

# Interaction of diffusion modes in the theory of localization

K. B. Efetov, A. I. Larkin, and D. E. Kheml'nitskiĭ

*L.D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR*

(Submitted 8 April 1980)

Zh. Eksp. Teor. Fiz. 79, 1120–1133 (September 1980)

The problem of electron conductivity in a random potential is reduced to the problem of thermodynamics of a nonlinear tensor  $\sigma$  model. The method of replicas is used and classical anticommutative fields are integrated. The symmetry of the tensor depends on the presence of magnetic and spin-orbit interactions. Renormalizability is demonstrated for the two-dimensional case and the Gell-Mann-Low function is calculated. This function is used to determine the dependence of the diffusion coefficient on the frequency or on the size of the system.

PACS numbers: 66.30.Dn

## 1. INTRODUCTION

A quantum particle subjected to a random potential may become localized or may escape to infinity.<sup>1</sup> In the three-dimensional case localization occurs only when the degree of disorder is sufficiently high. In the one- and two-dimensional cases this effect may occur for disorder which can be as weak as we please.<sup>2-5</sup> This is due to the dependence of the nature of classical diffusion on the number of dimensions. In the one- and two-dimensional cases a diffusing particle may pass very near the initial point after a sufficiently long time. In the three-dimensional case the probability of return is low. Therefore, in the former two cases a particle is scattered many times by the same center no matter how weak is the random potential. In the three-dimensional case such multiple scattering is important only if the potential is sufficiently strong. In all cases when multiple scattering is important we can expect quantum interference to result in localization.

Direct calculations of the quantum corrections to the conductivity were reported by Gor'kov and two of the present authors<sup>5</sup> and by Anderson *et al.*<sup>6</sup> These calculations showed that such corrections appear as a result of the interaction of the diffusion modes. Therefore, it is important to provide a description of the interaction of diffusion modes by an effective Lagrangian. Attempts to introduce such a Lagrangian phenomenologically have been made many times.<sup>7-10</sup> All these authors have used the replica method<sup>11</sup> in which a random system is replaced by  $N$  thermodynamic systems, the partition function is averaged over the random potential, all the necessary quantities are calculated, and then  $N$  is allowed to tend to zero.

All these investigations<sup>7-10</sup> differ in respect of the symmetry of the collective variable  $Q_{\alpha\beta}$  which describes diffusion. In a recent paper, Wegner<sup>10</sup> showed that  $Q_{\alpha\beta}$  is a  $2N \times 2N$  matrix and the effective Lagrangian has a high symmetry  $O(2N)/O(N) \times O(N)$ . Therefore, the Lagrangian describes the interaction of  $N^2$  Goldstone modes. The principal weakness of these investigations<sup>7-10</sup> is that at the intermediate stage after averaging over the random potential the procedure yields the Hamiltonian of a gas of Bose particles with an attractive interaction. Therefore, all the quantities are described formally by diverging integrals.

We shall use the Grassman variables at the intermediate stage so that after averaging we obtain the Hamiltonian of a Fermi gas with attraction. This theory is free of the shortcomings of Refs. 7–10. The effective Lagrangian of the diffusion modes can now be derived in the same way as the Ginzburg-Landau functional in the BCS theory. This Lagrangian has the  $S(2N)/S(N) \times S(N)$  symmetry [ $S(N)$  is a symplectic group]. We shall find the effective Lagrangian allowing for an external magnetic field, as well as for scattering by magnetic impurities and the spin-orbit interaction. In all these cases we shall show that the theory is renormalizable in a two-dimensional space and we shall calculate the Gell-Mann-Low functions in the one- and two-loop approximations.

## 2. DERIVATION OF THE FUNCTIONAL OF INTERACTING DIFFUSIONS

The nature of immersion of an electron in a random field of impurities can be studied simply by finding the correlation function of the density  $\rho(\mathbf{r}, t)$ :

$$\langle \rho(\mathbf{r}, t) \rho(\mathbf{r}', t') \rangle - \langle \rho^2 \rangle = \int d\omega \exp(i\omega(t-t')) \int \frac{n_\epsilon - n_{\epsilon+\omega}}{\omega} K(\omega) d\epsilon, \quad (1)$$

where  $\langle \dots \rangle$  denotes averaging over the distribution of impurities;  $K(\omega)$  is a two-particle Green function which can be expressed in the usual way in terms of retarded  $G^R$  and advanced  $G^A$  one-particle Green functions:

$$K(\omega) = G_{\epsilon+\omega}^R(\mathbf{r}, \alpha; \mathbf{r}', \beta) G_{\epsilon}^A(\mathbf{r}', \beta; \mathbf{r}, \alpha), \quad (2)$$

$$G_{\epsilon}^{(R,A)}(\mathbf{x}, y) = \sum_{\mathbf{k}} \frac{\varphi_{\mathbf{k}}^*(\mathbf{x}) \varphi_{\mathbf{k}}(\mathbf{x}')}{\epsilon - E_{\mathbf{k}} \pm i\delta}.$$

Here,  $\mathbf{x} = \mathbf{r}, \alpha$ , where  $\mathbf{r}$  is the coordinate and  $\alpha$  is the spin variable;  $E_{\mathbf{k}}$  and  $\varphi_{\mathbf{k}}$  are the eigenvalues and the eigenfunctions of an electron in an impurity field:

$$H\varphi_{\mathbf{k}} = E_{\mathbf{k}}\varphi_{\mathbf{k}}, \quad H = H_0 + H_1, \quad \langle H_1 \rangle = 0. \quad (3)$$

The problem is to average Eq. (2) over the distribution of impurities. Usually,<sup>12</sup> the Green functions are expanded as a perturbation theory series in terms of the random potential  $H_1$  and each term of the series is averaged. Diffusion is then obtained by summation of ladder diagrams. The effects of the interaction of diffusions can be calculated by summing a rapidly growing number of diagrams, the majority of which cancel out. Moreover, in an analysis of a diagram series we cannot employ the usual field theory methods, because it is

necessary to ensure the absence of closed loops in the diagrams. The method of replicas<sup>11</sup> allows us to use field theory methods. Then, diffusions behave as Goldstone modes in the "replica" Hamiltonian. In many cases it is found that the important distance exceeds the mean free path. These cases are described by the Lagrangian of interacting diffusions.

