mathbf{r})$ is the hydrodynamic velocity with which the random inhomogeneities of the magnetic field that are frozen in the plasma are frozen in the plasma are transported.

If the particles are intensively scattered by random inhomogeneities of the magnetic field, so that their direction distribution becomes close to isotropic, it is convenient to change over to the diffusion approximation, i.e., to describe the evolution of the particle distribution with the aid of the moments of the distribution function

$$N(\mathbf{r},p,t)=\int d\Omega\,F(\mathbf{r},\mathbf{p},t), \tag{3}$$

$$\mathbf{J}(\mathbf{r},p,t)=\int d\Omega\,\mathbf{v}F(\mathbf{r},\mathbf{p},t), \tag{4}$$

where $N(\mathbf{r},p,t)$ is the density of particles with given value of the momentum, $\mathbf{J}(r,p,t)$ is the vector of the particle-flux density in space, and the integration in (3) and (4) is carried out over the solid-angle element $d\Omega$ in momentum space.

Using the definitions (3) and (4), we can obtain from (1) the transport equation for the particle density $N(\mathbf{r},p,t)$ (Ref. 1):

$$\partial N/\partial t=\nabla_\alpha\varkappa_{\alpha\lambda}\nabla_\lambda N-(\mathbf{u}\nabla)N+(\nabla\mathbf{u})p\partial N/3\partial p \tag{5}$$

and the expression for the vector of the particle flux density in space:

$$J_\alpha=-\varkappa_{\alpha\lambda}\nabla_\lambda N-u_\alpha p\partial N/3\partial p, \tag{6}$$

where $\varkappa_{\alpha\lambda}(\mathbf{r},\mathbf{p})$ is the tensor of the particle diffusion coefficients in space.

The equation for the charge-particle energy density

$$E(\mathbf{r},t)=\int d\mathbf{p}\,\varepsilon(\mathbf{p})F(\mathbf{r},\mathbf{p},t)=\int_0^\infty dp\,p^2\varepsilon(p)N(\mathbf{r},p,t) \tag{7}$$

corresponding to the transport equation (5) is of the form[10]

$$\frac{\partial E}{\partial t}+\nabla\mathbf{q}={}^1/_3\int_0^\infty dp\,p^3v(\mathbf{u}\nabla)N\equiv Q(\mathbf{r},t), \tag{8}$$

where

$$\mathbf{q}(\mathbf{r},t)=\int d\mathbf{p}\,\mathbf{v}\varepsilon(\mathbf{p})F(\mathbf{r},\mathbf{p},t)=\int_0^\infty dp\,p^2\varepsilon(p)\mathbf{J}(\mathbf{r},p,t) \tag{9}$$

is the vector of the charge-particle energy flux density, and is equal to the amount of energy carried through a unit area per unit time. As seen from (9), the presence of the energy flux is due to the presence of a particle flux $\mathbf{J}(\mathbf{r},p,t)$ through the surface of the considered volume.

Equation (8) is of the form of a continuity equation with with a source in the right-hand side. According to (8), the change in the particle energy density is due to the presence of an energy flux from the given volume, as well as to the exchange of energy between the charged particles and the moving inhomogeneities of the magnetic field. If the scalar product $(\mathbf{u}\cdot\nabla)N>0$, then $Q(\mathbf{r},t)$ represents the energy acquired by the particles in a unit volume per unit volume per unit time as they interact with the moving inhomogeneities of the magnetic field. In the opposite case, when $(\mathbf{u}\cdot\nabla)N<0$, the particles give up energy to the inhomogeneities of the magnetic field. Thus, the character of the energy exchange between the charged particles and the magnetic inhomogeneities, at a given law of variation of the velocity of the medium in space, is determined by the character of the particle distribution function. In particular, for galactic cosmic rays (whose radial gradient is positive), with radial outflow of the solar wind, the plasma dissipates energy continuously on the high-energy charged particles—cosmic rays—as they are repeatedly scattered by the random magnetic-field inhomogeneities that are frozen in the moving plasma. When cosmic rays of solar origin (which have a negative radial gradient) propagate in the solar wind, the charged particles at each point of space give up energy to the moving inhomogeneities of the magnetic field. A similar conclusion concerning the character of the energy exchange between the charged particles in the magnetic inhomogeneities follows directly from the transport equation (5). We write down the transport equation (5) in a form corresponding to the law of conservation of the number of particles with given momentum[10]

$$\frac{\partial N}{\partial t}+\nabla \mathbf{J}+\frac{1}{p^2}\frac{\partial}{\partial p}p^2 J_p=0, \tag{10}$$

where

$$J_p(\mathbf{r},\ p,\ t)={}^1/_3 p(\mathbf{u}\nabla)N \tag{11}$$

is the particle-flux density in the space of the absolute values of the momentum, i.e., the number, per unit volume, of particles whose absolute value of the momentum, when changed by collision with the inhomogeneities of the magnetic field, goes through a given value in a unit time. We note that the operator

$$\frac{1}{p^2}\frac{\partial}{\partial p}p^2$$

is that part of the divergence operator in momentum space which depends on the modulus of the momentum.

