Thus, even if M_2 is 10–100 times less than M_1 it can be determined in principle by measuring the rotation of the polarization plane of the light in a uniformly rotating liquid.

Relatively recently the rotation of the polarization plane of a light wave propagating through a rotating transparent medium (Pockels-glass SF-57) was detected experimentally.[10] In the experiment of Jones[10] $\Omega = 100$ Hz, $n = 1.84$, the light path-length in the medium $l = 40$ cm, the rotation of polarization plane $\varphi \sim 10^{-6}$ rad, which corresponds to $\beta \approx 2.5 \times 10^{-8}$ rad/cm. It is seen from this that in this case M_2 is of the order of $M_2 \approx 10^{-(2-3)} M_1 \sim 10^{-(14-16)}$ sec, where the values of M_1 values were taken for simple liquids in the transparency region.[8]

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The role of solitons in strong turbulence

S. F. Krylov and V. V. Yan'kov

I. V. Kurchatov Institute of Atomic Energy
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We show that in those cases where it is difficult to subdivide the scale of the turbulence, equilibrium turbulence is possible. Stratification into phases is typical for such turbulence; the role of liquid drops is played by solitons. We find the parameters for the weak-turbulence spectrum which is in equilibrium with a soliton of a given amplitude. We consider the example of a system which allows soliton solutions, but which in not stratified into phases.

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INTRODUCTION

We clarify in the present paper the reason why in strong turbulence the particular solutions in the form of solitons play such an important role. We show that the solitons, though particular solutions, are important solutions, since in a number of situations the typical behavior of strong turbulence consists in a decomposition into two phases-weakly non-linear spectrum and solitons.

The dispersion of the waves for which soliton solutions exist is usually such that the energy transfer to the short-wavelength region is difficult, and the wave system is able to evolve for a long time without damping. This fact makes it possible to arrive at a state of thermodynamic equilibrium, characterized by a collective temperature, and one can obtain exactly the shape of the spectrum in equilibrium with the solitons. The solitons play here the role of the liquid phase droplets, while the binding energy of the waves in the soliton plays the role of the vaporization heat. The reason why the soliton solution can be singled out is that the soliton guarantees the largest binding energy and, hence, also the largest entropy of the weakly nonlinear spectrum. We stress that in such considerations we are dealing with thermodynamic equilibrium only in the wave degrees of freedom; there is no equilibrium between the waves and the medium.

The picture painted above is illustrated by the example of waves close to sound waves. In those cases where a subdivision of the scales is allowed and thermodynamic equilibrium has no meaning, solitons are not necessarily attractive solutions, as is illustrated by the example of turbulence in the Rudakov-Tsytovich equation.[1]

EQUILIBRIUM OF WAVES WITH SOLITONS

We shall assume that the waves have a frequency $\omega_k = c_s|\mathbf{k}| + \delta\omega_k$, where c_s is the wave velocity when we neglect dispersion, while the dispersion correction $\delta\omega_k$ is small ($\delta\omega_k \ll \omega_k$). Initially we consider weakly non-linear spectra for which the non-linear corrections are smaller than the dispersive ones. When the dispersion is almost that of sound, only almost collinear triads of waves interact, so that there appears apart from the energy and momentum conservation laws a new conservation law—we shall call it the law of conservation of "longitudinal momentum"

$$p_{\parallel} = \sum |\mathbf{k}| n_k,$$

where n_k is the occupation number. When multiplied by c_s, the longitudinal momentum is nearly the same as the energy

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$$\varepsilon = c_s \sum |\mathbf{k}| n_k,$$

so that it is more convenient to use one of these quantities and their small difference (the correction to the energy is

$$\varepsilon' = \sum n_k \delta\omega_k).$$

The momentum conservation law is also satisfied,

$$\mathbf{p} = \sum \mathbf{k} n_k.$$

For the equilibrium distribution function we have at once (Ref. 2, p. 25 of original)

$$\ln f = -(\varepsilon' + c\varepsilon + \mathbf{pu}) T^{-1} \equiv -n_k (\delta\omega_k + u_1 |\mathbf{k}| + \mathbf{uk}) T^{-1}. \qquad (1)$$