In the replica method a Green function is expressed in terms of a continuum integral of boson fields. Averaging over the impurity distribution gives the Hamiltonian of a Bose gas with attraction. This Hamiltonian does not have a stable ground state. The use of the continuum integral of Fermi fields makes it possible to avoid this difficulty. Classical fermion fields are described by the following Grassman anticommutative variables<sup>13,14</sup>

$$\{\chi_i, \chi_j\} = \{\kappa_i, \kappa_j\} = \{\chi_i, \kappa_j\}. \quad (4)$$

Integration with respect to the Grassman variables is described by

$$\int d\chi_i = \int d\kappa_i = 0; \quad \int \chi_i d\chi_i = \int \kappa_i d\kappa_i = 1. \quad (5)$$

It follows from the above formulas that

$$(e-E)^{-1} = i \int \kappa_k \chi_k \exp(-i\chi_k(e-E)\kappa_k) d\chi_k d\kappa_k \times \left( \int \exp(-i\chi_k(e-E)\kappa_k) d\chi_k d\kappa_k \right)^{-1}. \quad (6)$$

Using the representation (6) we can express a Green function in terms of a continuum integral

$$G^{(R,A)}(x, x') = i \int \kappa(x) \chi(x') \exp\left(-i \int dy \chi(y) (e - \hat{H} \pm i\delta) \kappa(y)\right) D\chi D\kappa \times \left\{ \int \exp\left[-i \int dy \chi(y) (e - \hat{H} \pm i\delta) \kappa(y)\right] D\chi D\kappa \right\}^{-1}; \quad (7)$$

$$\chi(x) = \sum_k \chi_k \varphi_k(x); \quad \kappa(x) = \sum_k \kappa_k \varphi_k(x).$$

The replica method makes it possible to describe the average (over the positions of impurities) of the ratio of two integrals in the form of the ratio of the averages

$$\langle f \rangle = \left\langle \int f(\chi, \kappa) \exp[-F(\chi, \kappa)] D\chi D\kappa \left\{ \int \exp[-F(\chi, \kappa)] D\chi D\kappa \right\}^{-1} \right\rangle = \lim_{N \rightarrow 0} \left\{ \left\langle \int f(\chi_n, \kappa_n) \exp\left[-\sum_{n=1}^N F(\chi_n, \kappa_n)\right] D\chi_n D\kappa_n \right\rangle \times \left[ \left\langle \int \exp\left[-\sum_{n=1}^N F(\chi_n, \kappa_n)\right] D\chi_n D\kappa_n \right\rangle \right]^{-1} \right\}. \quad (8)$$

The correlation function of the density  $K(\omega)$  can be expressed in the following form using Eqs. (2), (7), and (8):

$$K(\omega) = \left\langle \int \kappa_1(x) \chi_1(x') \kappa_{N+1}(x') \chi_{N+1}(x) e^{-i\omega t} D\chi_n D\kappa_n \right\rangle \times \left( \left\langle \int e^{-i\omega t} D\chi_n D\kappa_n \right\rangle \right)^{-1}; \quad (9)$$

$$L = i \int dy \sum_{n=1}^{2N} \chi_n(y) \left[ E - \hat{H} + \left( \frac{\omega}{2} + i\delta \right) \lambda_n \right] \kappa_n(y), \quad (10)$$

$$\lambda_n = \begin{cases} +1, & n \leq N, \\ -1, & n > N. \end{cases}$$

Instead of a double set of variables  $\chi$  and  $\kappa$ , it is convenient to introduce a spinor  $\psi$ :

$$\sqrt{2} \psi_n = \begin{pmatrix} \chi_n \\ \kappa_n \end{pmatrix}, \quad \sqrt{2} \bar{\psi}_n = (-\kappa_n, \chi_n), \quad \bar{\psi} = (C\psi)^\tau, \quad (11)$$

where  $C$  is the charge conjugacy matrix

$$C_{nm} = \delta_{nm} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (12)$$

In terms of spinor notation, Eq. (10) can be rewritten in the form

$$L = i \int dx \bar{\psi}(x) (H_0 + U(r) + \frac{1}{2} \omega \Lambda) \psi(x), \quad (13)$$

where  $\Lambda$  is the diagonal matrix with the elements  $\lambda_n$ ,

$$H_0 = \epsilon + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2},$$

and  $U(r)$  is the impurity potential. In the calculation of the averages over the distribution of impurities the potential  $U$  is regarded as a random quantity, distributed in accordance with the Gaussian  $\delta$ -correlated law

$$\langle U(r) U(r') \rangle = \frac{1}{2\pi\nu\tau} \delta(r-r'), \quad (14)$$

where  $\nu$  is the density of states and  $\tau$  is the transit time. This assumption is obeyed quantitatively in the case of small-radius impurities with a weak potential. In general, we can show that all the results remain valid also after a redefinition of the constants.

After averaging, Eq. (13) can be rewritten in the form

$$L = \int \mathcal{L} dr,$$

where

$$\mathcal{L} = i\bar{\psi} H_0 \psi + \frac{1}{4\pi\nu\tau} (\bar{\psi}\psi)^2 + \frac{i\omega}{2} \bar{\psi} \Lambda \psi. \quad (15)$$

The Lagrangian (15) is similar to that which is used in the theory of superconductivity. The metallic phase in the theory of localization corresponds to the superconducting phase. The order parameter in the theory of localization is the zero-trace tensor  $Q \sim \langle \psi \bar{\psi} \rangle$ . Diffusion modes correspond to slow changes in  $Q$ . These slow changes in  $Q$  can be calculated by rewriting the interaction term in the form

$$\sum_{p_1+p_2=p_1+p_2} (\bar{\psi}_{p_1} \psi_{p_2}) (\bar{\psi}_{p_2} \psi_{p_1}) \approx \sum_{p_1, p_2, q} \{ (\bar{\psi}_{p_1} \psi_{p_2}) (\bar{\psi}_{p_1+q} \psi_{p_2-q}) + (\bar{\psi}_{p_1} \psi_{p_2}) (\bar{\psi}_{p_2-q} \psi_{p_1+q}) + (\bar{\psi}_{p_1} \psi_{p_2-q}) (\bar{\psi}_{p_2+q} \psi_{p_1}) \}, \quad (16)$$

where regions with small values of  $q$  are treated separately.

