Exact solutions of the Landau-Lifshitz equations for weak ferromagnets

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A class of exact solutions of the Landau-Lifshitz equations is obtained for stationary-profile waves (moving domain walls and solitons) for the case of a uniaxial two-sublattice weak ferromagnet. Explicit equations are obained for the limiting velocities of domain walls and solitons as functions of the structure parameters of the weak ferromagnet. Solutions in analytic form, corresponding to moving domain walls, are obtained. It is shown that in the moving-domain-wall model of Zvezdin [JETP Lett. 29, 553 (1979)] and of Bar'yakhtar *et al.* [Sov. Tech. Phys. Lett. 5, 351 (1979)] the solution of the boundary-value problem ceases to be unique wth increasing magnetic field, and instability sets in.

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1. Earlier[1,2] investigations of the solutions of the Landau-Lifshitz equations have shown that the problem of stationary-profile waves (moving domain walls and solitons) is completely integrable in the case of a uniaxial single-sublattice ferromagnet characterized by an anisotropy energy

$$U_a=-K(\mathbf{mn})^2. \quad (1.1)$$

Here m is the magnetic-moment unit vector, n is the anisotropy-axis unit vector, and K is the anisotropy-energy constant. The first two integrals that exist in this case (the conservation laws) have made it possible to obtain a complete classification of the stationary-profile waves, to determine the dependence of the soliton amplitude on the velocity, and to determine the limiting velocities of the "slow" and "fast" waves.[1,2]

It is shown in the present paper that the system of Landau-Lifshitz equations for a uniaxial two-sublattice weak ferromagnet admits of separation of an important class of solutions that lead to completely integrable problems dealing with stationary-profile waves (domain walls and solitons). This makes it possible to present for a selected class of solutions a complete classification of the stationary-profile waves, to obtain exact expressions for the limiting velocities of domain walls and solitons in weak ferromagnets, as well as to write down the explicit form of the solutions for moving domain walls.

The method of separating completely integrable problems of stationary-profile waves in the case of a uniaxial two-sublattice weak ferromagnet is simple. Namely, the phase space of the stationary-profile-wave problem is four-dimensional in the case of a uniaxial single-sublattice ferromagnet, and the solutions of the Landau-Lifshitz equations for the anisotropy energy (1.1) correspond, in terms of the concepts of mechanics, to the problem of the motion of a charged material point over the surface of a unit sphere in electric and magnetic fields. In this case the problem is completely integrable for an electric field defined by a homogeneous quadratic potential. The complete phase space of the stationary-profile-wave problem for a two-sublattice weak ferromagnet is eight-dimensional. It is therefore necessary to separate the class of exact solutions of the system of Landau-Lifshitz equations belonging to a four-dimensional phase subspace and corresponding to the problem of the motion of a material point over the surface of a sphere in three-dimensional configuration space.

The proposed method was realized using as an example a uniaxial two-sublattice weak ferromagnet with free energy (see, e.g., Ref. 3)

$$\mathcal{F}=D\mathbf{m}^2+(\mathbf{m}')^2+(\mathbf{l}')^2-(\mathbf{mn})^2-(\mathbf{ln})^2+2d[\mathbf{lm}]\mathbf{v}. \quad (1.2)$$

Here m and l are the vectors of the magnetic moment and of the antiferromagnetism, and satisfy the conditions

$$\mathbf{m}^2+\mathbf{l}^2=1 \quad \mathbf{ml}=0; \quad (1.3)$$

D is the homogeneous exchange constant and d is the Dzyaloshinskiĭ constant, both referred to a constant uniaxial anisotropy energy K, while n and $\boldsymbol{\nu}$ are the unit vectors of the directions of the anisotropy axis and of the Dzyaloshinskiĭ interaction. The primes in (1.2) denote differentiation with respect to the independent variable $x-ut$ (the spatial variable is referred to the characteristic dimension of the Bloch wall with exchange energy constant A and with anisotropy constant K, while the wave velocity u is referred to the characteristic velocity $2|\gamma|(AK)^{1/2}/M_0$, where γ is the gyromagnetic ratio and M_0 is the saturation magnetization of each of the sublattices).

The main result is that such important classes of solutions as

$$\mathbf{l}(l_x, l_y, 0); \quad \mathbf{m}(0,0,m_z), \quad \mathbf{n}(0,0,1), \quad \boldsymbol{\nu}(\nu_x,\nu_y,0) \quad (1.4)$$

or

$$\mathbf{l}(l_x, 0, l_z), \quad \mathbf{m}(0,m_y,0), \quad \mathbf{n}(0,0,1), \boldsymbol{\nu}(1,0,0), \quad (1.5)$$

lead, after diagonalization of the matrix that defines the potential energy and after renormalization of the spatial variable and of the velocity, to a previously investigated fully integrable problem.[1,2] This means that for the class of exact solutions (1.4) and (1.5) the classification and structure of the self-localized solutions (domain walls and solitons) is the same as in the case of a uniaxial single-sublattice ferromagnet, the difference being that the new anisotropy axis does not coincide with the initial anisotropy axis n.

