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Quantum statistics of multimode lasing and noise in intracavity laser spectroscopy

V. R. Mironenko and V. I. Yudson

Spectroscopy Institute, USSR Academy of Sciences
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The statistics of a multimode laser with equidistant modes is investigated during all stages of lasing evolution. The distribution functions of the photon numbers in individual modes and of the total photon number are obtained. The distribution of the photons in an individual mode agrees in all lasing stages with the thermal distribution, but the time of correlation of the number of photons in the laser mode is much longer than the corresponding time for thermal emission. The total photon distribution function is obtained also in the dynamic stage of the lasing and in the stationary regime. The limitations of the intracavity laser spectroscopy method, due to the statistical character of the multimode laser radiation, are considered. The sensitivity threshold of this method is obtained.

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INTRODUCTION

In a number of situations connected with the operation or application of a multimode laser, the description of the laser radiation in terms of the mean values of the photon occupation numbers or fields is insufficient. In these cases it is necessary to take into account the statistical properties of the laser radiation. Thus, e.g., the onset of an individual ultrashort pulse in the multimode radiation is not a determined process and requires a probabilistic description. This, as is well known, is due to the statistical character of the radiation during the initial stage of lasing.[1] Allowance for the statistics of multimode radiation is important also in the method of intracavity laser spectroscopy (ILS). The fluctuations of the photon occupation numbers about the mean values produce noise (shot noise due to the emission and absorption of the photons), which will be shown below to limit the capabilities of this method.

In the first example, an important role in the description of the system is played by the phase relations between the fields of the individual modes. In the photon-occupation-number representation this means allowance for the off-diagonal elements of the density matrix. In the second case, however, interest attaches usually to the spectroscopic aspect of the problem, i.e., to the distribution of the photons over the laser modes. To describe the statistical properties of the field, it suffices here to know the photon distribution function, i.e., the diagonal part of the density matrix of the photon sub-system. Since the spectrum is usually recorded over times that are long compared with the duration of the initial (linear) lasing stage, it is of interest to investigate the properties of the photon distribution function not only during the initial lasing stage, but also during all the succeeding ones. There is at present no such published analysis of the statistics of multimode laser emission. From among the papers closest in scope, we note that of Letokhov *et al.*[2,3] who obtained, for a special model of a laser with identical modes, control equations for the distribution functions of the number of photons in an individual mode and of the total number of photons, and obtained their stationary solutions.

In the present paper we consider a model of laser with unlike modes and with an inhomogeneous gain contour, a model more appropriate for standard multimode lasers. Inasmuch as in this laser the inversion depends only on the total number N of the photons in the resonator, the gain saturation leads to narrowing of only the distribution N, whereas the photon numbers in the individual modes turn out to have a broad Rayleigh (thermal) distribution. The absence of factors that stabilize the number of photons in an individual mode causes the fluctuations of the number of photons to be long-lived, and the correlation time to be long compared with the case of a single-mode laser. The results of the investigation can be used to analyze the role of fluctuations in ILS. The threshold of the sensitivity of the ILS method is obtained.

§1. CONTROL EQUATION FOR THE PHOTON SUBSYSTEM AND ITS SOLUTION FOR THE INITIAL LASING STAGE

1. We consider the interaction of two-level atoms of an active medium with an assembly of laser modes. We assume that the luminescence-line contour of the medium is homogeneous and the transverse relaxation time $T_2 \ll T_1$, where T_1 is the longitudinal-relaxation

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time. Letokhov[2] formulated for this model of the active medium, using the approach developed in Ref. 4 for the single-mode case, an equation for the density matrix of the system "two-level atoms +multimode radiation." With account taken of the off-diagonal elements of the density matrix in first order in the small parameter $(T_2/T_1)^{1/2}$, and neglecting the effects of spatial inhomogeneity of the laser field, Letokhov[2] obtained for the diagonal elements of the density matrix a control equation that generalizes the single-mode equation of Ref. 4 to include the case of many modes with identical frequencies. We write down this equation, which is the starting point for our investigation, in a form that is suitable for the case of unlike modes.

$$\dot{P}_m(\mathbf{n},t)=\sum_l k_l(n_l+1)[(m+1)P_{m+1}(\mathbf{n}+1_l)-(M-m)P_m(\mathbf{n})]$$
$$+\sum_l k_l n_l[(M-m+1)P_{m-1}(\mathbf{n}-1_l)-mP_m(\mathbf{n})]+\sum_l \gamma_l[(n_l+1)P_m(\mathbf{n}+1_l)$$
$$-n_lP_m(\mathbf{n})]+S(M-m+1)P_{m-1}(\mathbf{n})-S(M-m)P_m(\mathbf{n})$$
$$+\mathcal{P}(m+1)P_{m+1}(\mathbf{n})-m\mathcal{P}P_m(\mathbf{n}). \tag{1}$$

Here $P_m(\mathbf{n})$ describes the probability of finding the system in a state with m atoms in the lower level and $M-m$ atoms in the upper level and with a distribution $\mathbf{n}=\{n_l\}$ of the photon numbers in the modes. Here l is the index of the mode, $\mathcal{P}$ is the rate of transition of the atoms from the lower level b to the upper level a as a result of the pumping, S is the rate of relaxation of the upper level, and γ_l is the rate of photon damping in the l-th mode. The quantity k_l characterizes the interaction of the atom with the l-th mode of frequency ω_l:

$$k_l=\frac{4\pi\omega_0}{\hbar T_2}\langle(\mu e^l)^2\rangle[(\omega_l-\omega_0)^2+T_2^{-2}]^{-1},\quad l=0,\pm1,\ldots, \tag{2}$$

where ω_0 is the frequency of the atomic transition, μ is the dipole-moment matrix element, $\mathbf{e}^l$ is the polarization vector of the l-th mode; the averaging in (2) is over the positions of the atoms and over the orientations of μ. For the equidistant-mode system of interest to us, Eq. (2) can be represented in the form

$$k_l=k\varkappa_l=k[1+\pi^2l^2/L_0^2]^{-1},\quad l=0,\pm1,\ldots, \tag{3}$$

where L_0 is the number of modes contained in the luminescence line: $L_0=\pi/(T_2\Delta\omega)$; $\Delta\omega$ is the frequency interval between modes.

2. We now obtain a closed control equation for the photon subsystem, eliminating the variable m from (1). We introduce, following Refs. 2 and 4, the conditional mean values

$$\eta_a=\frac{1}{MP(\mathbf{n},t)}\sum_m(M-m)P_m(\mathbf{n},t);\quad \eta_b=\frac{1}{MP(\mathbf{n},t)}\sum_m mP_m(\mathbf{n},t), \tag{4}$$

where $P(\mathbf{n},t)=\sum_m P_m(\mathbf{n},t)$. We assume that the rate $1/T_1\sim\mathcal{P}+S$ of the reaction of the atomic subsystem is large compared with the relaxation rate of the photon subsystem: $1/T_1\gg\gamma_l$. Thanks to this adiabaticity condition, the atomic subsystem manages to attune itself to the photon subsystem, so that the values of η_a and at η_b need be expressed only in terms of the photon variables $\mathbf{n}$, regardless of the considered instant of time t. For an approximate equation for the distribution function of the atoms over the levels, at a fixed value of the photon variables n, can be obtained by replacing in (1) the terms of the type $(n_l\pm1)P_m(\mathbf{n}\pm1_l)$ and $n_lP_m(\mathbf{n})$; this replacement was used in Refs. 2 and 4. Making this replacement and using the fact that the numbers m and $M-m$ are large, we obtain a Fokker-Planck equation in the variable m:

$$\dot{P}_m(\mathbf{n},t)=-\frac{\partial}{\partial m}\{[M(S+kN)-m(\mathcal{P}+S+2kN)]P_m(\mathbf{n})$$
$$-\frac{\partial}{\partial m}[(S+kN)(M-m)P_m(\mathbf{n})]\Big\}, \tag{5}$$

which contains the photon variables only in the form of the parameter N:

$$N=\sum_l\varkappa_l n_l. \tag{6}$$

If the main contribution to the sum (6) is made by modes with $l\ll L_0$, then N is approximately equal to the total number of photons $\sum_l n_l$. In the general case there is no such equality, but for brevity we shall call the quantity N the total number of photons.