In accordance with Eq. (10), the change in the number of particles with a given momentum per unit volume and per unit time is determined both by the presence of the particle flux **J** and by the presence of particle "migration" over the energy spectrum, due to the presence of the flux J_p. The flux J_p is in this case positive if the particle energy increases, and negative if it decreases. As follows from (11), at $(\mathbf{u}\cdot\nabla)N>0$ we have $J_p>0$, i.e., the particles are shifted in the spectrum towards higher momenta p, so that the particle energy increases in accordance with the conclusion that follows from Eq. (8) for the charged-particle energy density.

In the phenomenological theory of the motion of charged particles in a random magnetic field the transport equation (5) was postulated on the basis of the concept of systematic energy loss by the particles as they interact with the radially moving inhomogeneities of the magnetic field. An incorrect expression was postulated in this case for the vector of the particle flux density in space[6-9]

$$J_\alpha=-\varkappa_{\alpha\lambda}\nabla_\lambda N+u_\alpha N, \tag{12}$$

and the particle flux in the space of the absolute values of the momenta was defined as

$$J_p=\langle dp/dt\rangle N, \tag{13}$$

where $\langle dp/dt\rangle$ was taken to mean the average change of the particle momentum per unit time and was calculated by invoking intuitive reasoning that makes use of the concept of systematic loss of particle energy. The expression used for $\langle dp/dt\rangle$

$$\langle dp/dt\rangle=-{}^1/_3 p(\nabla\mathbf{u}) \tag{14}$$

is based on the analogy between the motion of particles in a medium with radially moving magnetic-field inhomogeneities, on the one hand, and the adiabatic decrease of the particle energy upon expansion of a given volume occupied by gas, on the other.

However, the analogy between the scattering of particles by moving magnetic-field inhomogeneities and the expansion of a given volume occupied by gas is incorrect. This is seen even from the fact that if the plasma velocity $\mathbf{u}(\mathbf{r})$ in an immobile coordinate frame is constant in time, then the average distance between the magnetic inhomogeneities remains unchanged.

Thus, even though the erroneous premises (12)–(14) yielded a perfectly correct form of the transport equation, the conclusion that the particles lose energy systematically when they become scattered by radially moving magnetic-field inhomogeneity (the adiabatic deceleration concept) is incorrect.

We note that a perfectly correct form of the transport equation, in which the fluxes of the particles in ordinary space **J** (Ref. 1) and in momentum space J_p (Ref. 10) have forms determined by relations (6) and (11), was obtained also in Ref. 12.

3. CHANGE OF ENERGY OF INDIVIDUAL PARTICLE AND CALCULATION OF SOURCE ENERGY ON THE BASIS OF THE KINETIC EQUATION

The average change of the energy of an individual particle per unit time can be calculated by using directly the equations of motion of the particle in a random magnetic field. In our case, when the expression for the diffusion coefficient in momentum space (2) is known, there is no need to resort to the equations of motion, since all the necessary information on the change of the energy of an individual particle is contained in the expression for $D_{\alpha\lambda}$. In fact, by definition

$$D_{\alpha\lambda}={}^1/_2\langle\Delta p_\alpha\Delta p_\lambda/\Delta t\rangle, \tag{15}$$

where $\Delta\mathbf{p}$ is the momentum change of a particle scattered by the moving magnetic-field inhomogeneity in a time t, and the angle brackets denote averaging over the statistical ensemble corresponding to the random magnetic field. The average rate of change of the particle momentum is given by the relation

$$\left\langle\frac{\Delta p_\alpha}{\Delta t}\right\rangle=\frac{\partial D_{\alpha\lambda}}{\partial p_\lambda}, \tag{16}$$

and the average rate of change of the particle energy in the course of multiple scattering by moving magnetic-field inhomogeneities is calculated with the aid of (15) and (16) as follows:

$$\left\langle\frac{\Delta\varepsilon(\mathbf{p})}{\Delta t}\right\rangle=\left\langle\frac{\Delta p_\alpha}{\Delta t}\right\rangle v_\alpha+{}^1/_2\left\langle\frac{\Delta p_\alpha\,\Delta p_\lambda}{\Delta t}\right\rangle\frac{\partial^2\varepsilon(\mathbf{p})}{\partial p_\alpha\,\partial p_\lambda}=\frac{\partial}{\partial p_\lambda}(v_\alpha D_{\alpha\lambda}). \tag{17}$$