If v is the renormalized stationary-profile wave (the renormalization will be defined below), then the three previously obtained characteristic velocities[1,2] are of the same form

$$v_\pm=(1+\varepsilon)^{1/2}\pm1,\quad v_0=(1+\varepsilon)^{1/4}. \tag{1.6}$$

In this case, however, the dependence of the positive parameter ε on the structure constants of the weak ferromagnet is different for the solutions (1.4) and (1.5), but the meaning of the characteristic velocities (1.6) is the same as in the case of a uniaxial single-sublattice ferromagnet.[1,2] Namely, v_- determines the separation boundary of the moving domain walls and solitons, v_+ the separation boundary of the solitons and the spin wave, and v_0 the characteristic curve on the (v, ε) plane, above which either precession or nutation about the new anisotropy axis can be excited.

The following inequalities hold for a number of weak ferromagnets

$$1\ll d\ll D. \tag{1.7}$$

(We recall that in our notation the constants d and D are referred to the anisotropy-energy constant K.) Allowance for relations (1.7) leads as $D\to\infty$ to the asymptotic values of the nonrenormalized (initial) characteristic velocities of the stationary-profile waves

$$u_+\sim u_-\sim D^{1/2} \tag{1.8}$$

for the classes of exact solutions investigated by us. The last expression coincides with the limiting velocity of domain walls in the approximate weak-ferromagnetism models.[4,5] However, for arbitrary finite values of the structure parameters of the weak ferromagnet, the difference

$$\frac{u_+-u_-}{D^{1/2}}\approx O\left(\frac{d}{D};\frac{d^2}{D}\right)\neq 0 \tag{1.9}$$

determines the gap of the soliton states in the stationary-profile wave-velocity spectrum.

In a reference frame with a polar axis directed along the new anisotropy axis, there is a known explicit form of the solutions for moving domain walls. The reason is that corresponding to the moving domain walls are solutions with one degree of freedom, namely, with a constant value of the azimuthal angle in a spherical coordinate system with polar axis along the new anisotropy axis. The explicit form of these solutions, in the representation of the vectors **l** and **m** will be given below. The exact solutions singled out in this manner are new analogs of the known exact solutions of Akhiezer and Borovik[12] for uniaxial ferromagnets.

In the last sections are given the results of an analysis of a simple model of the motion of a domain wall in a weak ferromagnet,[4,5] defined by an equation close to the sine-Gordon equation. We note that model equations of similar structure arise, for example, in investigations of spin states of the B phase of He^3 and of the propagation of optical pulses in a degenerate nonlinear medium.[6]

We present below a number of exact results of an analysis of the simple model of Refs. 4 and 5, pointing to a distinctly unique behavior of moving domain walls in strong magnetic field. We indicate, in particular, the solution of the problem of stationary motion of a domain wall ceases to be unique, that the solution used by Zvezdin[4] and Bar'yakhtar *et al.*[5] is unstable, and that solutions of the type of "fast" moving domain walls appear when the external magnetic field is increased. Even though the critical field that leads to these phenomena can exceed the fields reached in the experiments,[7,8] and the metastable phase loses stability, we regard the investigation of the simple model of Refs. 4 and 5 as necessary. It is important that, with increasing external field, the model equation of Refs. 4 and 5 generates solutions that have no counterparts among the solutions of the sine-Gordon equation (for example, the dissipative term leads to violation of the Lorentz invariance and to a possibility of existence of "fast" motions of the domain walls).

Finally, notice must be taken of the analogy between the situations that arise in the problem of the motion of domain walls, on the one hand, and the classical problem of Kolmogorov, Petrovskiĭ, and Piskunov[9] dealing with stationary-profile waves for the nonlinear diffusion equation.

2. The system of Landau-Lifshitz equations for a two-sublattice weak ferromagnet with free energy (1.2) can be written in the form

$$\begin{aligned}\frac{\partial \mathbf{m}}{\partial t}&=\left[\mathbf{l}\times\frac{\delta\mathcal{F}}{\delta\mathbf{l}}\right]+\left[\mathbf{m}\times\frac{\delta\mathcal{F}}{\delta\mathbf{m}}\right],\\ \frac{\partial \mathbf{l}}{\partial t}&=\left[\mathbf{l}\times\frac{\delta\mathcal{F}}{\delta\mathbf{m}}\right]+\left[\mathbf{m}\times\frac{\delta\mathcal{F}}{\delta\mathbf{l}}\right].\end{aligned} \tag{2.1}$$