The adiabaticity condition allows us to solve Eq. (5) neglecting the term $\dot{P}_m(\mathbf{n})$. The solution takes the form

$$P_m(\mathbf{n},t)=\frac{1}{\sqrt{2\pi}\sigma_m}\exp\left\{-\frac{(m-\bar{m})^2}{2\sigma_m^2}\right\}P(\mathbf{n},t), \tag{7}$$

where

$$\bar{m}=M\frac{S+kN}{\mathcal{P}+S+2kN};\quad \sigma_m^2=2M\frac{(S+kN)(\mathcal{P}+kN)}{(S+\mathcal{P}+2kN)^2}.$$

We now derive the equation of interest to us for the photon-subsystem distribution function $P(\mathbf{n},t)$. To this end we sum Eq. (1) over m, and calculate the conditional mean values (4) obtained by summation with the aid of the obtained distribution (7). As a result we obtain the equation

$$\dot{P}(\mathbf{n},t)=\sum_l\{[Mk_l(n_l+1)\eta_b(N)P(\mathbf{n}+1_l)-Mk_l\eta_a(N)(n_l+1)P(\mathbf{n})$$
$$+\gamma_l(n_l+1)P(\mathbf{n}+1_l)]-[Mk_ln_l\eta_b(N)P(\mathbf{n})$$
$$-Mk_ln_l\eta_a(N)P(\mathbf{n}-1_l)+\gamma_ln_lP(\mathbf{n})]\}, \tag{8}$$

where

$$\eta_a(N)=\frac{\mathcal{P}+kN}{\mathcal{P}+S+2kN},\quad \eta_b(N)=\frac{S+kN}{\mathcal{P}+S+2kN}. \tag{9}$$

Expressions (9) coincide formally with those obtained by Letokhov[2] (we have neglected the terms $kL_0\ll\mathcal{P}+S$), but the quantity N determined by formula (6) has generally speaking a different meaning.

Equation (8) is fundamental for the subsequent analysis of the statistics of multimode laser emission. We make now two remarks concerning its derivation.

Allowance for the terms discarded in (5) when the substitution $(n_l\pm1)\to n_l$ is made introduces in Eq. (8), corrections that correspond to fluctuations of the number of excited atoms of the active medium and lead to an additional contribution to the noise. The influence of this noise on the statistical properties of the radiation in individual modes is negligibly small compared with the action of the photon shot noise, but when the fluctuations of the total number of photons are considered in the case of a large excess above the lasing threshold, both types of noise are important. In order not to clutter up the analysis that follows, we confine ourselves to the case of a small excess above threshold ($\zeta - 1 \ll 1$), when Eq. (8) is sufficiently accurate.

The second remark likewise pertians to the applicability of the initial equation (1). It is derived by discarding from the off-diagonal elements of the density matrix a number of terms that oscillate with frequencies of the intermode beats and describe combined interaction of the modes. This approach is adequate for the balance approximation widely used in dynamic theory of multimode lasers and in particular in the ILS theory. The purpose of the present paper is to obtain a statistical description of the laser system precisely within the framework of this approximation. We note also that in the case of small excess above the threshold the influence of combination processes is small.

3. We are interested in the development of the lasing, starting with the instant when the laser is turned on. The initial condition for Eq. (8), when the problem is so formulated, is

$$P(\mathbf{n}, t=0) = \delta_{\mathbf{n},0}. \quad (10)$$

As already noted, during the initial lasing stage, when N is small compared with its stationary mean value $\bar{N}_\infty$, it is possible to neglect in (8) the dependence of η_a and η_b on N. The solution of Eq. (8) with initial condition (10) then takes the form

$$P(\mathbf{n}, t) = \prod_l P_l(n_l, t), \quad (11)$$

$$P_l(n_l, t) = [\bar{n}_l(t)]^{n_l}[1+\bar{n}_l(t)]^{-(1+n_l)}, \quad (12)$$

where $\bar{n}_l(t)$ is defined by the equation

$$\dot{\bar{n}}_l(t) = a_l(0)\bar{n}_l(t) + b_l \quad (13)$$

with initial condition $\bar{n}_l(0) = 0$. In Eq. (13) we introduced the gain of the l-th mode (with allowance for the losses)

$$a_l(N) = k_l M[\eta_a(N) - \eta_b(N)] - \gamma_l \quad (14)$$

and the mean value of the spontaneous noise

$$b_l = k_l M \eta_a(N).$$

According to (11), during the initial (linear) lasing stage the modes are independent. The distribution of the number of photons in an individual mode coincides with the thermal distribution. The effective "temperatures" depend here on the time and are different for the different modes.

The linear lasing stage is restricted by the condition $\bar{N} \ll \bar{N}_\infty$, i.e., to times $t \lesssim t_{sat}$. For t_{sat} we obtain with the aid of (3) and (13) the relation

$$[a_0(0)t_{sat}]^{-1/2} \exp[a_0(0)t_{sat}] \sim (\mathscr{P}+S)(\zeta-1)^{3/2}(kL_0)^{-1}, \quad (15)$$

where $\zeta = kM(\mathscr{P} - S)[\gamma(\mathscr{P}+S)]^{-1}$ is the parameter of the excess above threshold. For numerical estimates we assume the following values of the system parameters: $L_0 \sim 10^5$; $\gamma \sim 10^7$ sec^{-1}; $k \sim 0.1$ sec^{-1}; $\zeta - 1 \sim 0.2$; $a_0(0) \sim (\zeta - 1)\gamma \sim 0.2\gamma$; $1/T_1 \sim (\mathscr{P}+S) \sim 10^8$ sec^{-1}. For t_{sat} we obtain from (15) the estimate $t_{sat} \sim 5\times10^{-6}$ sec. For the total number of photons in the stationary regime we shall obtain below the estimate $\bar{N}_\infty \sim 10^8$. Using the smallness of the parameter $(\zeta - 1)$, we neglect the dependence of b_l on N, assuming in the estimates $b_l \approx \gamma$.

At times $t \gtrsim t_{sat}$ it is necessary to take into account the saturation of the gain.