The tensor $D_{\alpha\lambda}$ is given, accurate to terms of order $u^2/v^2\ll 1$, by the relation

$$D_{\alpha\lambda}=a(\delta_{\alpha\lambda}-g_\alpha g_\lambda)+b\{g_\alpha\eta_\lambda+g_\lambda\eta_\alpha-(\mathbf{g}\boldsymbol{\eta})(\delta_{\alpha\lambda}-g_\alpha g_\lambda)\}$$
$$+\rho\{\delta_{\alpha\lambda}+g_\alpha g_\lambda-(\mathbf{g}\boldsymbol{\eta})^2(\delta_{\alpha\lambda}+3g_\alpha g_\lambda)-2(\mathbf{g}\boldsymbol{\eta})(g_\alpha\eta_\lambda+g_\lambda\eta_\alpha)-2\eta_\alpha\eta_\lambda\}, \tag{18}$$

where a, b, and ρ are certain scalar functions of the momentum p

$$a=\frac{v\rho^2}{2\Lambda},\quad b=a\frac{u}{v},\quad \rho=\frac{b^2}{2a},\quad \mathbf{g}=\frac{\mathbf{v}}{v},\quad \boldsymbol{\eta}=\frac{\mathbf{u}}{u}. \tag{19}$$

For the average rate of change of the energy we obtain from (17)

$$\langle\Delta\varepsilon/\Delta t\rangle=(p/\Lambda)\{-(\mathbf{u}\mathbf{v})+[1-(\mathbf{g}\boldsymbol{\eta})^2]u^2\}, \tag{20}$$

from which it follows that the rate of change of the energy of an individual particle is determined by the relative directions of the vectors **u** and **v**. Consequently the particle acquires energy in the case of head-on collisions and loses energy in rear collisions. This result is natural: formula (20) is simply a logical consequence, for the considered case, of the known relation on which the Fermi acceleration mechanism is based, and the change of energy of the individual particle is determined exclusively by the relative change of the vectors u and v and does not depend on the concrete character of the change of the velocity of the medium in space [in contrast to relation (14) used in the formulation of the adiabatic deceleration hypothesis]. Even the very structure of (20) indicates that energy exchange between the charged particles and the magnetic inhomogeneities cannot be independent of the form of the particle distribution function, and this dependence should naturally manifest itself in the behavior of the macroscopic quantities (the energy flux, the energy source) that characterize the energy-exchange process.

To demonstrate the circumstance, we carry out an independent calculation of the source of the particle energy in Eq. (8), starting directly from Eq. (20), which determines the change of energy of an individual particle. In accordance with the definition of the particle-energy source $Q(\mathbf{r},t)$ as the amount of energy acquired (given up) per unit volume and per unit time by the charged particles as they interact with the moving magnetic-field inhomogeneities, we can write

$$Q(\mathbf{r},t)=\int d\mathbf{p}\,\langle\Delta\varepsilon(\mathbf{p})/\Delta t\rangle F(\mathbf{r},\mathbf{p},t). \tag{21}$$

If the particle distribution is close to isotropic, then, confining ourselves in the expansion of $F(\mathbf{r},\mathbf{p},t)$ in spherical harmonics to the first moments (3) and (4) of the distribution function, we obtain from (21)

$$Q(\mathbf{r},t)=\int dp p^3 v(\mathbf{u}\nabla)N(\mathbf{r},p,t), \tag{22}$$

i.e., we arrive at the same expression for the particle-energy source as in Eq. (8) for the energy density. In analogy with the definition (4) for the particle flux density vector in the space J_α (6), we can connect with the distribution function $F(\mathbf{r},\mathbf{p},t)$ also the particle flux in the space of the absolute values of the momentum (11). From (1) it follows that

$$\Pi_\alpha=-D_{\alpha\lambda}\partial F/\partial p_\lambda \tag{23}$$

is the particle-flux density vector in momentum space. The particle flux in the space of the absolute values of the momentum (11) is connected with Π by the obvious relation

$$J_p=\int d\Omega g_\alpha \Pi_\alpha(\mathbf{r},\mathbf{p},t). \tag{24}$$

In the weak-anisotropy approximation, when two terms of the expansion in spherical harmonics are sufficient, we obtain from (24)

$$J_p=\sigma(\eta\nabla)N, \tag{25}$$

in which the coefficient σ is expressed in terms of the tensor $D_{\alpha\lambda}$ by the relation

$$\sigma=vp\int d\Omega\eta_\alpha g_\lambda D_{\alpha\lambda}\Big/\int d\Omega(\delta_{\alpha\lambda}-g_\alpha g_\lambda)D_{\alpha\lambda}. \tag{26}$$

If we use in (26) the concrete form of $D_{\alpha\lambda}$, we obtain $\sigma=pu/3$ and, consequently, an expression that coincides with (11) for J_p (25).