Using the operation of charge conjugacy and applying the anticommutation rules, we can show that the first two terms on the right-hand side of Eq. (16) are equal. The third term alters slightly the value of  $\epsilon$ . The exponential function of the fourth power of  $\psi$  can be written conveniently in the form of a Gaussian integral

$$\exp\left\{-\frac{1}{4\pi\nu\tau} \int (\bar{\psi}\psi)^2 dx\right\} = \int DQ \exp\left[-\frac{1}{2\tau} \int (\bar{\psi} Q \psi + \frac{\pi\nu}{2} \text{Tr} Q^2) dy\right] / \int DQ \exp\left[-\frac{\pi\nu}{4\tau} \int \text{Tr} Q^2 dy\right], \quad (17)$$

where the matrix  $Q$  satisfies the conditions

$$Q = Q^\dagger, \quad C^\tau Q^\tau C = Q^\dagger. \quad (18)$$

The Hermitian nature appears always for arbitrary values of  $\psi$  and  $\bar{\psi}$  and the second condition in Eq. (18) follows from the fact that  $\psi$  and  $\bar{\psi}$  are linked by the charge conjugacy operations.

The elements  $Q_{nm}$  are quaternions, i.e., they are  $2 \times 2$  matrices of the  $Q = Q^a \tau_a$  type, where

$$\tau_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \quad (19)$$

The conditions (18) mean that all the quantities  $Q_{mn}^a$  are real numbers and  $Q_{nm}$  can be written in the form

$$Q_{nm} = \begin{pmatrix} D_{nm} & \Delta_{nm} \\ -\Delta_{nm}^* & D_{nm}^* \end{pmatrix}, \quad (20)$$

$$D_{nm} = D_{mn}^*, \quad \Delta_{nm} = -\Delta_{mn}^*. \quad (21)$$

Bearing in mind that  $Q$  varies slowly, we can integrate fermion fields and obtain an expression for the correlation function of the density in the form of a continuum integral with respect to  $Q$

$$K(\omega) = \int DQ e^{-F} \text{Tr} \{ (1-\Lambda)(1-\tau_3)Q(r) (1+\Lambda)(1+\tau_3)Q(r') \} \times \left( 16N^2 \int DQ e^{-F} \right)^{-1} \Big|_{r=r_0}. \quad (22)$$

The free energy  $F$  is

$$F = \int dr \left\{ -\text{Tr} \ln \left( -iH_0 - \frac{i\omega}{2} \Lambda - \frac{Q}{2\tau} \right) + \frac{\pi\nu}{4\tau} \text{Tr} Q^2 \right\}. \quad (23)$$

It is clear from Eq. (23) that  $\text{Tr} Q$  gives rise to a small correction to  $\varepsilon$ . If we ignore these corrections, we can regard  $Q$  as a trace-free matrix. If  $\varepsilon\tau/\hbar \gg 1$ , then the free-energy minima are important in the integral (22). The conditions for the minima are

$$Q = \frac{1}{\pi\nu} \int \hat{G}(p) dp, \quad (24)$$

$$\hat{G}(p) = \left( iE - i\frac{p^2}{2m} + \frac{i\omega}{2} \Lambda + \frac{Q}{2\tau} \right)^{-1}. \quad (25)$$

The solution of Eq. (24) for  $\omega \neq 0$  is of the form  $Q = \Lambda$ .

In the case of low values of  $\omega$  the free energy for all  $Q$ 's satisfying the condition  $Q^2 = 1$  differs little from the minimum value. All such zero-trace matrices can be written in the form

$$Q = U^{-1} \Lambda U, \quad (26)$$

where  $U$  is an arbitrary unitary quaternion-real matrix. If we assume that  $Q$  is homogeneous over a large region and expand it in terms of small deviations from equilibrium, we obtain the following expression for the free energy

$$F = F_0 + \frac{1}{4\tau^2} \text{Tr} \int \frac{dk}{(2\pi)^d} \left\{ \int \frac{dp}{(2\pi)^d} [G_p \delta Q_k G_{p+k} \delta Q_{-k}] + \frac{\pi\nu}{4\tau} \delta Q_k \delta Q_{-k} \right\} + \frac{i\omega\pi\nu}{2} \int \text{Tr} \Lambda Q dr. \quad (27)$$

We have to distinguish longitudinal variations of  $Q$  which disturb the condition  $Q^2 = 1$  and transverse (rotational) variations which conserve this condition and obey

$$Q \cdot \delta Q + \delta Q \cdot Q = 0. \quad (28)$$

The longitudinal variations of  $Q$  alter greatly the free energy and, therefore, we can ignore the longitudinal

fluctuations.