For stationary-profile waves, with account taken of the explicit form (1.2) of the free energy, the system (2.1) takes the form

$$\begin{aligned}u\mathbf{m}'&=-(\boldsymbol{\mu}_l+\boldsymbol{\mu}_m)'-(\mathbf{ln})[\mathbf{l}\times\mathbf{n}]-(\mathbf{mn})[\mathbf{m}\times\mathbf{n}]+d\{(\mathbf{l}\boldsymbol{\nu})\mathbf{m}-[\mathbf{m}\boldsymbol{\nu})\mathbf{l}\},\\ u\mathbf{l}'&=-\boldsymbol{\mu}_{lm}'-(\mathbf{ln})[\mathbf{m}\times\mathbf{n}]-(\mathbf{mn})[\mathbf{l}\times\mathbf{n}]\\ &\quad+d\{(\mathbf{m}\boldsymbol{\nu})\mathbf{m}-(\mathbf{l}\boldsymbol{\nu})\mathbf{l}+(\mathbf{l}^2-\mathbf{m}^2)\boldsymbol{\nu}\}+D[\mathbf{l}\times\mathbf{m}].\end{aligned} \tag{2.2}$$

We have introduced here the following notation for the rotation vectors:

$$\boldsymbol{\mu}_l=[\mathbf{l}\times\mathbf{l}'],\quad \boldsymbol{\mu}_m=[\mathbf{m}\times\mathbf{m}'],\quad \boldsymbol{\mu}_{lm}=[\mathbf{l}\times\mathbf{m}']+[\mathbf{m}\times\mathbf{l}']. \tag{2.3}$$

The system (2.2) is characterized by four degrees of freedom (i.e., by an eight-dimensional phase space). To separate the class of exact solutions corresponding to two degrees of freedom we call attention to the fact that the second relation of (1.3) is identically satisfied for any choice of the antiferromagnetism vector **l** with one zero projection, and of the magnetic moment **m** with two zero projections, corresponding to nonzero projections of the antiferromagnetism vector. In such cases the configuration space of the system is the surface of a unit sphere in the space of three vectors of the form

$$(l_x, l_y, m_z),\quad (l_x, m_y, l_z),\quad (m_x, l_y, l_z). \tag{2.4}$$

The diagonal rotation moment $\boldsymbol{\mu}_l+\boldsymbol{\mu}_m$ is here a single-component vector, and the nondiagonal rotation moment $\boldsymbol{\mu}_{lm}$ is its complementary two-component vector.

The next step is to separate mutual orientations of the anisotropy axis **n** and of the Dzyaloshinskiĭ-interaction vector $\boldsymbol{\nu}$ such that the system equations (2.2) corresponding to zero projections of the rotation moments $\boldsymbol{\mu}_l+\boldsymbol{\mu}_m$ and $\boldsymbol{\mu}_{lm}$ are satisfied identically. For the first triad in (2.4) this is realized by choosing the orthogonal pair

$$\mathbf{n}(0,0,1);\quad \boldsymbol{\nu}(\nu_x,\nu_y,0). \tag{2.5}$$

The system (2.2) then degenerates into the system

$$\begin{aligned} ul_x' &= -\mu_x' - (1-D) m_z l_y + d\{(l_y^2 - m_z^2)\nu_x - l_x l_y \nu_y\}, \\ ul_y' &= -\mu_y' + (1-D) m_z l_x + d\{(l_x^2 - m_z^2)\nu_y - l_x l_y \nu_x\}, \\ um_z' &= -\mu_z' + d(l_x\nu_x + l_y\nu_y) m_z. \end{aligned} \tag{2.6}$$

We have used here for the angular momenta the notation

$$\mu_x = (\boldsymbol{\mu}_{lm})_x, \quad \mu_y = (\boldsymbol{\mu}_{lm})_y, \quad \mu_z = (\boldsymbol{\mu}_l + \boldsymbol{\mu}_m)_z. \tag{2.7}$$

For the second triad of (2.4) the exact solutions with two degrees of freedom is realized by choosing the orthogonal pair

$$\mathbf{n}(0, 0, 1), \quad \boldsymbol{\nu}(1, 0, 0), \tag{2.8}$$

and the system (2.2) leads to the equations

$$\begin{aligned} ul_x' &= -\mu_x' - (1+D) l_z m_y + d(l_z^2 - m_y^2), \\ um_y' &= -\mu_y' + l_x l_z + d m_y l_z, \\ ul_z' &= -\mu_z' + D l_x m_y - d l_x l_z. \end{aligned} \tag{2.9}$$

Similar classes of exact solutions can be distinguished in the analysis of other additional pairs of vectors **l** and **m** and mutual orientations of the vectors **n** and $\boldsymbol{\nu}$.