A similar calculation for the particle-flux density vector in space J, on the basis of the definition (4), leads to the general expression for J

$$\mathbf{J}(\mathbf{r},p,t)=-\varkappa\nabla N-\sigma\eta\partial N/\partial p, \tag{27}$$

in which σ is determined from relation (26), and the diffusion coefficient $\varkappa$ is expressed in terms of the tensor $D_{\alpha\lambda}$ by the formula

$$\varkappa={}^4/_3\pi(vp)^2\left\{\int d\Omega(\delta_{\alpha\lambda}-g_\alpha g_\lambda)D_{\alpha\lambda}\right\}^{-1}. \tag{28}$$

It is seen therefore from the results presented in the present section that the conclusion that the energy exchange between charged particles and moving magnetic-field inhomogeneities depends on the form of the distribution function, a conclusion based on macroscopic equations (8) and (10), is in full agreement with the "microscopic" definitions of the macroscopic quantities (energy source, particle flux J_p) which are a direct consequence of the kinetic theory.

4. STATIONARY CASE. GENERAL RELATIONS

Before we proceed to investigate the character of the deformation of the space-energy distribution of the particles, we consider certain general consequences of the transport equation in the stationary case. The transport equation (5) corresponds to the law of particle-number conservation per unit volume:

$$\frac{\partial n}{\partial t}+\nabla\mathbf{I}=0, \tag{29}$$

where

$$n(\mathbf{r},t)=\int d\mathbf{p}F(\mathbf{r},\mathbf{p},t)=\int_0^\infty dp p^2 N(\mathbf{r},p,t) \tag{30}$$

and

$$\mathbf{I}(\mathbf{r},t)=\int d\mathbf{p}\mathbf{v}F(\mathbf{r},\mathbf{p},t)=\int_0^\infty dp p^2\mathbf{J}(\mathbf{r},p,t) \tag{31}$$

are respectively the particle density and the density vector of the flux of particles with all energies.

In the stationary case $\mathbf{I}(r)=0$, meaning the absence of a flux of particles with all energies. On the other hand, in the stationary case ($\partial E/\partial t=0$) integration of the equation for the energy density (8) leads to the relation

$$\oint dSq_n(\mathbf{r})={}^1/_3\int d\mathbf{r}(\mathbf{u}\nabla)\int_0^\infty dp p^3 vN(\mathbf{r},p,t) \tag{32}$$

(q_n is the q component normal to the surface); this relation shows that at a constant number of particles in a given volume and at a particle-energy density constant in time there exists in the system the stationary energy flux (32). Thus, in the steady state, the entire energy transferred to the particles in a unit volume per unit time through scattering by moving magnetic-field inhomogeneities is carried out of the volume because of the energy flux. Integrating the transport equation (10) over the given volume, we obtain the relation

$$\oint dSJ_n(\mathbf{r},p)=-{}^1/_3\int d\mathbf{r}(\mathbf{u}\nabla)\frac{\partial}{\partial p}(p^3N), \tag{33}$$

from which it follows that in the stationary case there exists in the system a particle flux $\mathbf{J}(\mathbf{r}, p)$ with a given value of the momentum. To establish the direction of this flux, we note that from the condition that the energy density of the charged particles (7) be finite it follows that at large values of the momentum p the density $N(\mathbf{r}, p)$ of particles with a given momentum should decrease with increasing p more rapidly than p^{-4}, and it should decrease more slowly than p^{-3} at small values of the momentum p. It follows thus from (33) that at a positive radial gradient $[(\mathbf{u}\cdot\nabla)N>0]$ the flux of the energetic particles is directed out of the system, and the flux of the low-energy particles is directed into the system (we note that at $N\propto p^{-3}$ the particle flux $\mathbf{J}=0$). This result is a direct consequence of the acceleration of the particles, whose distribution is characterized by a positive radial gradient. Indeed, in the stationary case at $(\mathbf{u}\cdot\nabla)N>0$ there exists an energy flux (connected with the particle flux $\mathbf{J}$ [see (9)], which is directed out of the system, and the flux of particles with all energies is equal to zero $[\mathbf{I}(\mathbf{r})=0]$. This is possible only when the flux of the low-energy particles is directed into the system, and the flux of the high-energy particles out of the system, thus ensuring the presence of an energy flux in this direction. We note that the experimental consequence of the presence of the flux of energy of cosmic rays in the stationary case can be the observation of different signs of the radial component of the diurnal variation for particles of high and low energy.

5. ENERGY SPECTRUM OF CHARGED PARTICLES

The notions developed concerning the character of the energy exchange between charged particles and moving magnetic-field inhomogeneities make it possible to examine the dynamics of formation of the energy spectrum of the charged particles in the course of multiple scattering. We consider the particle propagation in a spherically symmetrical region filled by a magnetized plasma that moves radially with constant velocity $\mathbf{u} = u\mathbf{r}/r$, in which random inhomogeneities of the magnetic field are frozen-in. At the boundary $r=r_0$ of the volume, the plasma velocity $\mathbf{u}=0$, and the scattering properties of the medium are characterized by a diffusion coefficient $\varkappa$, assumed to be constant. The propagation of the particles in such a model is described by the stationary transport equation (5)

$$\frac{1}{x}\frac{\partial}{\partial x}x^2\frac{\partial N}{\partial x}-\mu\frac{\partial N}{\partial x}+\frac{2}{3}\frac{\mu}{x}p\frac{\partial N}{\partial p}=0 \tag{34}$$

with boundary conditions on a sphere of radius $x=1$

$$N(x=1,\ p)=N_0(p), \tag{35}$$

where $x=r/r_0$ is the dimensionless coordinate and $\mu = ur_0/\varkappa$.