Homogeneous transverse fluctuations do not alter the free energy at all in the limit of low frequencies and only the gradients  $Q$  contribute to the free energy. In the case of small gradients, it follows from Eqs. (26) and (27) that

$$F = F_0 + \frac{\pi\nu}{4} \int [D \text{Tr} (\nabla Q)^2 + 2i\omega \text{Tr} (\Lambda Q)] dr, \quad (29)$$

where  $D = \nu^2 \tau$  is the diffusion coefficient. The expression (29) is the functional of the free energy of interacting diffusion modes. The matrix  $U$  in Eq. (26) can be written in the form

$$U = e^{W/2}, \quad W = \begin{pmatrix} 0 & B \\ -B^+ & 0 \end{pmatrix}, \quad Q = \Lambda e^W. \quad (30)$$

In Eq. (30),  $B$  is an arbitrary quaternion-real matrix. If the interaction between the modes is unimportant, then  $U$  differs little from unity and the free energy can be expanded as a series in powers of  $B$ . Retaining only the quadratic terms, we obtain

$$F \approx F_0 + \frac{\pi\nu}{2} \int dr (D \text{Tr} (\nabla B \nabla B^+) - i\omega \text{Tr} (B B^+)). \quad (31)$$

Using Eqs. (31) and (22), we find that the correlation function of the density is

$$K(\omega, q) = 2\pi\nu / (Dq^2 - i\omega). \quad (32)$$

### 3. LOWERING OF SYMMETRY

The free energy of Eq. (29) is symmetric under the transformations of Eq. (26). The high symmetry is due to invariance under time reversal, when a particle is replaced with a hole. The symmetry is manifested by the invariance of Eq. (29) under unitary transformation of each quaternion. If we allow for the spin degrees of freedom, we have to add to the symmetry also the invariance under rotations in the spin space. The symmetry under time reversal is broken by an external magnetic field or by the scattering on magnetic impurities. Magnetic impurities and the spin-orbit interaction break the symmetry under rotations in the spin space. The Hamiltonian allowing for all these interactions has the form

$$H_0 = E - e \left( \hat{p} - \frac{ie}{c} A \tau_3 \right) + \sigma [\hat{p} \times \nabla V_{so}], \quad (33)$$

$$H_1 = U_0 + U_s \sigma + \sigma [\hat{p} \times \nabla U_{so}]. \quad (34)$$

Here,  $A$  is the vector potential;  $\tau_3$  is defined by Eq. (19) and it appears because the  $\kappa, \chi$  Hamiltonian is expressed in terms of  $\psi$  and  $\bar{\psi}$ ;  $V_{so}$  is the spin-orbit interaction in the absence of impurities;  $U_{so}$  is the impurity spin-orbit interaction;  $U_s$  is a random magnetic field created by magnetic impurities.

For convenience, we shall assume that all these interactions are weaker than conventional scattering and can therefore be considered independently. Repeating the derivation of Eq. (29) in the presence of an external magnetic field, we find that the expression for the free energy becomes

$$F = \frac{\pi\nu}{4} \int dr \text{Tr} \left\{ D \left( \nabla Q + \frac{e}{\hbar c} A [Q, \tau_3] \right)^2 + 2i\omega \Lambda Q \right\}. \quad (35)$$

It is clear from Eq. (35) that in addition to the gradient term the free energy contains a constant term proportional to  $A^2$  which does not vanish for homogeneous  $Q$ . If we substitute Eq. (20) in Eq. (35), we can see that no constant terms appear in the matrix  $D$ . This corresponds to the case when the magnetic field does not destroy diffusion. In  $\Delta$  the constant term does not vanish and this means that fluctuations of  $\Delta$  can be ignored in the range of the longest wavelengths.

When the Hamiltonian is independent of the spins, all the above results can be applied separately to the spin-up and spin-down particles. In general, it is necessary to regard  $\kappa$  and  $\chi$  as spinors and  $\psi$  and  $\bar{\psi}$  as bispinors, described by the formulas

$$\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi \\ i\sigma_y \kappa \end{pmatrix}, \quad \bar{\psi} = \frac{1}{\sqrt{2}} (-\kappa, -i\chi\sigma_y) = (C\psi)^T, \quad (36)$$

$$C = \begin{pmatrix} 0 & i\sigma_y \\ i\sigma_y & 0 \end{pmatrix}.$$

The matrix  $Q$  satisfies Eq. (18) where the charge conjugacy matrix is defined by Eq. (36). In the case of scattering by magnetic impurities it is necessary to average over the random fields  $U_s$ . After such averaging we can show that Eq. (15) contains an additional term

$$\mathcal{L}_s = \frac{1}{12\pi\nu\tau_s} (\bar{\psi}\Sigma\psi)^2, \quad \Sigma_{nm} = \delta_{nm} \otimes \tau_s, \quad (37)$$

$$\langle U_s^i(\mathbf{r}) U_s^j(\mathbf{r}') \rangle = \frac{\delta_{ij}}{6\pi\nu\tau_s} \delta(\mathbf{r} - \mathbf{r}').$$

After integration over the Fermi fields, this term in  $\mathcal{L}$  gives rise to the following constant term in the free energy

$$F_s = \frac{\pi\nu}{6\tau_s} \int d\mathbf{r} \text{Tr} (Q\Sigma Q\Sigma). \quad (38)$$

Equation (38) can be written in a more convenient form if we use spin quaternions  $Q = Q_0 + \mathbf{Q}\sigma$ :

$$F_s = \frac{\pi\nu}{6\tau_s} \int d\mathbf{r} \text{Tr} \left\{ -\frac{3}{2} [Q_0\tau_s]^2 + \frac{1}{2} [Q\tau_s]^2 + 4Q^2 \right\}. \quad (39)$$

Equation (39) shows that at long wavelengths we can ignore fluctuations of  $Q$ , and also fluctuations of  $\Delta_0$ . We are then again left with conventional diffusion.

In the case of a sufficiently weak spin-orbit interaction we find that Eq. (15) acquires an additional term

$$\mathcal{L}_{so} = -\frac{1}{2} \left\langle \left( \int d\mathbf{r} \bar{\psi}(\mathbf{r}) \sigma [\nabla(V_{so} + U_{so})\mathbf{p}] \psi(\mathbf{r}) \right)^2 \right\rangle.$$

After integration with respect to  $\psi$  the function  $\mathcal{L}_{so}$  transforms into an expression for the additional spin-orbit contribution to the free energy

$$F_{so} = \frac{\pi\nu}{2} \sum_{i=1}^3 \frac{1}{\tau_{so}^{(i)}} \text{Tr} (Q\sigma^i)^2, \quad (40)$$

$$\frac{1}{\tau_{so}^{(i)}} = 2\pi\nu \langle \{ ([\nabla U_{so} \times \mathbf{p}]^i)^2 + [\nabla V \times \mathbf{p}]^i \} \rangle.$$

Equation (40) is derived allowing for the possible anisotropy of the spin-orbit interaction.