3. Equations (2.6) or (2.9) can be written in the form of the system of dynamics of a material point

$$\begin{aligned} \mathbf{M}' &= [\hat{G}\mathbf{q} \times \mathbf{q}] - u\mathbf{q}', \\ \mathbf{M} &= [\mathbf{q} \times \mathbf{q}'], \quad \mathbf{q}^2 = 1. \end{aligned} \tag{3.1}$$

Here **q** is a unit vector defined, for example, by one of the triads of (2.4), $\hat{G}$ is the matrix corresponding to the potential energy represented by the homogeneous quadratic form

$$U = {}^1/_2(\mathbf{q}\hat{G}\mathbf{q}). \tag{3.2}$$

The explicit form of the matrix G is determined by comparing the system (3.1) with the system (2.6) or (2.9).

We compare now Eqs. (3.1) with the equations of motion of a charged material point in an electric field **E** and in a magnetic field **H**, and located on the surface of a unit sphere

$$\ddot{\mathbf{r}} = \frac{e}{m}\mathbf{E} + \frac{e}{mc}[\dot{\mathbf{r}} \times \mathbf{H}], \quad \mathbf{r}^2 = 1. \tag{3.3}$$

The system (3.3) can be written in the form

$$\begin{aligned} \dot{\mathbf{M}} &= \frac{e}{m}[\mathbf{r} \times \mathbf{E}] + \frac{e}{mc}(\mathbf{r}\mathbf{H})\dot{\mathbf{r}}, \\ \mathbf{M} &= [\mathbf{r} \times \dot{\mathbf{r}}]. \end{aligned} \tag{3.4}$$

Comparison of (3.4) and (3.1) points to an analogy between the stationary-profile-wave problem for the Landau-Lifshitz equations and the problem of the motion of a charged material point in an electric field defined by the gradient of the potential energy (3.2),

$$\frac{e}{m}\mathbf{E} = -\hat{G}\mathbf{q} \tag{3.5}$$

and in a magnetic field of a monopole defined by the relation

$$\frac{e}{mc}\mathbf{r}\mathbf{H} = -u. \tag{3.6}$$

According to (3.6), the magnetic field is normal to the surface of the sphere at each of its points, and is determined by the velocity of the stationary-profile wave.

For the completely integrable problem of stationary-profile waves in a uniaxial single-sublattice ferromagnet, which was investigated by us earlier,[1,2] the matrix $\hat{G}$ is diagonal

$$\hat{G} = \mathrm{diag}\,(-1-\varepsilon;\ -1;\ 0) \tag{3.7}$$

with a parameter $\varepsilon = 2\pi M_0^2/K$. Consequently, diagonalization of the matrices $\hat{G}$ defined by the exact-solution classes distinguished above, and the reduction of the diagonal matrices to the form (3.7) enables us to reduce completely the problem of stationary-profile waves in a uniaxial two-sublattice weak ferromagnet to the previously investigated problem.[1,2] Diagonalization of the matrix $\hat{G}$ calls for rotations of the reference frame, and determines the orientation of the new anisotropy axis, while the reduction of the diagonal matrices to the canonical form (3.7) corresponds to renormalization of the independent variable and of the stationary-profile-wave velocity. These transformations determine the explicit dependence of the parameter ε on the structural parameters of the weak ferromagnet. We note that the direction of the new anisotropy axis coincides with the direction of a vector defined by one of the triads of (2.4) for the corresponding homogeneous solutions of the Landau-Lifshitz equations.

For the class of solutions (1.4) the position of the new anisotropy axis and accordingly of the polar axis of the spherical coordinate system is determined by the rotation of the vector **n** around the direction of the vector $\boldsymbol{\nu}$ through the angle

$$\psi = \frac{1}{2}\arccos\frac{1-D}{[(1-D)^2 + 4d^2]^{1/2}}. \tag{3.8}$$

The characteristic parameter of the weak ferromagnet is then

$$\varepsilon = \frac{[(1-D)^2 + 4d^2]^{1/2} - (1-D)}{[(1-D)^2 + 4d^2]^{1/2} + (1-D)}. \tag{3.9}$$

For the class of solutions (1.5), the direction of the new anisotropy axis is determined by rotation of the vector **n** around the vector $\boldsymbol{\nu}$ through the angle

$$\psi = \frac{1}{2}\arccos\frac{1+D}{[(1+D)^2 + 4d^2]^{1/2}}. \tag{3.10}$$

In this case the characteristic weak-ferromagnetism parameter is

$$\varepsilon = \frac{[(1+D)^2 + 4d^2]^{1/2} + (D-1)}{[(1+D)^2 + 4d^2]^{1/2} - (D-1)}. \tag{3.11}$$

The renormalized velocity v is connected with the stationary-profile wave velocity u by the relation

$$u^2 = N^2 v^2. \tag{3.12}$$

Here

$$2N^2 = [(1-D)^2 + 4d^2]^{1/2} + 1 - D \tag{3.13}$$

for the solutions of class (1.4) and

$$2N^2 = [(1+D)^2 + 4d^2]^{1/2} + 1 - D \tag{3.14}$$

for solutions of class (1.5).