The boundary-value problem (34), (35) was considered for different forms of the limiting septcrum $N_0(p)$ in Refs. 1, 8, 13, 14. We shall not describe the detailed solution, which can be found in Ref. 14, and confine ourselves to a brief discussion of the end results. The numerical calculations of the energy spectrum of cosmic rays show that the experimental data agree best with the dependence of the limiting spectrum on the dimensionless momentum $\zeta=p/mc$, in the form

$$N_0(\zeta)=A\zeta^{-1}(1+\zeta^2)^{-(\gamma-1)/2}, \tag{36}$$

where

$$A=(\gamma-3)(mc)^{-3}n_0, \tag{37}$$

γ is the exponent of the spectrum, and n_0 is the density of the particles with all energies at $x=1$.

The figure shows a plot of the function

$$f(\zeta)=\frac{N(z,p)}{n(z)}p^2\frac{dp}{d\zeta}, \tag{38}$$

$$z=ur/\varkappa, \tag{39}$$

which constitutes the ratio, normalized to unity, of the number of particles in the interval from ζ to $\zeta+d\zeta$ and the total number of particles $n(z)$ per unit volume, as a function of the dimensionless momentum ζ for values $z=\mu=3$ (curve 1), $z=\mu/2$ (curve 2), and $z=0$ (curve 3). The parameter is $\varkappa=3$ for $u=4\times10^7$ cm/sec, $r_0=1.5\times10^{15}$ cm, and $\varkappa=2\times10^{22}$ cm²/sec. It is seen from the figure that at a constant diffusion coefficient $\varkappa$ the energy spectrum of the charged particles is deformed in such a way that the maximum of the spectrum shifts towards low particle energies on going towards the center of the region filled with the plasma. However, this character of the space-energy distribution of the particles is not proof of slowing down of the high-energy particles and cannot be treated as a decrease of the energy of an individual particle in the course of multiple scattering by radially moving magnetic-field inhomogeneities. In the considered stationary case the particle energy source $Q>0$ and the energy is transferred from the moving magnetic inhomogeneities to the charged particles. The entire energy acquired by the particles in the given volume flows out through the surface of the volume. The formation of the energy spectrum corresponds fully in this case to the general picture of stationary modulation, considered in the preceding section. The flux of the particles of all energies is $\mathbf{I}=0$ and the particle flux density vector $\mathbf{J}(\mathbf{r}, p)$ for the high-energy particles is directed outward from the system, while for the low energy particles, to the contrary, it is directed towards the center of the system. Accordingly, the resultant energy flux is always directed out of the system and increases in absolute value with increasing z.[14] On the other hand, the considered behavior of the energy spectrum, which manifests itself in a shift of the maximum of the spectrum towards

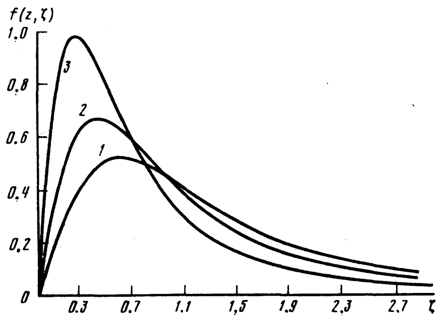

FIG. 1.

lower energies, is not universal, and is typical only for a particle-propagation model with a constant diffusion coefficient. Goldstein, Fisk, and Ramaty[15] calculated numerically the energy spectrum of the cosmic rays near the earth's orbit from a spectrum given on the boundary of the modulation region, using a propagation model with a diffusion coefficient in the form of a monotonically increasing function of the particle momentum. The maximum of the energy spectrum turned out to be shifted towards higher energies, so that the internal regions of the volume occupied by the plasma turn out to be enriched with high-energy particles.