§ 2. THE FOKKER-PLANCK AND LANGEVIN EQUATIONS. LASING DYNAMICS

1. Shortly after the start of the lasing ($t \gtrsim [a_0(0)]^{-1}$), the average number $\bar{n}_l$ of the photons in the modes becomes large: $n_l \gg 1$. This enables us to change from the difference equation (8) to the Fokker-Planck differential equation. Using the expansion

$$\dot{P}(\mathbf{n}, t) = \sum_l \{[\ldots]_{\mathbf{n}} - [\ldots]_{\mathbf{n}-1_l}\} = \sum_l \left\{\frac{\partial}{\partial n_l}[\ldots] - \frac{1}{2}\frac{\partial^2}{\partial n_l^2}[\ldots] + \ldots\right\},$$

Neglecting in the curly brackets the derivatives of $[\ldots]$ of order higher than the first, which are small because of the inequality $\bar{n}_l \gg 1$, and expanding the expression in the square bracket[1)] in terms of the derivative with respect to n_l, we get the equation

$$\dot{P}(\mathbf{n}, t) = -\sum_l \frac{\partial}{\partial n_l}\left[(a_l(N)n_l + b_l)P(\mathbf{n}, t) - \frac{\partial}{\partial n_l}(b_l n_l P(\mathbf{n}, t))\right]. \quad (16)$$

Since the numbers n_l become large even during the linear stage of the lasing (at $t \sim t_{sat}$, $\bar{n}_l \sim \bar{N}_\infty/L_0 \sim 10^3$), the region of applicability of Eq. (16) overlaps the region in which the solution (11), (12) obtained above is valid. At $\bar{n}_l \gg 1$ Eqs. (11) and (12) take the form

$$P(\mathbf{n}, t) = \prod_l P_l(n_l, t) = \prod_l \frac{1}{\bar{n}_l(t)} \exp[-n_l/\bar{n}_l(t)] \quad (17)$$

and determine also the solutions of Eq. (16) during the linear stage $t \lesssim t_{sat}$. Using expression (17) for $P(\mathbf{n}, t)$, we find that the distribution of N during the initial stage is Gaussian with a variance $\sigma^2 \sim \bar{N}^2/L_0$; $\bar{N} = \sum_l \varkappa_l \bar{n}_l$.

At $t \gtrsim t_{sat}$ the dependence of the gains a_l on N becomes significant.

2. It is convenient to proceed with the analysis by using the system of Langevin equations, which can be obtained by a standard procedure (see, e.g., Ref. 5) from the Fokker-Planck equation (16)

$$\dot{n}_l(t) = a_l[N(t)]n_l(t) + b_l + F_l(\mathbf{n}, t). \quad (18)$$

The random forces F_l in (18) have a Gaussian character, $\langle F_l \rangle = 0$, and

$$\langle F_l(\mathbf{n}, t)F_{l'}(\mathbf{n}, t')\rangle = 2b_l n_l \delta_{ll'}\delta(t-t'). \quad (19)$$

The initial condition for the system (18) is[2)] $n_l(0) = 0$. The solutions of (18) constitute a set of trajectories in the configuration space of the system. Each of these trajectories starts out from the origin and corresponds to a concrete realization of the random forces $\{F_l(t)\}$.

3. We investigate first the dynamics of the lasing, i.e., the motion of the system along the trajectory $\{\bar{n}_l(t)\}$ corresponding to zero random forces $\{F_l(t) \equiv 0\}$. This trajectory satisfies the usual system of rate equations

$$\dot{\bar{n}}_l(t) = a_l[\bar{N}(t)]\bar{n}_l(t) + b_l. \quad (20)$$

During the linear stage of lasing, the equation (20) coincides with (13); the solution in this stage is of the form

$$\bar{n}_l(t) = b_l[\exp(a_l(0)t) - 1]/a_l(0). \quad (21)$$

We now describe the known[6,7] stationary solution of the system (20). The stationary value $\bar{N}_\infty$ is determined from the approximate equality $a_0(\bar{N}_\infty) = -b_0/\bar{n}(\infty) \approx 0$:

$$\bar{N}_\infty = \frac{\mathscr{P}+S}{2k}(\zeta-1). \quad (22)$$

At the chosen values of the laser parameters we get $\overline{N}_\infty \approx 10^8$. The stationary-lasing spectrum has (at $\gamma_l \equiv \gamma$) a Lorentz shape

$$\bar{n}_l(\infty) = \bar{n}_0(\infty)\,[1 + \pi^2 l^2 \bar{n}_0(\infty)/L_0^2]^{-1}. \tag{23}$$

The value of $n_0(\infty)$ is obtained from the equality $\overline{N}_\infty = \Sigma \varkappa_l \bar{n}_l(\infty)$; $\bar{n}_0(\infty) = (\overline{N}_\infty/L_0)^2 \sim 10^6$. The width of the stationary lasing spectrum

$$L_\infty = [L_0^2/\bar{n}_0(\infty)]^{1/2} = L_0^2/\overline{N}_\infty \tag{24}$$

is equal to $L_\infty \approx 10^2$. The time of establishment of stationary lasing is of the order of $t_\infty \sim |a_0(\overline{N}_\infty)|^{-1} \sim n_0(\infty)/\gamma \sim 0.1$ sec.

During the intermediate lasing stage, bounded by the conditions $\bar{n}_l \gg 1$ and $t \ll t_\infty$, the term b_l is small compared with the first term in the right hand side of (20). Neglecting b_l, we obtain for $n_l(t)$ the expression

$$\bar{n}_l(t) = \bar{n}_l(t_0)\, U_l(t, t_0), \tag{25a}$$

where

$$U_l(t, t_0) = \exp\{\varphi(t, t_0) - (1 - \varkappa_l)[(t - t_0)\gamma + \varphi(t, t_0)] - (t - t_0)\delta\gamma_l\} \tag{25b}$$

and

$$\varphi(t, t_0) = \int_{t_0}^{t} a_0[\overline{N}(t')]\,dt'. \tag{26}$$

In the derivation of these formulas we used an identity that follows from (3) and (14):

$$a_l = a_0 - (1 - \varkappa_l)(a_0 + \gamma) - \delta\gamma_l; \quad \delta\gamma_l \equiv \gamma_l - \gamma; \quad \gamma_0 \equiv \gamma.$$

We choose t_0 to be an instant of time in the linear stage of the lasing, $t_0 \le t_{sat}$, so that we can substitute (21) for $\bar{n}_l(t_0)$ in (25a). Neglecting at $\bar{n}_l(t_0) \gg 1$ the number 1 in (21), we obtain for $n_l(t)$:

$$\bar{n}_l(t) = (b_l/a_l(0)) \exp\{\bar{\varphi}(t, 0) - (1 - \varkappa_l)[\gamma t - \bar{\varphi}(t, 0)] - \delta\gamma_l t\}. \tag{27}$$

We are interested in times $t_{sat} \le t \ll t_\infty$. The function $\bar{\varphi}(t, 0)$, and consequently also $\overline{N}(t)$, is determined by the self-consistency equation (we assume that the selective losses $\delta\gamma_l$ are relatively small and neglect the corresponding changes in $\bar{\varphi}$ and $\overline{N}$)

$$\overline{N}(t) = \sum_l \varkappa_l \bar{n}_l(t) = e^{\bar{\varphi}(t,0)} \frac{b_0 L_0}{a_0(0)\sqrt{\gamma t \pi}} [1 + O(1/t)]. \tag{28}$$

In the derivation of (28) we used the assumption $\bar{\varphi}(t, 0) \ll \gamma t$, which will be subsequently verified. Using (26) and (28), we obtain the asymptotic form of $\bar{\varphi}(t, 0)$:

$$\varphi(t, 0) = \ln\left[\frac{\overline{N}_\infty a_0(0)}{L_0 b_0}\right] + {}^1/_2 \ln(\gamma t \pi) + \chi(t), \tag{29}$$

where $\chi(t) \to 0$ at $t \ge t_{sat}$. Substituting (29) in (27) we obtain at $t_{sat} \le t \le t_\infty$:

$$\bar{n}_l(t) = \sqrt{\pi\gamma t}\, \frac{\overline{N}_\infty}{L_0} \exp\left\{ -\frac{\pi^2 l^2}{[L(t)]^2} - \delta\gamma_l t \right\}, \tag{30}$$

where

$$L(t) = L_0/\sqrt{\gamma t}. \tag{31}$$

We similarly obtain from (28)

$$\delta\overline{N}(t) \equiv \overline{N}(t) - \overline{N}_\infty = [2a_0'(\infty)t]^{-1}, \tag{32}$$

where

$$a_0'(t) = \left[\frac{\partial a_0(N)}{\partial N}\right]_{N = \overline{N}(t)}. \tag{33}$$

For $a_0'(t)$ at $t \ge t_{sat}$ we have the estimate $|a_0'(t)| \sim a_0(0)/\overline{N}_\infty$.