Relations (1.6) and (3.12) together with the expressions given above for ε and N determine the explicit dependence of the characteristic velocities $u_\pm$ and u_0 on the structure parameters of the weak ferromagnet for the distinguished classes of the exact solutions. In a spherical reference frame with polar axis directed along the new anisotropy axis, the classification and structure of the self-localized solutions (domain walls and solitons) in a weak ferromagnet coincide exactly with the classification and structure of the self-localized solutions in a uniaxial ferromagnet at $\varepsilon = \varepsilon(D, d)$. The same remark holds also for the explicit form of the two first integrals (the conservation laws) obtained previously.[2]

Moreover, the two previously obtained first integrals correspond in the general case to two divergent forms

$$\frac{\partial}{\partial t}\{\varkappa^2-(\mathbf{q}\hat{G}\mathbf{q})\}+2\frac{\partial}{\partial x}\left\{\varkappa^2\tau-\left(\frac{\partial \mathbf{q}}{\partial x}[\hat{G}\mathbf{q}\times\mathbf{q}]\right)\right\}=0, \quad (3.15a)$$

$$\frac{\partial\tau}{\partial t}+\frac{\partial}{\partial x}\left\{\tau^2-\frac{1}{\varkappa}\frac{\partial^2\varkappa}{\partial x^2}-\frac{1}{2}\varkappa^2+\frac{1}{2}(\mathbf{q}\,\hat{G}\mathbf{q})-\frac{1}{\varkappa^2}\left(\frac{\partial \mathbf{q}}{\partial x}\hat{G}\frac{\partial \mathbf{q}}{\partial x}\right)\right\}=0. \quad (3.15b)$$

Here

$$\varkappa^2=\left(\frac{\partial \mathbf{q}}{\partial x}\frac{\partial \mathbf{q}}{\partial x}\right);\quad \varkappa^2\tau=\left(\mathbf{q}\left[\frac{\partial \mathbf{q}}{\partial x}\times\frac{\partial^2 \mathbf{q}}{\partial x^2}\right]\right), \quad (3.16)$$

and the unit vector $\mathbf{q}(x, t)$ coincides with the magnetic-moment unit vector $\mathbf{m}$ for the case of a single-sublattice ferromagnets or with one of the triads of (2.4) for the case of a two-sublattice weak ferromagnet. The derivation of the second divergent form is based on a generalization of a method proposed by Lakshman *et al.*,[10] who confined themselves, however, to the case $\hat{G}\equiv 0$.

When account is taken of the explicit expressions for the rotation angles ψ and the renormalization coefficients N, it is easy to obtain solutions for $\mathbf{m}$ and $\mathbf{l}$. We demonstrate this using as an example the solutions corresponding to the motion of domain walls. For the class of solutions (1.4) we find that

$$\begin{aligned} m_z&=\cos\psi\,\mathrm{th}\,\xi+\sin\psi\cos\varphi_c\,\mathrm{ch}^{-1}\xi,\\ l_x&=(\nu_x\sin\varphi_c-\nu_y\cos\psi\cos\varphi_c)\,\mathrm{ch}^{-1}\xi-\nu_y\sin\psi\,\mathrm{th}\,\xi,\\ l_y&=(-\nu_y\sin\varphi_c+\nu_x\cos\psi\cos\varphi_c)\,\mathrm{ch}^{-1}\xi-\nu_x\sin\psi\,\mathrm{th}\,\xi. \end{aligned} \quad (3.17)$$

Here

$$\xi=N(1+\varepsilon\cos^2\varphi_c)^{1/2}(x-ut), \quad (3.18)$$

and the angle φ_c is determined from the equation

$$u^2=N^2\frac{\varepsilon^2\cos^2\varphi_c\sin^2\varphi_c}{1+\varepsilon\cos^2\varphi_c}. \quad (3.19)$$

The parameters ε and N are defined here by (3.9) and (3.13), and the angle ψ by (3.8).

For the class of solutions (1.5), the solution corresponding to a moving domain wall is

$$\begin{aligned} m_y&=-\sin\psi\,\mathrm{th}\,\xi-\cos\psi\cos\varphi_c\,\mathrm{ch}^{-1}\xi,\\ l_x&=\sin\varphi_c\,\mathrm{ch}^{-1}\xi,\\ l_z&=-\sin\psi\cos\varphi_c\,\mathrm{ch}^{-1}\xi-\cos\psi\,\mathrm{th}\,\xi. \end{aligned} \quad (3.20)$$

The independent variable ξ and the angle φ are defined here as before by relations (3.18) and (3.19). The parameters ε and N and the angle ψ, however, are determined by expressions (3.11), (3.14), and (3.10).