At the present time there are no analytic methods of solving boundary-value problems for the transport equation with a momentum-dependent diffusion coefficient. For large values of the particle momentum, however, it is possible to obtain an approximate solution of the problem by using the smallness of the parameter $\mu=ur_0/\varkappa$. If the parameter $\mu\ll 1$, then it is convenient to seek the solution of the boundary-value problem (34), (35) in the form of an expansion in powers of μ. Confining ourselves to terms of order μ^2,

$$N(x, p)=N_0(p)+\mu N_1(x, p)+\mu^2 N_2(x, p), \tag{40}$$

we obtain from (34)

$$N_1(x,p)=(1-x)\frac{p}{3}\frac{\partial N_0}{\partial p}, \tag{41}$$

$$N_2(x,p)=(1-x^2)\frac{p}{18}\frac{\partial N_0}{\partial p}+(2-3x+x^2)\frac{p}{27}\frac{\partial}{\partial p}p\frac{\partial N_0}{\partial p}. \tag{42}$$

We consider now the distance dependence of the average energy per particle:

$$\langle\varepsilon(x)\rangle=E(x)/n(x)=\int_0^\infty dpp^2\varepsilon N(x,p)\Big/\int_0^\infty dpp^2N(x,p). \tag{43}$$

The quantity $\langle\varepsilon(x)\rangle$ calculated on the basis of (40)–(42) takes the form

$$\langle\varepsilon(x)\rangle=\langle\varepsilon(1)\rangle\left\{1-\frac{1-x}{3n_0}\int_0^\infty dpp^3\left(1-\frac{\varepsilon(p)}{\langle\varepsilon(1)\rangle}\right)\mu(p)\frac{\partial N_0}{\partial p}\right\}. \tag{44}$$

If the diffusion coefficient is independent of the momentum, then

$$\langle\varepsilon(x)\rangle=\langle\varepsilon(1)\rangle-\frac{1-x}{3n_0}\mu\int_0^\infty dpp^3vN_0(p), \tag{44a}$$

from which we see that the average energy per particle decreases with increasing distance from the boundary of the modulation region, i.e., the maximum of the energy spectrum is shifted towards lower particle energies.

If the limiting spectrum $N_0(p)$ is defined as in (36), and the diffusion coefficient $\varkappa(p)$ increases with increasing particle momentum p rapidly enough, then the average energy $\langle\varepsilon(x)\rangle$ (44) is an increasing function of the distance x, i.e., the spectrum of the particles within the system is enriched with high-energy particles.

If the particle diffusion coefficient $\varkappa$ is independent of the momentum, the flux density of particles with a specified momentum is given by

$$J(x,p)=\frac{\varkappa\mu^2}{27r_0}x\frac{1}{p^2}\frac{\partial}{\partial p}p^4\frac{\partial N_0}{\partial p}, \tag{45}$$

from which it is seen that the flux of the high-energy particles is directed away from the system, while the low-energy particles flow into the system at zero flux $I(x)$ of particles with all energies. Thus, in the steady state, low-energy particles enter continuously the plasma-filled region, while the high-energy particles leave this region, so that a stationary charged-particle flux is present in the system (at a particle energy density that is constant with time), and the presence of this stationary flux is a direct consequence of the energy transfer from the moving magnetic-field inhomogeneities to the charged particles. The character of the dependence of the average energy $\langle\varepsilon(x)\rangle$ on the distance at constant $\varkappa$ is determined in the stationary state not by the adiabatic deceleration, but by the specific features of the convective particle outflow, whose intensity depends on the particle energy.

To clarify the dynamics of formation of the energy spectrum of the particles and the character of the change of the average energy per particle, it is necessary to consider the time evolution of the particle distribution function. The initial distribution is assumed to be the particle spectrum at the boundary of the plasma-filled region. The particle density $N(x,p,t)$ will vary and approach a stationary distribution, satisfying thereby the nonstationary transport equation at $t<\infty$

$$\frac{\partial N}{\partial\tau}=\frac{1}{x^2}\frac{\partial}{\partial x}x^2\frac{\partial N}{\partial x}-\mu\frac{\partial N}{\partial x}+\frac{2}{3}\frac{\mu}{x}p\frac{\partial N}{\partial p},\quad \tau=\frac{\varkappa t}{r_0^2}, \tag{46}$$

the boundary

$$N_1(1, p, \tau)=N_0(p),\ N(0, p, \tau)<\infty \tag{47}$$

and the initial condition

$$N(x, p, 0)=N_0(p). \tag{48}$$

We seek the solution of the problem (46)–(48) in the form

$$N(x, p, \tau)=N_0(p)+\mu N_1(x, p, \tau). \tag{49}$$

For the function N_1 we obtain the equation

$$\frac{\partial N_1}{\partial\tau}=\frac{1}{x^2}\frac{\partial}{\partial x}x^2\frac{\partial N_1}{\partial x}+\frac{2}{3}\frac{\mu}{x}p\frac{\partial N_0}{\partial p}, \tag{50}$$

with boundary conditions

$$N_1(1, p, \tau)=0,\ N_1(0, p, \tau)<\infty \tag{51}$$

and initial condition

$$N_1(x, p, 0)=0. \tag{52}$$

The solution of Eq. (50) with the additional conditions (51) and (52) is of the form

$$N_1(x,p,\tau)=\varphi(x,\tau)\frac{p}{3}\frac{\partial N_0}{\partial p}, \tag{53}$$

$$\varphi(x,\tau)=1-x-4\sum_{n=1}^\infty\frac{1-\cos\pi n}{\pi^3n^3}\frac{\sin\pi nx}{x}\exp(-\pi^2n^2\tau). \tag{54}$$