It follows from (30) and (31) that the lasing spectrum in the absence of selective losses has a Gaussian profile with a width that decreases like $1/\sqrt{\gamma t}$ (this behavior of the multimode lasing spectrum profile was established by Ambartsumian, Kryukov and Letokhov[8]). It is seen from (30) that the intensity at the maximum of the lasing spectrum increases like $\sqrt{\gamma t}$, thus approximately ensuring constancy of $\overline{N}(t)$. The difference $\delta\overline{N}(t)$ tends to zero like $1/t$ (see the figure).

The foregoing analysis of the laser dynamics allows us to proceed to the investigation of the statistics of the multimode emission during the nonlinear stage.

§3. THE FUNCTION $P(\mathbf{n}, t)$ IN THE DYNAMIC APPROXIMATION

1. We choose in the linear stage an arbitrary instant of time t_0 ($t_0 < t_{sat}$) satisfying the condition $\bar{n}_0(t_0) \gg 1$. Then the function $P(\mathbf{n}, t_0)$ is determined by Eq. (17). In the present section the further evolution of the function $P(\mathbf{n}, t)$ is considered with the aid of the system (18) in the dynamic approximation, i.e., neglecting during the stage (t_0, t) the random forces in (18).[3)] In the following sections we investigate the corrections to this approximation and the restrictions on the range of its validity.

In the dynamic approximation, the function $P(\mathbf{n}, t)$ is connected with the function $P(\mathbf{n}, t_0)$ in the following manner:

$$P(\mathbf{n}, t) = \int \left\{ \prod_l \delta[n_l - n_l^{d}(t|t_0, \mathbf{n}^0)] \right\} P(\mathbf{n}^0, t_0) \prod_l dn_l^0, \tag{34}$$

where the determined functions $n_l^{d}(t|t_0, \mathbf{n}^0)$ satisfy the system (18) and the initial conditions $n_l^{d}(t_0|t_0, \mathbf{n}^0) = n_l^0$. We emphasize that in contrast to the situation dealt with in the analysis of the lasing dynamics in Sec. 2, we must now find the solutions of the dynamic problem for arbitrary initial conditions.

2. To determine the functions $n_l^{d}(t|t_0, \mathbf{n}^0)$ we use the fact that the variance of the quantity N^0, in accordance with the expression (17), is relatively small: $\sigma(t_0)/\overline{N}^0 \sim L_0^{-1/2} \ll 1$. Therefore, despite the large scatter of the initial values n_l^0, it turns out that for the trajectories that make the largest contribution to the integral (34) the $N^{d}(t|t_0, \mathbf{n}^0)$ curves pass on the (N, t) plane close to the curve $N = N^{d}(t|t_0, \mathbf{n}^0) \equiv \overline{N}(t)$ investigated in Item 3 of Sec. 2 (see the figure). The functions $n^{d}(t|t_0, \mathbf{n}^0)$ are of the form[4)]

$$n_l^{d}(t|t_0, \mathbf{n}^0) = n_l^0 \exp\{\varphi(t, t_0) - (1 - \varkappa_l)[(t - t_0)\gamma + \varphi(t, t_0)] - (t - t_0)\delta\gamma_l\}. \tag{35}$$

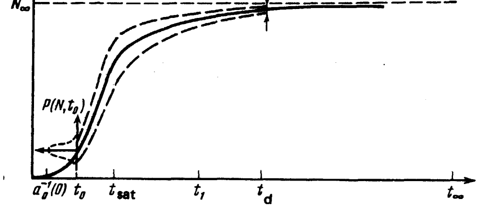

FIG. 1. Plots of $\overline{N}(t)$ (solid line) and plots of $N^{d}(t\,|\,t_0, \mathbf{n}^0)$ (dashed lines).

Here

$$\varphi(t,t_0)=\overline{\varphi}(t,t_0)+\delta\varphi(t,t_0)=\int_{t_0}^{t} a_0[\overline{N}(t')+\Delta N^d(t'|t_0,\mathbf{n}^0)]dt', \tag{36}$$

where $\Delta N^d=N^d-\overline{N}$, and $\overline{\varphi}(t,t_0)$ is determined by (26). Using the relative smallness of $\Delta N^d(t\,|t_0,\mathbf{n}^0)$, we obtain from (36) an approximate expression for $\delta\varphi(t,t_0)$:

$$\delta\varphi(t,t_0)=\int_{t_0}^{t} a_0'(t')\Delta N^d(t'|t_0,\mathbf{n}^0)dt'. \tag{37}$$

We neglect in the square brackets of the argument of the exponential in (35) the term $\delta\varphi\ll\overline{\varphi}\ll\gamma t$. Taking (25b) and (29) into account, we get

$$n_l^d(t|t_0,\mathbf{n}^0)=n_l^0U_l(t,t_0)\exp[\delta\varphi(t,t_0)]. \tag{38}$$

Using (37) and (38), we obtain a differential equation for $\delta\varphi(t,t_0)$:

$$N^d(t|t_0,\mathbf{n}^0)=\overline{N}(t)+\frac{1}{a_0'(t)}\frac{\partial}{\partial t}[\delta\varphi(t,t_0)]=\sum_l \varkappa_l n_l^0U_l(t,t_0)\exp[\delta\varphi(t,t_0)]. \tag{39}$$

Obtaining the solution of this equation with the initial condition $\delta\varphi(t_0,t_0)=0$ that follows from (37), and substituting this solution in (38), we obtain the function $n_l^d(t\,|t_0,\mathbf{n}^0)$ in explicit form

$$n_l^d(t|t_0,\mathbf{n}^0)=n_l^0U_l(t,t_0)\left\{1-\sum_l[n_l^0-\bar{n}_l(t_0)]D_l(t,t_0)\right\}^{-1}, \tag{40}$$

where

$$D_l(t,t_0)=\int_{t_0}^{t} a_0'(t')\varkappa_lU_l(t',t_0)\exp\left[\int_{t'}^{t} a_0'(t'')\overline{N}(t'')dt''\right]dt'. \tag{41}$$

3. We substitute expression (17) for $P(\mathbf{n},t_0)$ and the obtained functions n_l^d in (34). Next, introducing auxiliary integration with respect to z and the corresponding δ function, we integrate with respect to $\{n_l^0\}$:

$$P(\mathbf{n},t)=\int dz\,\delta\left\{z-1+\sum_l D_l(t,t_0)[zn_lU_l^{-1}(t,t_0)-\bar{n}_l(t_0)]\right\} \tag{42}$$

$$\times\prod_l\frac{z}{U_l(t,t_0)\bar{n}_l(t_0)}\exp[-zn_l/U_l(t,t_0)\bar{n}_l(t_0)]. \tag{43}$$

Integrating now in (42) with respect to z and using (25), we obtain an expression for $P(\mathbf{n},t)$

$$z(t,\mathbf{n})=\left\{1+\sum_l\varkappa_ln_l\int_{t_0}^{t} a_0'(t')U_l^{-1}(t,t')\right.$$

$$\left.\times\exp\left[\int_{t'}^{t} a_0'(t'')\overline{N}(t'')dt''\right]dt'\right\}^{-1}\exp\left[\int_{t_0}^{t} a_0'(t')\overline{N}(t')dt'\right]. \tag{44}$$

Equation (44) depends formally on the choice of the instant of time t_0. Actually, however, with the choice $t_0<t_{sat}$ (when, properly speaking, the simple form (17) of $P(\mathbf{n},t_0)$ can be used), there is no dependence on t_0, since $\overline{N}(t_0)a_0'(t_0)\approx 0$.