The obtained exact solutions are in essence the analogs of the well-known solutions of the Landau-Lifshitz equations with one degree of freedom, corresponding to the motion of Bloch or Néel domain walls in a uniaxial ferromagnet.[3]

We note in conclusion that the results can be generalized to include the case of non-uniaxial anisotropy defined by a more general homogeneous quadratic form compared with that used before,[1,2] as well as to the case when account is taken of internal magnetic fields.

4. As shown in Refs. 4 and 5, the Landau-Lifshitz equations for a two-sublattice weak ferromagnet lead under certain conditions, in the limit as $D\to\infty$, to a simple model equation in the form

$$\phi_{tt}-\phi_{xx}+\sin\phi=-2h\sin(\phi/2)-\alpha\phi_t, \quad (4.1)$$

where ϕ is the angle variable that characterizes the domain wall, h is the external magnetic field, α is the damping parameter, and the characteristic velocity $\sim D^{1/2}$ is normalized to unity.

As $h\to 0$ and $\alpha\to 0$, Eq. (4.1) degenerates to the known sine-Gordon equation. It was shown[4,5] that one of the main solutions of the sine-Gordon equation

$$\phi=4\,\mathrm{arctg}\exp\frac{x-vt}{(1-v^2)^{1/2}}, \quad (4.2)$$

which corresponds to a domain wall moving with velocity v, is also the exact solution of (4.1) provided that the following constraint is imposed on the parameters v, h, and α:

$$v^2=\frac{h^2}{h^2+\alpha^2}. \quad (4.3)$$

According to Refs. 4 and 5, relation (2.4) determines correctly qualitatively the dependence of the domain-wall velocity on the external field until the region of limiting velocities ($v^2\sim 1$) is reached. Relation (4.3), however, hardly leads to a qualitatively correct description of the motion of the domain wall in sufficiently strong magnetic fields.

In fact, Eq. (2.1), for solutions of the type

$$\phi(x, t)=\phi(x-vt) \quad (4.4)$$

with the boundary conditions

$$\begin{aligned} &\lim\phi=0 \quad \text{as} \quad x-vt\to-\infty,\\ &\lim\phi=2\pi \quad \text{as} \quad x-vt\to+\infty, \end{aligned} \quad (4.5)$$

corresponding to the moving domain wall, has a unique solution (4.2), (4.3) only subject to the additional condition

$$h<1. \quad (4.6)$$

On going to strong external magnetic fields ($h>1$) the solution (4.2), (4.3) ceases to be unique and the boundary value problem (2.6) acquires a continuous set of solutions characterized by a continuous velocity spectrum

$$0<v^2<1. \quad (4.7)$$

In fact, Eq. (1) for the stationary-profile waves (4.4) takes the form

$$(v^2-1)\phi''-(\alpha v)\phi'+\sin\phi+2h\sin(\phi/2)=0, \quad (4.8)$$

where ϕ' denotes differentiation with respect to the independent variable $(x-vt)$. At $h<1$, the singular points

$$\phi'=0,\quad \phi=0, \quad (4.9)$$
$$\phi'=0,\quad \phi=2\pi, \quad (4.10)$$

on the (ϕ', ϕ) phase plane correspond to singular points of the saddle type, while the solution of the boundary-value problem (2.6) corresponds to a unique common separatrix. The third singular point Eq. (2.9) corresponds at $h<1$ to a singular point

$$\phi'=0, \quad \phi=2\arccos(-h) \tag{4.11}$$

of the type of node or focus.

With increasing external field h, the singular point (4.11) moves towards the singular saddle point (4.10) and reaches the latter at $h=1$. The coalescence of the singular points at $h\geqslant 1$ leads to the onset of a singular point of the node or focus type, and to the advent of a continuous velocity spectrum for the moving domain walls. The situation at $h>1$ is analogous to that investigated by Kolmogorov *et al.*[9] Moreover, if we impose on the angle variable ϕ that characterizes the domain wall the condition

$$0\leqslant\phi\leqslant 2\pi, \tag{4.12}$$

then the continuous spectrum of the velocities of the moving domain walls is bounded from below at $h>1$, namely,

$$v_{min}^2=\frac{4(h-1)}{4(h-1)+\alpha^2}\leqslant v^2<1. \tag{4.13}$$

One of the characteristic features of Eq. (4.8) is that when dissipation is taken into account ($\alpha\neq 0$) and at $h>1$ there exist solutions that satisfy the boundary conditions (4.5) at $v^2>1$.