The value of the energy density

$$E(x,\tau)={}^1/_3\varphi(x,\tau)\int_0^\infty dpp^3\varepsilon\frac{\partial N_0}{\partial p} \tag{55}$$

corresponding to the solution (54) decreases with time, from the initial value to the value $E(x, \tau \to \infty)$ that characterizes the steady state. A similar change takes place in the energy per particle:

$$\langle \varepsilon(x,\tau)\rangle = \langle \varepsilon(1)\rangle - \frac{\mu}{3}\frac{\varphi(x,\tau)}{n_0}\int_0^\infty dpp^3vN_0, \tag{56}$$

which decreases from the initial value $\langle\varepsilon(1)\rangle$ to its stationary value (44a). To determine the causes of the decrease of $\langle\varepsilon(x,\tau)\rangle$ with time we calculate the particle flux

$$J_1(x,\tau) = {}^4/_3 up\frac{\partial N_0}{\partial p}\sum_{n=1}^\infty \frac{1-\cos\pi n}{\pi^3n^3}(\pi nx\cos\pi n - \sin\pi nx)\exp(-\pi^2n^2\tau). \tag{57}$$

In the case of a decreasing particle spectrum ($\partial N_0/\partial p < 0$) the flux of particles of any energy is positive, and since the of the high-energy particles with given momentum decreases more rapidly, the particle flux normalized to the function $N_0(p)$ exceeds at large values of the momentum the flux of particles with small momentum. Consequently, the particles leave the system on account of the convective outflow, which turns out to be more effective at high energies. The presence of the particle flux (57) leads to a decrease of the density $n(x,\tau)$ of particles with any energy, as well as of the energy density of the particles $E(x,\tau)$. The decrease with time of the average energy per particle ($\partial\langle\varepsilon(x,t)\rangle/\partial t < 0$) is not a criterion of the energy loss by the particles when they are scattered by the radially moving magnetic-field inhomogeneities. To illustrate the last statement, we consider the case when the charged-particle spectrum inside the plasma-filled region is poor in high-energy particles at the initial instant of time. The initial condition for the transport equation (46) with boundary condition (47), corresponding to this requirement, is of the form

$$N(x,p,0) = N_0(p) + \xi\mu p(1-x)\partial N_0/3\partial p, \tag{58}$$

where $\xi < 1$.

The solution of the boundary-value problem (46), (47) with the initial condition (48), in the approximation linear in μ, is of the form

$$N_1(x,p,\tau) = \varphi_\xi(x,\tau)p\partial N_0/3\partial p, \tag{59}$$

$$\varphi_\xi(x,\tau) = 1-x+4(\xi-1)\sum_{n=1}^\infty\frac{1-\cos\pi n}{\pi^3n^3}\frac{\sin\pi nx}{x}\exp(-\pi^2n^2\tau), \tag{60}$$

which coincides with (53) at $\xi = 0$.

The particle energy density at $\xi < 1$

$$E(x,\tau) = {}^1/_3\varphi_\xi(x,\tau)\int_0^\infty dpp^3\varepsilon\partial N_0/\partial p \tag{61}$$

increases with time and approaches a stationary value. The average energy per particle

$$\langle\varepsilon(x,\tau)\rangle = \langle\varepsilon(1)\rangle - \frac{\mu}{3}\frac{\varphi_\xi(x,\tau)}{n_0}\int_0^\infty dpp^3vN_0, \tag{62}$$

also increases with time at $\xi > 1$.

Consequently, under the initial condition (58), the spectrum inside the plasma-filled region becomes enriched with high-energy particles. In the approximation linear in μ, this effect is due to the presence of a particle flux in a direction that depends on the particle energy:

$$J_\xi = (1-\xi)J_1. \tag{63}$$

At $\xi > 1$ the flux is negative, i.e., the particles are "drawn" into the system, and the relative number of high-energy particles entering the system exceeds the number of low-energy particles. As a result, the average energy per particle increases with time. Therefore in this case, too, the character of the variation $\langle\varepsilon(x,\tau)\rangle(\partial\langle\varepsilon(x,\tau)\rangle/\partial\tau > 0)$ with time is not a consequence of an acquisition of energy by the particles, but is due to the presence of the energy-dependent particle flux in space.