We now obtain the distribution function $P(N,t)$ of the total number of photons. It is more convenient here to start from the integral representation (34) of the function $P(\mathbf{n},t)$. Using (34) and (40) we obtain for $P(N,t)$ the following chain of equations:

$$P(N,t)=\int d\mathbf{n}\,\delta\left[N-\sum_l\varkappa_ln_l\right]P(\mathbf{n},t)=\int d\mathbf{n}^0\left\{1-\sum_l D_l(t,t_0)\right.$$

$$\left.\times[n_l^0-\bar{n}_l(t_0)]\right\}\delta\left\{N-\overline{N}(t)-\sum_l G_l(t,t_0)[n_l^0-\bar{n}_l(t_0)]\right\}P(\mathbf{n}^0,t_0), \tag{45}$$

$$G_l(t,t_0)=\varkappa_lU_l(t,t_0)+ND_l(t,t_0). \tag{46}$$

Using the expansion of the δ function in a Fourier integral in ω, and integrating in (45) with respect to $\mathbf{n}^0$, we obtain

$$P(N,t)=\int\frac{d\omega}{2\pi}\left[1+i\omega\sum_l\frac{D_l(t,t_0)\bar{n}_l^2(t_0)G_l(t,t_0)}{1+i\omega G_l(t,t_0)\bar{n}_l(t_0)}\right]$$

$$\times\exp\left\{i\omega\left[N-\overline{N}(t)+\sum_l G_l(t,t_0)\bar{n}_l(t_0)\right]-\sum_l\ln[1+i\omega G_l(t,t_0)\bar{n}_l(t_0)]\right\}. \tag{47}$$

It can be shown that the values of ω that make the main contribution to the integral (47) satisfy the inequality

$$|\omega G_l(t,t_0)\bar{n}_l(t_0)|\lesssim 1/\sqrt{L(t)}\ll 1. \tag{48}$$

Therefore, expanding the argument of the exponential in (47) up to second-order terms in the parameter (48), we get

$$P(N,t)=\int\frac{d\omega}{2\pi}\exp\left\{i\omega[N-\overline{N}(t)]-\frac{\omega^2}{2}\sigma_d^2(t)\right\}, \tag{49}$$

$$\sigma_d^2(t)=\sum_l[G_l(t,t_0)\bar{n}_l(t_0)]^2. \tag{50}$$

In the derivation of (49) we omitted the second term in the square brackets of (47), which is small compared with unity relative to the parameter $[L(t)]^{-1/2}$. With the aid of (41), (46), (25) and (29) we obtain the asymptotic (at $t_{sat}\ll t$) value of $G_l(t,t_0)\bar{n}_l(t_0)$:

$$G_l(t,t_0)\bar{n}_l(t_0)=n_l(t)[1/2t-(1-\varkappa_l)\gamma](|a_0'(\infty)|\overline{N}_\infty)^{-1}. \tag{51}$$

Substituting (51) in (50), we get

$$\sigma_d(t)=\frac{1}{4}\left(\frac{9\pi}{2}\right)^{1/4}\frac{1}{|a_0'|t\sqrt{L(t)}}. \tag{52}$$

As seen from (49), the main contribution to the integral (47) is made by values $\omega\lesssim\sigma_d^{-1}(t)$. From this inequality and from (51) it follows that the estimate (48) is correct.

Integrating in (49), we obtain for $P(N,t)$ a Gaussian distribution

$$P(N,t)=(\sqrt{2\pi}\sigma_d(t))^{-1}\exp\{-[N-\overline{N}(t)]^2/2\sigma_d^2(t)\} \tag{53}$$

with variance $\sigma_d^2(t)$ determined by (52). Analogous operations yield, accurate to the discarded quantities that are small in the parameter $[L(t)]^{-1/2}$, the photon distribution function in an individual mode

$$P_l(n_l,t)=(\bar{n}_l(t))^{-1}\exp[-n_l/\bar{n}_l(t)]. \tag{54}$$

The functions $P_l(n_l,t)$ might have been obtained in this approximation directly from (38) via the substitution $\exp[\delta\varphi]\to 1$. With this substitution, however, the function $P(\mathbf{n},t)$ would turn out to be a product of the distribution functions of independent modes. This, in turn, would lead to a large variance of N: $\sigma^2(t)\sim\overline{N}_\infty^2/L(t)$ (cf. (52)], and in accordance with (37) $\delta\varphi(t,t_0)$ would turn out to be not small, in contrast to the assumption.

The results of the present section will be used next to investigate the noise.

§4. NOISE AT LONG TIMES ($t>t_{sat}$)

1. We investigate this solution of the system of Langevin equations (18) with the initial condition $n_l(t_1)=n_l^1$ at the instant of time $t_1\geq t_{sat}$. We represent this solution by the sum

$$n_l(t)=n_l^d(t)+n_l^n(t), \tag{55}$$

where the definite function $n_l^d(t)$ is a simpler designation of the function $n^d(t|t_1, n^1)$ used in Sec. 3. For the random function $n_l^n(t)$ we obtain from (18) the equation

$$\dot{n}_l^n(t) = a_l[N^d(t)]n_l^n(t) + a_l'(t)N^n(t)n_l(t) + F_l(t) \qquad (56)$$

with initial condition $n_l^n(_{-1}) = 0$. We have used here an inequality for $N^n = \sum_l \varkappa_l n_l^n$: $\langle (N^n)^2 \rangle \ll \bar{N}_\infty^2$, which will be proved below. For $N^n(t)$ we have

$$\dot{N}(t) = \sum_l \varkappa_l a_l [N^d(t)] n_l^n(t) + a_0' \bar{N}_\infty N^n(t) + F(t), \qquad (57)$$

where $F(t) = \sum_l \varkappa_l F_l(t)$, and, as a result of (19), $\langle F \rangle = 0$ and

$$\langle F(t)F(t') \rangle = 2b_0 \bar{N}_\infty \delta(t-t'). \qquad (58)$$

In the second term of the right-hand side of (57) and in the correlator (58) we have neglected the difference between sums of the type $\sum_l \varkappa_l n_l$ and the quantity $\bar{N}_\infty$.