Equation (40) shows that all fluctuations of the vector components  $Q$  are unimportant at long wavelengths. However, in addition to fluctuations of  $D_0$ , fluctuations

of  $\Delta_0$  are also important. This is due to the fact that the spin-orbit interaction conserves the symmetry under time reversal. The matrix  $Q$  then becomes

$$Q = Q_0\tau_0 + iQ_1\tau_1 + iQ_2\tau_2 + Q_3\tau_3, \quad (41)$$

where all the  $Q_i$ 's are unit matrices in the spin space. It follows from Eq. (41) that in the spin-orbit case the matrix  $Q$  is not real with respect to quaternions.

We thus find that in the language of Goldstone modes a magnetic field, magnetic impurities, and spin-orbit interaction act as characteristic anisotropies of different symmetry. In the range of moderately long wavelengths the number of Goldstone modes is large and when the wavelength is increased some of them lose their Goldstone nature so that at the longest wavelength these modes are represented by diffusion and possibly by fluctuations of  $\Delta_0$ .

#### 4. INTERACTION OF DIFFUSION MODES

At the end of Sec. 2 we have shown that the correlation function of the density has the diffusion form in the Gaussian approximation. Corrections to the correlation functions due to the interactions can be allowed for by including terms of higher order in  $B$ . Using Eq. (30) and the expression for the free energy functional (29), we obtain the free energy to within  $B^2$ :

$$F = \frac{\pi\nu}{2} \int d\mathbf{r} \left\{ D \text{Tr} \left[ \partial B \partial B^* + \frac{1}{12} (\partial B B^* \partial B B^* + \partial B^* B \partial B^* B) - \partial B \partial B^* B B^* - \partial B^* \partial B B^* B \right] - i\omega \text{Tr} \left( B^* B - \frac{1}{12} (B^* B)^2 \right) + \frac{1}{2\tau_{so}^{(i)}} \text{Tr} (|B, \sigma^i| |B^*, \sigma^i|) + \frac{1}{6\tau_s} \text{Tr} [B, \Sigma] |B^*, \Sigma| \right\};$$

$$\partial B = \nabla B - \frac{e}{\hbar c} \mathbf{A} [B, \tau_3]. \quad (42)$$

In further calculations it is convenient to expand the matrix  $B$  in terms of the complete set of  $B$  matrices

$$B = B_{\alpha\alpha}\tau_\alpha \otimes s_\alpha, \quad (43)$$

where  $\tau_\alpha$  is given by Eq. (19) and

$$s_0 = 1, \quad s_{1,2,3} = i\sigma_{1,2,3}.$$

The condition of charge conjugacy  $C^T B^T C = B^*$  with the matrix  $C$ , given by Eq. (36), means that  $B_{\alpha\alpha}$  and  $B_{3\alpha}$  are real, whereas  $B_{1\alpha}$  and  $B_{2\alpha}$  are imaginary matrices.

Using the expansion (43) we can write the part  $F_2$  quadratic in  $B$  in the form

$$F_2 = 2\pi\nu \sum_{\mathbf{r}} \text{Tr} (-i\omega + E_\alpha + X_{\alpha\alpha}) B_{\alpha\alpha}^E B_{\alpha\alpha}^{E*}. \quad (44)$$

In Eq. (44),  $B_{\alpha\alpha}^E$  are the coefficients in the expansion of the functions  $B_{\alpha\alpha}(\mathbf{r})$  in terms of eigenfunctions of the Hamiltonian describing a particle in a magnetic field and  $E_\alpha$  are the eigenenergies:

$$E_0 = E_s = p^2 D, \quad E_1 = E_2 = \frac{4D\hbar e H}{c} \left( n + \frac{1}{2} \right), \quad (45)$$

where  $H$  is the magnetic field.

Expanding the commutators in Eq. (42), we can calculate the quantities ( $i = 1, 2, 3$ )

$$X_{00} = X_{30} = 0, \quad X_{0i} = X_{3i} = \frac{4}{3\tau_s} + \frac{2}{\tau_{so}} - \frac{2}{\tau_{so}^{(i)}}, \quad (46)$$

$$X_{10} = X_{20} = \frac{2}{\tau_s}, \quad X_{1i} = X_{2i} = \frac{2}{3\tau_s} + \frac{2}{\tau_{so}} - \frac{2}{\tau_{so}^{(i)}}.$$

In the first approximation, the correlation functions are governed by Gaussian integrals and are given by

$$2\pi\nu\langle(B_{i\alpha}^+)^{\alpha\alpha}(B_{i\beta}^+)^{\beta\beta}\rangle = \frac{\delta_{\alpha\beta}\delta_{\alpha\beta}\delta_{ii}\delta_{\beta\beta}}{-i\omega + E_{\alpha} + X_{\alpha\alpha}}. \quad (47)$$