Thermodynamic equilibrium is characterized by the parameters T, u_1, and $\mathbf{u}$. Their number equals the number of integrals of motion. Changing to the average occupation number, we get a modified Rayleigh-Jeans law:

$$N_k \equiv \int n_k f(n_k)\, dn_k \left(\int f(n_k)\, dn_k \right)^{-1} = T(\delta\omega_k + u_1 |\mathbf{k}| + \mathbf{uk})^{-1}. \qquad (2)$$

In the particular case $\mathbf{u} = 0, u_1 = c_s$, Eq. (2) gives the Rayleigh-Jeans distribution. When $|\mathbf{u}| \ll u_1$ the spectrum is almost isotropic, and when $u_1 = |\mathbf{u}|, \delta\omega_k \ll u_1 |\mathbf{k}|$ it is quasi-one-dimensional.

We now consider the interaction of solitons with the weakly non-linear spectrum starting with the isotropic case $\mathbf{u} = 0$. The soliton has no internal degrees of freedom and if the distance between the solitons is much larger than their size, their entropy can be neglected in comparison with the entropy of the weakly non-linear spectrum. The entropy of the weakly non-linear spectrum is determined by the integrals of motion and in equilibrium with the solitons reaches a maximum.

To find the equilibrium we note that the way the phase volume $\Delta\Gamma$ depends on the integrals of motion ε and ε' (in the isotropic case the momentum integral is immaterial) is given by the formula

$$\Delta\Gamma(\varepsilon, \varepsilon') = f^{-1}(\varepsilon, \varepsilon') = \exp(\varepsilon' + u_1 c_s^{-1} \varepsilon) T^{-1}, \qquad (3)$$

i.e., the phase volume increases with increasing energy ε and increasing of the integral ε' in the spectrum. If part of the waves change to a soliton the energy in the spectrum diminishes and thus the phases volume decreases. At the same time a soliton leads to the splitting off of the integral ε' which increases the phase volume. At equilibrium the variation of the phase volume (3) vanishes and we must thus take into account that in the soliton ε and ε' are rigidly connected:

$$\varepsilon' = -F(\varepsilon), \quad \delta\varepsilon' = -(dF/d\varepsilon)\,\delta\varepsilon.$$

Varying (3) and using this relation we find

$$dF/d\varepsilon = u_1/c_s. \qquad (4)$$

The function $F(\varepsilon)$ is, as a rule, concave so that for a soliton which is stronger than follows from Eq. (4) it is profitable to absorb waves—this increases the phase volume of the weakly non-linear spectrum and, hence, the entropy. Weaker solitons must gradually be dissipated under the impacts of weakly non-linear waves. We do not aim to evaluate $F(\varepsilon)$, which gives the dependence of the binding energy of the waves in the soliton on the total energy in them, for any actual cases. For the important case of two-dimensional solitons which are close to Koretweg-de Vries solitons one can obtain this dependence by using Petviashvili's work[3] and for the one-dimensional ion-sound solitons by using Karpman's work.[4]

For the consideration of the anisotropic case we note that only waves with almost collinear wave vectors interact significantly, so that for each direction one can carry out a quasi-one-dimensional consideration, projecting the conservation laws onto the chosen axis and introducing a coefficient $u_1(\theta)$ which depends on the angle. It is clear from the analysis of Eq. (2) that $u_1(\theta)$ is a minimum for those directions in which the largest number of waves is moving. This means according to Eq. (4) that the threshold for soliton formation is smallest in that direction. The growth of the solitons moving in that direction is accompanied by an isotropization of the weakly non-linear spectrum.

DIFFERENCES OF THE EQUILIBRIUM BETWEEN SOLITONS AND WEAKLY NON-LINEAR WAVES AND OF EQUILIBRIUM BETWEEN A LIQUID AND ITS VAPOR

The equilibrium between solitons and weakly non-linear waves differs from the liquid-vapor equilibrium because the attractive forces between the waves do not saturate, i.e., for the usually considered solitons $dF/d\varepsilon$ grows without bound with increasing ε. In the case of a molecular system the vaporization heat, which is the analog of $dF/d\varepsilon$, although it depends on the size of the droplet, tends to a finite limit when the size increases. Therefore in the case of non-linear waves the role of the nucleation process increases strongly—for each weakly non-linear spectrum we can indicate a soliton with an amplitude which is so large that it grows, absorbing the spectrum. The role of the dew effect grows—the weak solitons dissipate with subsequent absorption of the weakly non-linear waves by the strong solitons.