To conclude this section, we present one illustrative example which shows that the spatial deformation of the energy spectrum of the particles does lead, by itself, to any conclusions concerning the character of energy exchange between the particles and the moving inhomogeneities of the magnetic field. We consider the interaction of particles with magnetic inhomogeneities frozen into a flux of an imcompressible plasma whose velocity field satisfies the condition $\nabla\cdot\mathbf{u} = 0$. In the spherically symmetrical case we have $u \propto r^{-2}$ and the boundary-value problem for the stationary transport equation (5) takes the form

$$\frac{1}{x^2}\frac{\partial}{\partial x}x^2\frac{\partial N}{\partial x} - \frac{\mu}{x^2}\frac{\partial N}{\partial x} = 0 \tag{64}$$

with the boundary condition

$$N(1,p) = N_0(p). \tag{65}$$

The solution of the boundary-value problem (64), (65)

$$N(x,p) = N_0(p)\exp[\mu(1-x^{-1})] \tag{66}$$

shows that the form of the energy spectrum of the particles is the same at all points of space, in accordance with the fact that in this case ($\nabla\cdot\mathbf{u} = 0$) the transport equation does not contain the derivative $\partial N/\partial p$.

A feature of the modulation process in this case is that, the conserved form of the spectrum, notwithstanding, energy is continuously transferred from the moving inhomogeneities to the charged particles. As seen from (66), the particle distribution is characterized by a positive radial gradient. This corresponds to "migration" of the particles over the spectrum with increasing momentum and to the presence of a positive source Q in Eq. (8) for the particle energy density. The particle flux

$$J(x,p) = -\frac{u}{3x^2}\exp[\mu(1-x^{-1})]\frac{1}{p^2}\frac{\partial}{\partial p}(p^3N_0(p)) \tag{67}$$

is directed out of the system in the case of high-energy particles and into the system for low-energy particles (see Sec. 4). If a spectrum defined by relation (36) is specified on the boundary of the region, then the particle flux (67) is positive at $p > \sqrt{2}(\gamma-3)^{1/2}mc$ and negative at $p < \sqrt{2}(\gamma-3)^{-1/2}mc$. The particle energy flux corresponding to (67), namely

$$q = (u/3x^2)\exp[\mu(1-x^{-1})] \tag{68}$$

is directed out of this system in accordance with the general premises that determine the energy balance of the particles in the stationary case (Sec. 4).

A feature of the modulation of the space-energy distribution of the particles in this case is that despite the preserved form of the energy spectrum, continuous energy transfer takes place from the moving inhomogeneities to the charged particles. The similarity, characteristic of this case ($\nabla \cdot \mathbf{u} = 0$), of the form of the energy spectrum of the particles in space is the consequence of the balance of the particle flux **J** in the space and the "transfer" of the particles over the J_p system.

6. CONCLUSION

Our analysis shows that the energy exchange between charged particles and moving magnetic-field inhomogeneities, at a given law of variation of the velocity of the medium in space, is determined by the concrete form of the particle distribution function. The considered illustrative typical boundary-value problems show that the character of the change of the charged-particle energy spectrum is not a criterion that determines the change of the charged-particle energy in multiple scattering by moving magnetic-field inhomogeneities. In the general case, only an analysis of the system of the moment equations can lead to definite conclusions concerning the energy dissipation in a system consisting of charged particles and magnetic inhomogeneities.

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Plasma hydrodynamics in a high-current channel

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The equilibrium configurations and low-frequency radial oscillations (ion sound) of intense charged-particle beams are considered in the approximation of the hydrodynamics of perfect electron and ion liquids interacting with each other via the electromagnetic field produced by the charges. The employed macroscopic approach is valid without any assumptions concerning the equations of state. The oscillation singularities due to the specifics of the equation of state of the medium are expressed in terms one macroscopic parameter, the speed of sound. In the intermediate low-current region, the ion-sound oscillations with wavelength exceeding a certain critical value increase exponentially. This instability is due to the tendency of the beam to split up into individual jets if the magnetic field of the current flowing through an individual channel is capable of preventing the charges from spreading radially. In a high-current beam, the instability of buildup of radial ion-sound oscillations is suppressed by the magnetic field of the current.

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1. INTRODUCTION

Research into self-compressing streams of charged particles, initiated by Bennet[1] and revived by Budker,[2] has become recently a separate branch of the physics of non-neutral plasma[3] and of strong electron-ion beams.[4] A far from complete list of the projects in which high-current devices are used—from suggestions aimed at solving the problem of controlled thermonuclear fusion[5] to the development of x-ray and gamma-ray lasers[6] and of collective accelerators for charged particles[7]—gives an idea of the degree of influence of this trend on the development of modern physics.

The study of pinch systems (see, e.g., Refs. 8–19) has shown that the most interesting experimental phenomena occur during the strong compression stage (x-ray flash, high temperature, density and multiplicity of atom ionization, acceleration of electrons and ions, the appearance of neutrons, the explosive character of the electron emission). A non-traditional approach to the pinching phenomenon,[20] based on an analysis of the equilibrium[21–24] and radiation[25,26] of a plasma in the magnetic field of the current itself, makes it possible

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