In this approximation the correlation function  $B$  is identical with the correlation function  $Q$ . Using Eq. (47) and going to the limit  $N=0$ , we can obtain correlation functions of the density  $\langle D_0 D_0^* \rangle$ , of the spin density  $\langle DD^* \rangle$ , and also of the triplet and singlet  $\Delta$ :

$$\langle \Delta\Delta^* \rangle, \quad \langle \Delta_0\Delta_0^* \rangle.$$

The energy  $F_{int}$  describing the interaction between modes has the following form in the absence of a magnetic field:

$$F_{int} = \frac{1}{6} \pi\nu (\delta_{\alpha\beta}\delta_{\alpha\beta} + \delta_{\alpha\beta}\delta_{\beta\alpha} - \delta_{\alpha\beta}\delta_{\beta\alpha})(\delta_{\alpha\beta}\gamma\gamma\delta + \delta_{\alpha\beta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\delta}) \\ \times \int dr \text{Tr} [\nabla B_{\alpha\alpha} B_{\beta\beta}^+ \nabla B_{\gamma\gamma} B_{\delta\delta}^+ + \nabla B_{\alpha\alpha} B_{\beta\beta}^+ \nabla B_{\gamma\gamma} B_{\delta\delta}^+ \\ - \nabla B_{\alpha\alpha} \nabla B_{\beta\beta}^+ B_{\gamma\gamma} B_{\delta\delta}^+ - \nabla B_{\alpha\alpha} \nabla B_{\beta\beta}^+ B_{\gamma\gamma} B_{\delta\delta}^+]. \quad (48)$$

We shall calculate the correlation function of the density using the theory of perturbations relative to  $F_{int}$ . Separating the most diverging (at large distances) terms and using the formulas in Eq. (47), we obtain

$$D = D_0 \left( 1 - \frac{1}{4\pi\nu} \sum_{\alpha,\beta} \frac{(2\delta_{\alpha\alpha}-1)(2\delta_{\beta\beta}-1)+N}{E_{\alpha} + X_{\alpha\alpha} - i\omega} \right). \quad (49)$$

Equation (49) contains all the known effects of the first order in respect of the interaction of diffusion modes. In the absence of symmetry-breaking interactions, the correction to the diffusion coefficient in the two-dimensional case is proportional to  $\ln\omega\tau$  (Refs. 4-6). An external field truncates these logarithms,<sup>15</sup> in the same way as magnetic impurities.<sup>16</sup> If allowance is made for the spin-orbit interaction, the sign of the correction to  $D$  is reversed.<sup>17</sup> In the three-dimensional case the contribution of long-wavelength fluctuations is proportional not to the logarithm but to the square root of the frequency.

## 5. RENORMALIZATION GROUP

In the preceding section we have shown that the interaction of diffusion modes is described in all the investigated cases by the nonlinear  $\sigma$  model<sup>18,19</sup>

$$F = \frac{1}{t} \int [\text{Tr}(\nabla Q)^2 - \bar{\omega} \text{Tr}(\Lambda Q)] dx, \quad (50)$$

where  $\bar{\omega} = \pi\nu t\omega/2$ .

The matrix  $Q$  in Eq. (50) obeys the conditions

$$Q^2 = 1, \quad \text{Tr} Q = 0, \quad Q = Q^+. \quad (51)$$

In the case of potential scattering by nonmagnetic impurities and spin-orbit scattering we can assume that the matrix  $Q$  consists of  $4N^2$  quaternions satisfying the conditions (18)-(20) and (41). In the model which allows for the scattering by magnetic impurities or for the interaction with a magnetic field the matrix  $Q$  consists of  $4N^2$  complex numbers. This form differs slightly from that used in the preceding section, where the spin structure of each element has been important. However, since only the diagonal element of the spin

matrices contribute at long wavelengths, this difference is removed by a suitable redefinition of  $t$ .

We shall investigate the model described by Eq. (50) in a space of dimensions  $2 + \varepsilon$  using the renormalization group method, similar to that employed by Polyakov<sup>18</sup> in the vector model. We shall represent the matrix  $Q$  in the form (22), where  $U$  is a unitary matrix consisting of  $rN^2$  complex numbers in the magnetic case or of  $4N^2$  quaternions of suitable type described by Eqs. (20) and (41) in the nonmagnetic case and in the spin-orbit scattering case.

Following the renormalization group method, we shall divide the matrix  $U(\mathbf{r})$  into a product of fast  $U_0(\mathbf{r})$  and slow  $\bar{U}(\mathbf{r})$  parts:

$$U(\mathbf{r}) = U_0(\mathbf{r})\bar{U}(\mathbf{r}), \quad (52)$$

where  $U_0(\mathbf{r})$  and  $\bar{U}(\mathbf{r})$  satisfy the same conditions as  $U$ .

Substituting Eq. (52) into Eq. (50), we reduce the expression for the free energy (50) to

$$F = \frac{1}{t} \int [\text{Tr}(\nabla Q_0)^2 + 2 \text{Tr}([\nabla Q_0, Q_0]\mathbf{A}) + \text{Tr}[Q_0, \mathbf{A}]^2 - \bar{\omega} \text{Tr}(\mathbf{C}\Lambda U^+ Q_0)] dx, \quad (53)$$

$$Q_0 = U_0^+ \Lambda U_0, \quad \mathbf{A} = \nabla U \cdot U^+.$$

We can easily demonstrate that the matrix  $Q_0$  satisfies the conditions (20) and (41).

We shall integrate Eq. (22) using the free energy given by Eq. (53) and use the fast variable  $Q_0(\mathbf{r})$ . After integration, the energy  $F$  in Eq. (53) becomes modified to  $F_1$ , describing slow fluctuations

$$F_1 = - \ln \int e^{-F} DQ_0. \quad (54)$$

The functional integral with respect to  $Q_0$  in Eq. (54) can be calculated subject to the restrictions of Eq. (51) by parametrization of Eq. (30), where  $B$  is an arbitrary matrix composed of  $N^2$  quaternions whose form is described by Eqs. (18)-(20) and (41) or of  $N^2$  complex numbers in the magnetic case.