2. In Item 4 below it will be shown that in the calculation of noise in a single mode we can neglect the second term of (56). The solution of (56) then takes the form

$$n_l^n(t) = \int_{t_1}^{t} F_l(t') \exp\left[\int_{t'}^{t} a_l[N^d(t'')]dt''\right] dt'. \qquad (59)$$

Using (19), (25b), (35), and (29), we obtain at $t \ll t_\infty$:

$$\langle [n_l^n(t)]^2 \rangle = \int_{t_1}^{t} 2b_l n_l^d(t') \exp\left\{2\int_{t'}^{t} a_l[N^d(t'')]dt''\right\} dt'$$

$$= 2b_l n_l^d(t) \int_{t_1}^{t} \exp[{}^1/_2 \ln(t/t') - (1-\varkappa_l)\gamma(t-t') - (t-t')\delta\gamma_l]dt'. \qquad (60)$$

In the last integral we have neglected the small difference between $a_0(N^d)$ and $a_0(\bar{N})$. The quantity $\langle [n_l^n(t)]^2 \rangle$ is of interest, naturally, only for modes that land in the lasing line contour at the instant of time t, i.e., modes that satisfy the estimate $(1-\varkappa_l)\gamma t \sim [l/L(t)]^2 \le 1$. The value of $\langle (n_l^n)^2 \rangle$ can then be estimated at $(t_{sat} \le t \ll t_\infty)$:

$$\langle [n_l^n(t)]^2 \rangle \leqslant 4 n_l^d \gamma t \ll [n_l^d(t)]^2. \qquad (61)$$

We present the explicit form of (60) for several particular cases. For central modes, i.e., at $(1-\varkappa_l)\gamma t \ll 1$, we have

$$\langle [n_l^n(t)]^2 \rangle = 4b_l n_l^d(t) \sqrt{t}[\sqrt{t} - \sqrt{t_1}]. \qquad (62)$$

For a small time interval $\Delta t \equiv t_1 \ll t_1$ we get from (60)

$$\langle [n_l^n(t)]^2 \rangle = 2b_l n_l^d(t) \Delta t. \qquad (63)$$

3. We investigate now the fluctuations of the total number of photons. The quantity N^n satisfies the equation (57). We separate in N^n the fast and slow components, $N^n = N_s^n + N_f^s$, which satisfy the equations

$$\dot{N}_s^n(t) = a_0' \bar{N}_\infty N_s^n(t) + \sum_l \varkappa_l a_l [N^d(t)] n_l^n(t), \qquad (64)$$

$$\dot{N}_f^n(t) = a_0' \bar{N}_\infty N_f^n(t) + F(t). \qquad (65)$$

The functions $n_l^n(t)$ in the right-hand side of (64) vary slowly with time, therefore the solution $N_s^n(t)$ can be obtained from (64) in the adiabatic approximation, neglecting the term $\dot{N}_s^n$:

$$N_s^n(t) = [|a_0'(\infty)|\bar{N}_\infty]^{-1} \sum_l \varkappa_l a_l [N^d(t)] n_l^n(t). \qquad (66)$$

The solution of (65) with the initial condition $N_f^n(t_1) = 0$ is

$$N_f^n(t) = \int_{t_1}^{t} F(t') \exp[-|a_0'(\infty)|\bar{N}_\infty(t-t')]dt'. \qquad (67)$$

For the mean squared value of N_f^n we get from (67) with the aid of (58) the expression

$$\langle [N_f^n(t)]^2 \rangle = \frac{b_0}{|a_0'(\infty)|}\{1 - \exp[-2|a_0'(\infty)|\bar{N}_\infty(t-t_1)]\}. \qquad (68)$$

At $\Delta t \ge a_0^{-1}(0)$ the quantity $\langle [N_f^n(t)]^2 \rangle$ reaches a stationary value

$$\langle [N_f^n(t)]^2 \rangle \equiv \sigma_n^2 = b_0/|a_0'(\infty)| \sim \bar{N}_\infty/(\zeta - 1). \qquad (69)$$

The order of magnitude of $\langle [N_s^n(t)]^2 \rangle$ is obtained from (66) using at $t \ll t_\infty$ the estimate (61):

$$\langle [N_s^n(t)]^2 \rangle \leqslant L(t) n^d(t) \gamma / [a_0(0)]^2 t \sim \gamma \bar{N}_\infty / [a_0(0)]^2 t. \qquad (70)$$

From (68) and (70) we find that $\langle (N_s^n)^2 \rangle$ is smaller than $\langle (N_f^n)^2 \rangle$ by a factor $a_0(0)t \gg 1$. At times $t \ge t_\infty$ we have for $\langle (N_s^n)^2 \rangle$ the estimate $\langle (N_s^n)^2 \rangle \sim (\zeta - 1)L_\infty$, which is less than $\langle (N_f^n)^2 \rangle$ by a factor $\bar{n}_0(\infty)(\zeta - 1) \gg 1$.

The foregoing comparison of the variances of N_f^n and N_s^n shows that the random function $N_s^n(t)$ can be neglected wherever correlators at equal times are considered. On the other hand, when integrals of correlators of the type $\langle N^n(t)N^n(t') \rangle$ are considered, it must be taken into account that $N_f^n(t)$ (67) has a short correlation time $\sim a_0^{-1}(0)$, and its contribution to the double integral with respect to t and t' turns out to be comparable with the contribution of $N_s^n(t)$.

4. We now justify the neglect of the term $a_l' N^n n_l$ in (56) when $\langle [n_l^n(t)]^2 \rangle$ was determined. The quantity $\langle [n_l^n(t)]^2 \rangle$ is expressed in terms of an integral of the correlators of products of the terms of the right-hand side, taken at two arbitrary times. In this case, as noted above, the contribution of the term $a_0' N^n n_l$ to the integral can be estimated by replacing it by the quantity $a_0' N_s^n n_l$. Comparing the variances of $a_0' N_s^n n_l$ with, e.g., the first term of (56), we obtain, using (70):

$$\langle [a_0' n_l N_s^n]^2 \rangle / \langle [a_l n_l^n]^2 \rangle \sim \frac{1}{L(t)} \ll 1.$$

§5. DISTRIBUTION FUNCTIONS FOR LONG TIMES ($t_{sat} \leqslant t$)

1. The distribution function of the total number of photons can be represented in the form

$$P(N,t) = \int \langle \delta[N - N(t|t_1, n^1, \{F\})] \rangle_{\{F\}} P(n^1, t_1) dn^1, \qquad (71)$$

where $N(t|t_1, n^1, \{F\})$ is a solution of the Langevin system (18) with the initial condition $n(t_1) = n^1$ for a concrete realization of the random forces $\{F\}$. The averaging $\langle \ldots \rangle_{\{F\}}$ in the integral of (71) is over the realizations of the random forces. Just as in Sec. 4, we represent $N(t|t_1, n^1, \{F\})$ as a sum of the definite function $N^d(t|t_1, n^1)$ investigated in Sec. 3 and the random component $N^n(t|t_1, n^1, \{F\})$ considered in the preceding section. As follows from the results of Sec. 4, the function N^n can be replaced by N_f^n defined by (67). The latter, under the condition $\Delta t = t - t_1 \gg a_0^{-1}(0)$, can be represented in the simple form

$$N_f^n(t) = F(t)[|a_0'(\infty)|\bar{N}_\infty]^{-1}, \qquad (72)$$

which does not depend on the initial condition at $t = t_1$. We then have for $P(N,t)$

$$P(N,t) = \int \frac{d\Omega}{2\pi} \exp(i\Omega N) \langle \exp[-i\Omega N_f^n(t)] \rangle_{\{F\}}$$
$$\times \int dn^1 \exp[-i\Omega N^d(t|t_1, n^1)] P(n^1, t_1). \qquad (73)$$