For a system of non-linear wave the concept of the phase separation boundary has no clear meaning and it is impossible to obtain an equilibrium condition by equating the temperature and the chemical potential in the soliton and outside it. For that reason one has so far not considered the equilibrium of solitons with weakly non-linear waves, although the analogy between the modulational instability and a phase transition has been noted long ago[5] and efforts have been made in that direction. Due to the absence of a phase separation boundary, it is also impossible to apply to the system of waves the proof of the impossibility of the existence of phases in one-dimensional systems, given in the last section of the book of Landau and Lifshitz.[2] Nonetheless it is apparently legitimate to call the soliton and the weakly non-linear waves phases as there is an equilibrium between them found from the condition that the phase volume be a maximum. In other words, the solitons appear then when their formation increases the entropy of the weakly non-linear waves.

We note that in our considerations we can replace the soliton by any stationary solution, for instance, a non-linear periodic wave.

STRONG TURBULENCE IN THE RUDAKOV-TSYTOVICH EQUATION

Here we wish to give an example of a system which has among its solutions stationary non-linear waves, but the typical solution does not break up into those waves and a weakly non-linear spectrum. A list of the oscillation modes described by the equation

$$(iE_t+|E|^2E)_{xx}=\alpha E, \tag{5}$$

is given in Ref. 1. One can consider Eq. (5) as basic for waves with a frequency which tends to a constant as $k\to\infty$. For that reason an entropywise profitable scale subdivision does not require energy, proceeds unimpeded, and there is no need for the formation of periodic non-linear waves[1] which would release energy.

The group velocity tends to zero when the size of the solutions is reduced, $\partial\omega/\partial k\sim k^{-3}$ so that $|E|^2$ is frozen-in, and the evolution is confined to an increase in the wavevector while the smooth envelope is conserved. This assumption was confirmed by solving (5) on a computer. A typical result for $|E|^2$ is shown in the figure where the smooth curve shows $|E|^2$ and the oscillating one ReE. The solution is given for $t=5$. We chose for the initial conditions

$$\mathrm{Re}\,E(0,\,x)=\sin x,\qquad \mathrm{Im}\,E(0,\,x)=\sin 3x.$$

The three-dimensional generalizations of Eq. (5) should behave similarly, and also waves with other kinds of non-linearity, but with the same dispersion—the main factor determining the nature of the evolution is that the frequency tends to a constant as $k\to\infty$.

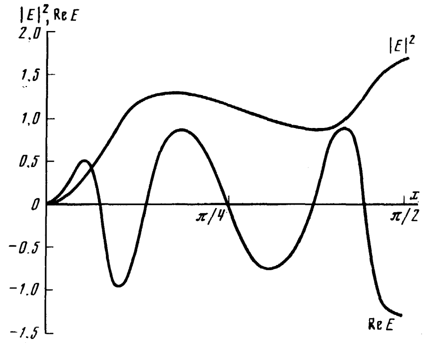

FIG. 1. Periodic solution of Eq. 5 for $\alpha=4$. We show one fourth of a period.

CONCLUSION

Valuable in the present paper are not the exact relations such as (2) and (4), but the qualitative statements about the tendencies of the evolution of turbulence which do not require complete equilibrium. The latter include the statement that the amplitude of the solitons growth as their number diminishes, that the weak solitons are absorbed by strong ones, and also that there are no stationary solutions for the evolution of waves for which a subdivision of scale is not forbidden. We have, in fact, expounded here a method for qualitatively analyzing the evolution of non-linear waves which sometimes—for the case of complete equilibrium—can be given an exact meaning. Equilibrium is impossible if there is a fast subdivision of scale (for instance, in the case of an ideal liquid or of three-dimensional Langmuir waves[6]) or if there is a "cryptolinearity," i.e., if one can by a transformation completely remove the interaction, a procedure that involves an infinite number of integrals of motion. Almost all equations of that kind are listed in Whitham's book.[7]

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