Expanding  $Q_0$  in terms of the matrix  $W$ , we reduce the energy functional  $F$  to

$$F = F_0 + F_1 + F_2 + \Phi; \\ F_0 = - \frac{1}{t} \int \text{Tr}(\nabla W)^2 dx = \frac{2}{t} \int \text{Tr}(\nabla B \nabla B^+) dx, \\ F_1 = - \frac{2}{t} \int \text{Tr}([\nabla W, W]\mathbf{A}) dx \approx \frac{4}{t} \int \text{Tr}(W \nabla W \mathbf{A}) dx, \quad (55) \\ F_2 = \frac{1}{t} \int [\text{Tr}(\Lambda \Lambda W^2 \mathbf{A}) + \text{Tr}(\Lambda W \mathbf{A})^2] dx, \\ F_0 = \frac{1}{t} \int [\text{Tr}(\nabla \bar{Q})^2 - \bar{\omega} \text{Tr} \bar{Q} \Lambda] dx.$$

In Eq. (55),  $\bar{Q}$  is given by

$$\bar{Q} = U^+ \Lambda U.$$

The quantity  $\Phi$  contains terms of higher order in  $W$ . There are no terms linear in  $W$  because  $W$  varies rapidly. It is assumed that  $W_k$  differs from zero for  $\lambda k_0 < k < k_0$  and that  $W_k = 0$  for  $k < \lambda k_0$ , where  $k_0$  is the truncation parameter and  $0 < \lambda < 1$ . Integration is carried out over the momentum range  $k^2 \gg \bar{\omega}$ .

In the lowest order in  $t$  the integral of the exponential function in Eq. (54) can be reduced to

$$F = -\frac{1}{2} \langle F_1^2 \rangle_0 + \langle F_2 \rangle_0 + F_0, \quad (56)$$

where  $\langle \dots \rangle_0$  denotes averaging with the functional  $F_0$  from Eq. (55).

The averages in Eq. (56) can be calculated using the formula

$$\tau_a \tau_b^+ + \tau_b \tau_a^+ = 2\delta_{ab}. \quad (57)$$

In the calculation of Gaussian integrals we can easily verify the relationship

$$\langle BPB^+ \rangle_0 = \frac{t}{2k^2} \left[ P_{ii} + \frac{\alpha^2}{2} (P_{ii}^+ - P_{ii}) \right], \quad (58)$$

$$\langle B^+PB \rangle_0 = \langle BPB \rangle_0 = \frac{\alpha t}{4k^2} P^+.$$

In Eq. (58),  $P$  is an arbitrary matrix consisting of  $P_{ij}$  quaternions of the type described by Eqs. (20) and (41) or of complex numbers in the magnetic case. The coefficient  $\alpha$  in Eq. (58) depends on the model and is given by

$$\alpha = \begin{cases} -1, & \text{potential scattering,} \\ 0, & \text{magnetic case,} \\ 1, & \text{spin-orbit scattering.} \end{cases} \quad (59)$$

The averages which include a large number of the matrices  $B$  are calculated using the Wick theorem. Applying the formulas in Eq. (58), we can now calculate  $\langle F_1^2 \rangle_0$  and  $\langle F_2 \rangle_0$ . It should be pointed out that each of them contains terms of a form different from the unrenormalized energy  $F_0$  but their sum reproduces exactly the free energy

$$F = \frac{1}{t} \int \left[ \text{Tr}(\nabla Q)^2 \left( 1 - \frac{2N-\alpha}{8} t \int \frac{dk}{k^2} \right) - \bar{\omega} \text{Tr} \Lambda \bar{Q} \left( 1 - \frac{Nt}{4} \int \frac{dk}{k^2} \right) \right] dr. \quad (60)$$

Equation (60) shows that our theory is renormalizable in the first order. This equation allows us to find the form of the Gell-Mann-Low function, which we shall write down for a space of the dimensions  $2 + \varepsilon$ :

$$\frac{dt}{d \ln k} = \beta(t) = \sum_{n=1}^{\infty} \beta_n t^n; \quad (61)$$

$$\beta_1 = -\varepsilon t, \quad \beta_2 = \frac{2N-\alpha}{8} t^2;$$

$$\frac{d \ln \bar{\omega}}{d \ln k} = \xi(t) = \frac{Nt}{4}. \quad (62)$$

It is clear from Eq. (62) that in the limit  $N \rightarrow 0$  in the first order with respect to  $t$  the function vanishes  $\xi(t) \rightarrow 0$ . This property is retained in all orders with respect to  $t$  for  $N \rightarrow 0$  and it is associated with the law of conservation of the number of particles in elastic collisions.

Following exactly the same procedure we can calculate also the Gell-Mann-Low function in higher orders. Terms of the order of  $t^3$  in this function can be found by counting the number of diagrams of the correct order in  $t$  shown in Fig. 1. Some of them include quadratic divergences which are canceled out by a contribution from the Jacobian. Logarithmic terms appear due to integrals of the type

$$\int \frac{dk_1 dk_2}{k_1^2 k_2^2}.$$

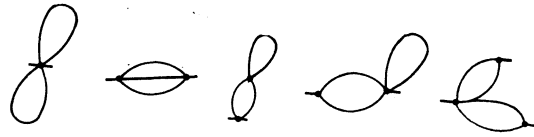


FIG. 1.

In some terms the integration is carried out over the range  $\lambda k_0 < k_1, k_2 < k_0$ , whereas in others it is carried out over the range  $\lambda k_0 < k_1, k_2, |\bar{k}_1 + \bar{k}_2| < k_0$ . Direct calculations demonstrate that terms of the order  $\ln^2 \lambda$  cancel out. The first order of  $\ln \lambda$  is absent from each of the calculated integrals. This corresponds to the case when terms of the order of  $t^3$  do not appear in the Gell-Mann-Low function of Eq. (61) and the coefficient  $\beta_3$  vanishes in Eq. (61). This result was obtained earlier by one of the authors<sup>20</sup> in the practically important case of  $N = 0$ .

It should be pointed out that this result is in conflict with the answer obtained by Brezin *et al.*<sup>19</sup> Since Wegner<sup>10</sup> refers to the work of Brezin *et al.*,<sup>19</sup> his  $\beta$  function of a phase-invariant ensemble also differs from the result  $\beta_3 = 0$ . We may put forward the hypothesis that  $\beta(t) \equiv 0$  for  $N = 0$  in the case of scattering by magnetic impurities and on application of an external magnetic field.