The integral with respect to $\mathbf{n}^1$ in this expression determines the characteristic function $\chi_d(\Omega, t)$ of the distribution obtained in Sec. 3 for the quantity N^d. The averaging over $\{F\}$, on the other hand, determines the characteristic function $\chi_n(\Omega, t)$ of the distribution of N_f^n. After finding the function $\chi^d(\Omega, t)$ with the aid of (53) and $\chi_n(\Omega, t)$ with the aid of (72) and (58), we obtain from (73) for $P(N, t)$ an expression valid for all times $t \gtrsim t_{sat}$:

$$P(N, t)=[(2\pi)^{1/2}\sigma(t)]^{-1}\exp\{-[N-\bar{N}(t)]^2/2\sigma^2(t)\}, \tag{74}$$

$$\sigma^2(t)=\sigma_d^2(t)+\sigma_n^2(t)=\frac{3}{16}\left(\frac{\pi}{2}\right)^{1/2}\frac{1}{L(t)(a_0' t)^2}+\frac{b_0}{|a_0'|}. \tag{75}$$

Allowance for the noise leads, as seen from (75), to a nonzero variance of N as $t\to\infty$ (cf. the dynamic expression (52)]. Under the condition $\sigma_d(t_d)\sim\sigma_n$ we obtain the time t_d that bounds from above the range of validity of the dynamic expressions (43) and (53) for $P(N, t)$ and $P(\mathbf{n}, t)$:

$$t_d\sim[(\zeta-1)a_0(0)]^{-1}(\bar{N}_\infty/L_0)^{2/3}\sim 5\cdot 10^2 a_0^{-1}(0). \tag{76}$$

2. We now obtain the photon distribution function $P_l(n_l, t)$ in an individual mode. In item 4 of Sec. 4 it was shown that, with a relative error $[L(t)]^{-1/2}\ll 1$, it is possible to neglect the term $a_l' N^n n_l$ in (18) when noise in an individual mode is investigated. Neglecting, with the same accuracy, the difference between $a_l(N^d)$ and $a_l(N)$, we obtain for $n_l(t)$ a linearized Langevin equation, corresponding to the following Fokker-Planck equation:

$$P_l(n_l, t)=-\frac{\partial}{\partial n_l}\left\{[a_l(\bar{N}(t))n_l+b_l]P_l(n_l, t)-\frac{\partial}{\partial n_l}[b_l n_l P_l(n_l, t)]\right\}. \tag{77}$$

The solution of this equation, which joins at $t_{sat}\ll t \ll t_d$ the function (54) obtained in the dynamic approximation, has the same form as (54). Thus, with relative error $[L(t)]^{-1/2}\ll 1$, the function $P_l(n_l, t)$ takes at all times t bounded from below by the condition $\bar{n}_l(t)\gg 1$ [i.e., $t\gtrsim a_0^{-1}(0)$] the form

$$P_l(n_l, t)=\frac{1}{\bar{n}_l(t)}\exp[-n_l/\bar{n}_l(t)]. \tag{78}$$

3. In the stationary regime it is also possible to obtain the total distribution function $P(\mathbf{n}, \infty)$. On the basis of the detailed balancing principle, we obtain for the stationary solution of the Fokker-Planck equation (16) the following relations:

$$[a_l(N)n_l+b_l]P(\mathbf{n}, \infty)-\frac{\partial}{\partial n_l}[b_l n_l P(\mathbf{n}, \infty)]=0. \tag{79}$$

Expanding the coefficient $a_l(N)$ near $\bar{N}_\infty$ in powers of $\Delta N = N-\bar{N}_\infty$ and using the relation $a_l(\infty)\bar{n}_l(\infty)+b_l=0$ that follows from (20), we seek the solution of this equation in the form

$$\ln P(\mathbf{n}, \infty)=(a_0'/2b_0)(\Delta N)^2+\chi(\mathbf{n}). \tag{80}$$

For the function $\chi(\mathbf{n})$ we have the equation

$$\frac{\partial\chi(\mathbf{n})}{\partial n_l}=-\frac{1}{\bar{n}_l(\infty)}+(1-\varkappa_l)\frac{a_0'}{b_0}\Delta N; \tag{81}$$

for the modes in the stationary-lasing contour, taking (75) into account, we can neglect the second term in the right-hand side of (81) in the parameter $\sigma^{-1}(\infty)\ll 1$. We then have $\chi(\mathbf{n})=-\sum_l n_l/\bar{n}_l(\infty)+\text{const}$. After determining the value of the constant from the normalization condition with the aid of relations (23) and (24), we obtain the final form of the total distribution function

$$P(\mathbf{n}, \infty)=\bar{N}_\infty\left(\frac{|a_0'|}{2b_0 L_\infty}\right)^{1/2}\exp\left[-\frac{(N-\bar{N}_\infty)^2}{2\sigma^2(\infty)}\right]\prod_l[\bar{n}_l(\infty)]^{-1}\exp\left[-\frac{n_l}{\bar{n}_l(\infty)}\right]. \tag{82}$$

It can be shown that the partial distribution functions $P(N, \infty)$ and $P_l(n_l, \infty)$ obtained from (82) coincide with the distributions (74) and (78) at $t=\infty$.

§6. NOISE IN THE INTRA-CAVITY LASER SPECTROSCOPIC METHOD

1. The foregoing investigation of the statistical properties of multimode laser radiation is directly applicable to the analysis of noise in the ILS method. Up to now, the theoretical description of the ILS, proposed in Refs. 10–12, was based on various modifications of the system of rate equations (see, e.g., Refs. 6, 10, 13–15). The gist of the ILS method, according to this description, is that an increase of the damping constant of some particular laser mode by an amount $\delta\gamma_l$ leads to an exponential decrease of the number of photons in this mode compared with the number of photons in the absence of the additional absorption. The smooth lasing spectrum of the multimode laser then acquires, at the frequency of the selected mode, a dip whose relative magnitude at times $t\ll t_\infty$ increases like $[1-\exp(-\delta\gamma_l t)]$ (30).

We note that this dynamic description of the ILS takes no account whatever of the statistical character of the multimode-lasing spectrum. There is therefore no lower bound on the registered dip, and the possibility of observing arbitrarily weak absorption lines is determined entirely by the precision of the measuring apparatus.

2. Stochastic processes, as shown in the present paper, lead in a concrete realization of the random forces to a lasing spectrum described by a frequency function that is not smooth at all, but strongly chopped up. The variance $\langle(\Delta n_l)^2\rangle$ of the number of photons n_l in each mode is equal, in accord with (78), to the square $\bar{n}_l^2$ of the average number of photons. Additional damping $\delta\gamma_l$ of one of the modes leads to a decrease of $\bar{n}_l$ in this mode. For relatively weak absorption lines, when $\delta\gamma_l\lesssim|a_0(t)|$ and the depth of the dip is $\delta\bar{n}_l/\bar{n}_l\lesssim\frac{1}{2}$ and is linearly connected with $\delta\gamma_l$, the value of $\delta\bar{n}_l$ turns out to be less than the mean squared scatter $[\langle(\Delta n_l)^2\rangle]^{1/2}\sim n_l$ of the number of photons in the individual mode. This means that the statistical character of the radiation makes it impossible to register a weak absorption line in an individual measurement of the lasing spectrum. The only lines that can be registered are those that decrease by many times the average number of photons in the given mode. The condition $\delta\bar{n}_l\sim[\langle(\Delta n_l)^2\rangle]^{1/2}$ determines the sensitivity threshold in the considered scheme of a single measurement of the lasing spectrum. The threshold values of the absorption coefficient $K=\delta\gamma_l/c$ are determined by the formulas

$$K=1/ct, \quad t\leqslant t_\infty, \tag{83a}$$

$$K=1/ct_\infty=\gamma/c\bar{n}_0(\infty), \quad t\geqslant t_\infty. \tag{83b}$$

For example, at $t\sim 3\times 10^{-3}$ sec an estimate yields $K=10^{-8}$ cm^{-1}. Thus, the statistical character of the multimode lasing limits in principle the sensitivity threshold of the ILS method in a single measurement.