In other words, the model in question admits of the existence of "fast" motions of the domain walls in strong magnetic fields. In particular, at $h\geqslant 1$ a domain wall can move with the limiting velocity ($v^2=1$). This motion is characterized by a finite width of the front at $\alpha\neq 0$. We note that as $v^2\to 1$ solutions of the type of moving domain walls go over continuously into the limiting solution $v^2=1$.

At $h>1$ there exists a moving domain wall that propagates with velocity $v=1$. According to (4.8) we have in this case

$$\alpha\phi'=\sin\phi+2h\sin(\phi/2). \tag{4.14}$$

Integrating the last equation, we find that

$$(h-1)\,\mathrm{th}^{h+1}\frac{\phi}{4}+(h+1)\,\mathrm{th}^{h-1}\frac{\phi}{4}=\exp\left[\frac{h^2-1}{\alpha h}(x-t)+\mathrm{const}\right]. \tag{4.15}$$

We note that the characteristic dimension of the transition layer (of the domain wall) is of the order of $\alpha h/(h^2-1)$. As $h\to 1$ from above, the solution acquires a peculiar asymptotic behavior. Namely, as $\phi\to 0$ the exponential approach to the homogeneous state with a characteristic length of the order of α is preserved, whereas as $\phi\to 2\pi$ the asymptotic behavior is algebraic

$$(2\pi-\phi)^2\sim\alpha/(x-t).$$

The implicit solution at $h=1$ is given by

$$\tfrac{1}{4}\,\mathrm{tg}^2(\phi/4)+\ln\mathrm{tg}^2(\phi/4)=(x-t)/\alpha+\mathrm{const}. \tag{4.16}$$

5. We investigate now the stability of those solutions of (4.8) which correspond to stationary motion of domain walls. Upon the following transformation of the space and time variables

$$x\to\xi=\frac{x-vt}{(1-v^2)^{1/2}},\quad t\to\tau=\frac{t-vx}{(1-v^2)^{1/2}}, \tag{5.1}$$

we find that in the reference frame moving with the domain wall the initial equation (4.1) takes the form

$$\phi_{\tau\tau}-\phi_{\xi\xi}+\sin\phi=-2h\sin\frac{\phi}{2}-\frac{\alpha}{(1-v^2)^{1/2}}(\phi_\tau-v\phi_\xi). \tag{5.2}$$

Assuming

$$\phi(\xi,\tau)=\phi_0(\xi)+\Psi(\xi,\tau), \tag{5.3}$$

where $\phi_0(\xi)$ is one of the solutions of the boundary value problem (4.5) for the equation (4.8), we find that in the linear approximation $\Psi(\xi,t)$ satisfies the equation

$$\Psi_{\tau\tau}-\Psi_{\xi\xi}+\frac{\alpha}{(1-v^2)^{1/2}}(\Psi_\tau-v\Psi_\xi)+\left(\cos\phi_0+h\cos\frac{\phi_0}{2}\right)\Psi=0. \tag{5.4}$$

The latter has solutions of the type

$$\Psi(\xi,\tau)=\Psi(\xi,\Gamma)e^{-\Gamma\tau} \tag{5.5}$$

and leads to the problem of the eigenvalues of the parameter Γ

$$\lim\Psi(\xi,\Gamma)=0\quad\text{as}\quad\xi\to\pm\infty \tag{5.6}$$

for the non-self-adjoint equation

$$-\Psi_{\xi\xi}-\frac{\alpha v}{(1-v^2)^{1/2}}\Psi_\xi+\left(\Gamma^2-\frac{\alpha\Gamma}{(1-v^2)^{1/2}}+\cos\phi_0+h\cos\frac{\phi_0}{2}\right)\Psi=0. \tag{5.7}$$

We note that a zero eigenvalue of the parameter Γ corresponds to the eigenfunction

$$\Psi_{\Gamma=0}=d\phi_0/d\xi. \tag{5.8}$$

For the previously considered[4,5] solution (4.2), (4.3) the equation (5.7) leads following the substitution

$$\Psi=e^{-h\xi/2}\Phi(\xi) \tag{5.9}$$

to the self-adjoint equation

$$\Phi_{\xi\xi}+\{E-U(\xi)\}\Phi=0. \tag{5.10}$$

Here

$$U(\xi)=-1-h\,\mathrm{th}\,\xi+2\,\mathrm{th}^2\xi, \tag{5.11}$$

and the parameter E is connected with the sought eigenvalue Γ by the relation

$$E=-\Gamma^2+(\alpha^2+h^2)^{1/2}\Gamma-\tfrac{1}{4}h^2. \tag{5.12}$$

The asymptotic boundary conditions (3.6) take in this case the form

$$\lim e^{-h\xi/2}\Phi(\xi)=0\quad\text{as}\quad\xi\to\pm\infty. \tag{5.13}$$

An investigation of the asymptotic behavior of the solutions of the boundary-value problem (5.10), (5.13) shows that, at least for $h>1$, the moving domain wall (4.2), (4.3) is unstable, since the eigenvalues of the parameter E that belong to the band

$$1-h-\tfrac{1}{4}h^2=E_-<E<E_+=-\tfrac{1}{4}h^2, \tag{5.14}$$

lead to negative values of Γ. More accurately speaking, at any value of the parameter E from the band defined by (5.14) the following three conditions are simultaneously satisfied:

(a) there is at least one solution of (5.10) satisfying the asymptotic boundary condition (5.13) as $\xi\to-\infty$;

(b) all the solutions of (5.10) satisfy the asymptotic boundary condition as $\xi\to+\infty$;

(c) the equation (5.12), which is quadratic in Γ, has one negative root.

It can be shown analogously for the solutions of (4.8) at $h>1$, which correspond to domain walls moving with arbitrary velocity $v<1$, that the band defined by relations (5.14) takes the form

$$1-h-(\alpha v)^2/4(1-v^2)=E_-<E<E_+=-(\alpha v)^2/4(1-v^2). \quad (5.15)$$

This leads again to negative Γ, i.e., points to instability of these solutions. The continuous set of negative eigenvalues is contained in the band

$$\Gamma_-<\Gamma<0,$$

where $\Gamma=\Gamma(E_-)$ is the negative root of (5.10).

One of us[11] has shown that a similar situation arises in an analysis (in the linear approximation) of the stability of the stationary-profile waves of the problem of Kolmogorov *et al.*[9]

6. A consequence of the initial equation (4.1) is the relation

$$(-{}^1/{}_2\phi_x^2-{}^1/{}_2\phi_t^2-\cos\phi-h\cos{}^1/{}_2\phi)_x+(\phi_t\phi_x)_t=-\alpha\phi_t\phi_x, \quad (6.1)$$

which degenerates as $\alpha\to0$ and $h\to0$ into one of the known divergent forms of the sine-Gordon equation. Under the boundary conditions

$$\phi(x,t)|_{x\to+\infty}=2\pi,\quad \phi(x,t)|_{x\to-\infty}=0 \quad (6.2)$$

relation (6.1) leads to the differential equation

$$dI/dt+\alpha I+8h=0 \quad (6.3)$$

for the functional

$$I(t)=\int_{-\infty}^{\infty}dx(\phi_t\phi_x). \quad (6.4)$$

The obvious solution of (6.3)

$$I(t)=-8h/\alpha+\{I(0)+8h/\alpha\}e^{-\alpha t} \quad (6.5)$$

shows that at any initial distribution of $\phi(x,t=0)$ that satisfies the boundary conditions (6.2), a stationary value of the functional (6.4) is established as $t\to\infty$:

$$\lim I(t)=-8h/\alpha \quad \text{as} \quad t\to\infty. \quad (6.6)$$

A consequence of the initial equation (4.1), besides relation (6.1), is the relation

$$({}^1/{}_2\phi_t^2+{}^1/{}_2\phi_x^2-\cos\phi-4h\cos{}^1/{}_2\phi)_t-(\phi_t\phi_x)_x=-\alpha\phi_t^2, \quad (6.7)$$

which leads at $\alpha=0$ to a differential form of energy conservation. At $\alpha\neq0$ and under the boundary condition (6.2), the integral form of (6.7) is

$$\frac{dE}{dt}=-\alpha\int_{-\infty}^{\infty}dx\,\phi_t^2. \quad (6.8)$$

Here

$$E(t)=\int_{-\infty}^{\infty}dx({}^1/{}_2\phi_t^2+{}^1/{}_2\phi_x^2-\cos\phi-4h\cos{}^1/{}_2\phi) \quad (6.9)$$

is the functional of the energy.

We note that for solutions satisfying the condition

$$\lim\phi(x,t)=\phi(x-vt) \quad \text{as} \quad t\to\infty, \quad (6.10)$$

where $\phi(x-vt)$ is one of the solutions of (4.8) with the boundary conditions (4.5), the energy functional (6.9) becomes unlimited at $t\to\infty$. Moreover, it can be shown that

$$\lim\frac{dE}{dt}=-8hv \quad \text{as} \quad t\to\infty. \quad (6.11)$$

Consequently, relations (6.11) and (6.6) coincide for the solutions that satisfy the condition (6.10).

Unfortunately, in contrast to the classical problem,[9] for which it is known that the time evolution of the initial distribution (under boundary conditions that are natural for a solitary stationary-profile wave) leads to formation of a solitary stationary-profile wave that propagates with a velocity equal to the lower boundary of the continuous velocity spectrum [the counterpart of $v_{\min}(h,\alpha)$, defined by (4.13)], there is at present no known solution for the problem considered by us.

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