In the potential scattering case we have an unstable fixed point

$$t^* = 8\varepsilon + O(\varepsilon^2),$$

which corresponds to the mobility threshold. In the limits  $\varepsilon \rightarrow 0$  and  $t^* \rightarrow 0$ , this clearly corresponds to localization of all the states in the two-dimensional case.

In an external magnetic field when allowance is made for the scattering by magnetic impurities there are no fixed points in the range of validity of the theory  $t \ll 1$ . Therefore, in the magnetic case an electron is localized in a two-dimensional system only when the degree of disorder is sufficiently high.

An interesting possibility appears also in the case of the spin-orbit interaction. In this case we can see from Eq. (61) that the effective charge tends to zero for  $\varepsilon = 0$  and  $N = 0$ :

$$t = 8 / \ln \bar{\omega}. \quad (63)$$

In Eq. (61) for the case of potential scattering of spin-free particles and spin-orbit scattering the coefficients in front of  $t^2$  and  $\beta(t)$  have the same absolute values for  $N = 0$ . If we compare the model spin-free Hamiltonian (50) with the Hamiltonian for real particles (23), we can see that in the spin-orbit scattering case the correction to the diffusion coefficient has the opposite sign and is half the value in the potential scattering case.

The same result was obtained by a different method by Hikami *et al.*<sup>17</sup> Anderson *et al.*<sup>21</sup> put forward the opposite hypothesis that, because of invariance under time reversal, the spin-orbit interaction should not affect electron localization. It follows from the

formulas in Eq. (49) that, in general, the corrections to the diffusion coefficient appear due to fluctuations with spins of zero and unity. The spin-orbit interaction suppresses spin fluctuations and, therefore, it influences the result.

## CONCLUSIONS

We have seen that the problem of electron conductivity in a random field of impurities is equivalent to the problem of thermodynamics of a tensor field which is shown to be renormalizable. The Gell-Mann-Low function is found for the two-dimensional case and this function can be used to determine the dependence of the effective diffusion coefficient (and, consequently, of the conductivity) on the frequency and size. At finite temperatures the frequency should be replaced with the reciprocal of the energy relaxation time.

An external magnetic field, magnetic impurities, and spin-orbit interaction alter the effective energy of a tensor field and its symmetry. Therefore, the conductivity depends on temperature, frequency, size, and magnetic field. The nature of this dependence changes in the presence of the spin-orbit interaction or of magnetic impurities.

One of the authors (A. I. L.) is grateful to S. Hikami for stimulating discussions.

- <sup>1</sup>P. W. Anderson, *Phys. Rev.* **109**, 1492 (1958).  
<sup>2</sup>N. F. Mott, *Adv. Phys.* **16**, 49 (1967).  
<sup>3</sup>V. L. Berezinskiĭ, *Zh. Eksp. Teor. Fiz.* **65**, 1251 (1973) [*Sov. Phys. JETP* **38**, 620 (1974)].  
<sup>4</sup>E. Abrahams, P. W. Anderson, D. C. Licciardello, and

- T. V. Ramakrishnan, *Phys. Rev. Lett.* **42**, 673 (1979).  
<sup>5</sup>L. P. Gor'kov, A. I. Larkin, D. E. Khmel'nitskiĭ, *Pis'ma Zh. Eksp. Teor. Fiz.* **30**, 248 (1979) [*JETP Lett.* **30**, 228 (1979)].  
<sup>6</sup>P. W. Anderson, E. Abrahams, and T. V. Ramakrishnan, *Phys. Rev. Lett.* **43**, 718 (1979).  
<sup>7</sup>A. Nitzan, K. F. Freed, and M. H. Cohen, *Phys. Rev. B* **15**, 4476 (1977).  
<sup>8</sup>A. Aharony and Y. Imry, *J. Phys. C* **10**, L487 (1977).  
<sup>9</sup>H. G. Schuster, *Z. Phys. B* **31**, 99 (1978).  
<sup>10</sup>F. Wegner, *Z. Phys. B* **35**, 207 (1979).  
<sup>11</sup>S. F. Edwards and P. W. Anderson, *J. Phys. F* **5**, 965 (1975).  
<sup>12</sup>A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, *Metody kvantovoi teorii polya v statisticheskoĭ fizike, Fizmatgiz, M.*, 1963 (*Methods of Quantum Field Theory in Statistical Physics*, Prentice-Hall, Englewood Cliffs, N. J., 1963).  
<sup>13</sup>F. A. Berezin, *Metod vtorichnogo kvantovaniya (Second Quantization Method)*, Fizmatgiz, M., 1965.  
<sup>14</sup>N. N. Bogolyubov and D. V. Shirkov, *Teoriya kvantovannykh poleĭ*, Nauka, M., 1976 (*Introduction to the Theory of Quantized Fields*, Interscience, New York, 1959).  
<sup>15</sup>B. L. Altshuler, D. E. Khmel'nitskiĭ, A. I. Larkin, and P. A. Lee, Preprint, Bell Laboratories, 1980.  
<sup>16</sup>P. A. Lee, Report of Eighth Intern. Conf. on Amorphous and Liquid Semiconductors, Cambridge, Mass., 1979, in: *J. Non-Cryst. Solids* **35-36**, 21 (1980).  
<sup>17</sup>S. Hikami, A. I. Larkin, and Y. Nagaoka, Preprint, Kyoto University, 1979.  
<sup>18</sup>A. M. Polyakov, *Phys. Lett. B* **59**, 79 (1975).  
<sup>19</sup>E. Brezin, S. Hikami, and J. Zinn-Justen, Preprint, Saclay, 1979.  
<sup>20</sup>D. E. Khmel'nitskiĭ, Preprint, Nordita, 1979.  
<sup>21</sup>P. W. Anderson, D. J. Thouless, E. Abrahams, and D. Fisher, Preprint, Bell Laboratories, 1980.

Translated by A. Tybulewicz