The sensitivity threshold can be increased by a factor $p^{1/2}$ by accumulating the single obtained by registering a series of p independent pulses.

3. In the preceding Item we considered the most widely used scheme of recording selective absorption in the ILS method, wherein one measurement is performed for an individual realization of the lasing-development process.

As follows from the investigation in Sec. 4, the difficulties connected with the large variance of the signal in individual realizations of the lasing spectrum can in principle be avoided if two measurements of the spectrum, at the instants of time t_1 and t_2 ($t_{sat} \le t_1 < t_2 \ll t_\infty$), are performed in a given lasing pulse. Despite the fact that the lasing spectra at t_1 and t_2 are described by random functions with a large variance of the number of photons in each mode, these spectra are strongly correlated. The reason is that the evolution of the individual modes during the considered stage (t_1, t_2) is well described by the dynamic approximation, and the noise is low and increases slowly (see Secs. 3 and 4). In accordance with Eqs. (25), (40), and (55) the selective loss is given by the formula

$$\exp[-(t_2-t_1)\delta\gamma_l]=\{n_l(t_2)n_{l'}(t_1)/n_l(t_1)n_{l'}(t_2)\} \times \{1+[n_{l'}^{n}(t_2)/n_{l'}(t_2)-n_l^{n}(t_2)/n_l(t_2)]\}, \tag{84}$$

where l is the index of the mode that has the additional damping $\delta\gamma_l$, and l' is one of the neighboring modes, which have no selective losses.[5] Here, too, use is made of the relative smallness of n_l^{n} and $n_{l'}^{n}$ (61). The term in the square brackets of (84) stems from noise and leads to the appearance of variance of the measured absorption coefficient. At low selective losses ($\delta\gamma_l(t_2 - t_1) \ll 1$), this variance, in accordance with (84), is given by

$$\langle(\Delta K)^2\rangle=2\frac{\langle[n_l^{n}(t_2)]^2\rangle}{[n_l(t_2)]^2}\frac{1}{c^2(t_2-t_1)^2}, \tag{85}$$

where $\langle(n_l^{n})^2\rangle$ is determined in the general case by (60). The minimum value of the absorption coefficient that can be registered with the aid of the described correlation procedure is determined by the condition $K_{corr} =[\langle(\Delta K)^2\rangle]^{1/2}$. In the simple particular case $t_1 \ll t_2$ we obtain for the central modes, using (62),

$$K_{corr}=\frac{1}{ct_2}[\gamma t_2/n_l(t_2)]^{1/2}\sim\frac{1}{ct_2}(t_2/t_\infty)^{1/4}. \tag{86}$$

Thus, the two-time correlation procedure of measurement makes it possible to decrease the threshold value of of the absorption coefficient registered in the pulse regime by a factor $(t_\infty/t_2)^{1/4} \gg 1$ compared with the procedure in which a single measurement of the spectrum is made in one lasing pulse, for which formula (83a) is valid.

CONCLUSION

We have investigated the statistics of radiation of a multimode laser with homogeneous gain contour and equidistant modes. The distribution functions of the number of photons in the individual modes and of the total number of photons were obtained for all lasing stages. The photon-number distribution (78) in an individual mode of multimode lasing coincides at all stages with the distribution for the thermal radiation.[6] However, the noise investigation reported in Sec. 4 shows that in contrast to thermal radiation the correlation time of the number of photons in a single mode is large: $\Delta t \sim t_\infty$. It is shown that the distribution of the total number of photons is Gaussian. The relative variance of this distribution decreases with time in power-law fashion, and reaches a stationary value at sufficiently large times. The decrease of the variance of this distribution is due to saturation of the gain. The relatively slow power-law character of the narrowing of the distribution of the total intensity is connected with the difference between the interactions of the individual modes with the active medium.[7] Also obtained is the photon distribution function during the dynamic stage of the lasing and for the stationary case.

The results of the investigations can be used to analyze the role of noise in the ILS method. In the literature on ILS the prevailing opinion is that the spectrum $\{n_l\}$ of the lasing of the employed multimode laser is a smooth function of the frequency and it is impossible to register against the background of this spectrum the dips corresponding to additional losses in individual modes. What is really smooth, however, is only the spectrum of the mean values $\{\bar{n}_l\}$. The real lasing spectrum $\{\bar{n}_l\}$ corresponds to a concrete realization of the action of random forces during the stages preceding the measurement. Therefore the lasing spectrum observed in a single measurement has a very choppy structure, the scatter of the number of photons in one mode is equal to the average number of photons in this mode (the strong choppiness of the lasing spectrum of solid-state lasers ($T_1 \gg \gamma$) was experimentally observed in Ref. 16). These natural fluctuations of the number of photons in individual modes lead to the appearance of a nonzero sensitivity threshold (83) of the ILS method (when the ILS is described in terms of the rate equations, no limitations are imposed in principle on the magnitude of the registered absorption).

We considered also a two-time correlation procedure for the measurement of selective absorption. We have shown that thanks to the large photon-number correlation time in an individual mode it is possible to decrease the scatter of the values of the measured signal and to lower the sensitivity threshold of the method.

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1) We replace thereby the exact expression $[b_l - a_l(N/2]n_l$ for the diffusion coefficient by the approximate value $b_l n_l$. This approximation is suitable because the role of the diffusion term is negligibly small during the lasing stage bounded by the conditions $\bar{n}_l \gg 1$ and $t \lesssim t_{sat}$; during the subsequent stages ($t \lesssim t_{sat}$), however, the following estimate is valid: a_l/b_l

$\lesssim 1/(\nu t) \ll 1$.

2)Just as the solutions of (16), the solutions of the system (18) are valid only at times such that $\bar{n}_l \gg 1$. They are used hereafter only for such times.

3)In the single-mode case the evolution of the initial distribution function was investigated in the dynamic approximation, e.g., by Baklanov *et al.*[9]

4)Here, as in Sec. 2, we neglect the terms bl in the dynamic equations at times $a_0^{-1}(0) \ll t \ll t_\infty$.

5)For simplicity we neglect in (84) the difference between the gains of the neighboring modes. In the lasers customarily used for ILS this difference is extremely small.

6)For a stationary distribution of photons in an individual mode, this correspondence was established by Ambartsymyan *et al.*[3] for the case of identical modes in a special laser model with nonresonant feedback. We note that in the case of a single-mode laser the distribution of the number of photons, which is thermal during the initial (linear) lasing stage, narrows down rapidly during the succeeding stages because of gain-saturation effects.

7)Using Eq. (20) of Letokhov's paper,[2] it is easy to show that in a laser with identical modes ($\bar{n}_l \equiv \bar{n}$), just as in a single-mode laser, the variance of the distribution of the total intensity narrows down exponentially rapidly to a stationary value.

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