density, polarization, and so forth) differ fundamentally from the results obtained by Baryshevskiĭ *et al.*[20] This is due to the erroneous nature of their work[20] (a detailed critical analysis of Ref. 20 is given in Refs. 21 and 22).

A substantial effect on the radiation of channeled particles is exerted by secondary processes which occur in the crystal (for example, multiple scattering, etc.). Here dechanneling of the particles and some broadening of the spectrum occur. Therefore a separate article will be devoted to the discussion of this question.

In conclusion the authors express their gratitude to Yu. V. Kononets for a helpful discussion and for a number of remarks which made possible improvement of this article.

[1)]The first indications of the existence of this effect for electrons were obtained by Agan'yants *et al.*[12] In the current Soviet-American experiment at the Stanford Linear Accelerator Center (SLAC) spontaneous γ radiation was observed for positrons with energy 1–14 GeV in planar channeling of positrons through a diamond crystal.[13] For electrons and positrons of low energies (E= 28–56 MeV) the effect was measured by Datz *et al.*[23]

[2)]The real potential differs from harmonic. For more accurate calculation of the radiation it is necessary to take into account the anharmonic part of the potential. Such a calculation was carried out for the first time in Refs. 4 and 6 (see for example Fig. 1 in Ref. 6). In Refs. 4 and 6 at the same time the authors carried out an averaging over the amplitudes of oscillation of the particles in evaluation of the intensity of radiation [see for example Eqs. (5.4)–(5.6) in Ref. 6].

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Stochastic self-oscillations in parametric excitation of spin waves

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We consider a situation wherein an external monochromatic pump excites parametrically a pair of primary spin waves, each of which breaks up in turn into two secondary waves. The dynamics of the system is simulated numerically and it is shown that instability of the phase trajectories is observed in it when the initial conditions are perturbed. The values of the Kolmogorov entropy are calculated for different values of the excess over the parametric resonance threshold.

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The irreversibility of the behavior of complicated dynamic systems, consisting of a large number of particles, is due to the instability of the phase trajectories of such systems relative to some arbitrarily weak perturbation of the initial conditions. Recent mathematical investigations[1] by Smale, Anosov, Sinaĭ, and others have shown that dynamics randomization due to such an instability is possible also in systems having a small number of degrees of freedom. The stochastic behavior in a real physical model, described by only three

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variables, was observed in 1963 by E. Lorenz[2] in a study of convective flow in the atmosphere (see also the review by Rabinovich[3]).

The distinguishing feature of small randomizing systems (SRS) is that, while they require a probabilistic description they do not admit of relaxation to an equilibrium thermal distribution, in constrast to the multiparticle systems investigated in kinetic theory; nor can they be analyzed with the aid of methods of nonequilibrium statistical mechanics,[4] which presuppose the presence of a large number of degrees of freedom. The development of a probabilistic theory of SRS is only beginning (see, e.g., Ref. 5).

Small randomizing systems are not rare exceptions: it turns out that a stochastic behavior is typical[1] of models described by systems of ordinary differential equations when the number of variables in the system is larger than or equal to three. Since, however, there are no exact analytic methods of detecting such systems, the conclusion concerning the stochasticity of each concrete dynamic system should be based on results of its numerical simulation and the number of systems for which such calculations are made remains relatively small.

In the present paper we consider a system of six interacting modes in parametric microwave pumping in an antiferromagnetic crystal. It is shown by numerical simulation that exponential instability of the phase trajectories takes place in this system.

1. FORMULATION OF THE MODEL

We consider a situation wherein an external monochromatic microwave pump excites parametrically in an antiferromagnet a pair of primary spin waves (SW), each of which decays in turn into secondary waves; in addition, nonlinear dynamic interaction takes place between the primary waves and corresponds to the "phase" mechanism of the post-threshold limitation (see Ref. 6). The Hamiltonian of the system is given by

$$\begin{aligned}\mathcal{H}=\omega_{\mathbf{k}_0}(|A_{\mathbf{k}_0}|^2+|A_{-\mathbf{k}_0}|^2)+[hVe^{-i\omega_p t}A^*_{\mathbf{k}_0}A^*_{-\mathbf{k}_0}+\text{c.c.}]\\ +{}^1/_2T(|A_{\mathbf{k}_0}|^4+|A_{-\mathbf{k}_0}|^4)+2S|A_{\mathbf{k}_0}|^2|A_{-\mathbf{k}_0}|^2\\ +{}^1/_2(\Phi A_{\pm\mathbf{k}_0}a^*_{\pm\mathbf{k}_1}a^*_{\pm\mathbf{k}_2}+\text{c.c.})+\omega_{\mathbf{k}_1}|a_{\pm\mathbf{k}_1}|^2+\omega_{\mathbf{k}_2}|a_{\pm\mathbf{k}_2}|^2.\end{aligned} \tag{1}$$

It describes the following: a) parametric excitation of a pair of primary SW with wave vectors $\pm\mathbf{k}_0$ by a homogeneous external pump with amplitude h and frequency ω_p(V is the coefficient of coupling of the external pump and the SW); b) nonlinear dynamic interaction between the primary SW (the interaction amplitudes are T and S); c) decay of each of the primary waves into a pair of secondary waves with wave vectors $\pm\mathbf{k}_1$, $\pm\mathbf{k}_2$ (Φ is the interaction amplitude). The following resonance condition is assumed satisfied[1)]

$$\mathbf{k}_0=\mathbf{k}_1+\mathbf{k}_2, \quad \omega_{\mathbf{k}_0}=\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2}. \tag{2}$$

When account is taken of the damping, the equations of motion for the canonical complex amplitudes of the waves are

$$\begin{aligned}\dot{A}_{\pm\mathbf{k}_0}=-\gamma_{\mathbf{k}_0}A_{\pm\mathbf{k}_0}-i\partial\mathcal{H}/\partial A^*_{\pm\mathbf{k}_0},\\ \dot{a}_{\pm\mathbf{k}_{1,2}}=-\gamma_{\mathbf{k}_{1,2}}a_{\pm\mathbf{k}_{1,2}}-i\partial\mathcal{H}/\partial a^*_{\pm\mathbf{k}_{1,2}}.\end{aligned} \tag{3}$$

Since six modes are present, the total dynamics of the system is described by 12 real variables. To simplify the analysis, we assume that the damping of all the waves is the same: $\gamma_{\mathbf{k}_0}=\gamma_{\mathbf{k}1,2}=\gamma$. Then, if the amplitudes $|A_{\mathbf{k}_0}|$ and $|A_{-\mathbf{k}_0}|$, as well as $|a_{\pm\mathbf{k}1}|$ and $|a_{\pm\mathbf{k}2}|$ coincide at the initial of instant of time, then their equality is preserved in all succeeding instants of time. We regard this assumption as satisfied.[2)] Under these conditions the Hamiltonian contains only two combinations of the wave phases, Ψ and χ, which are defined by the relations

$$\begin{aligned}\chi=\phi_1+\phi_2-\delta, \quad \Psi=\Psi_0+\tilde{\Psi}, \quad \Psi_0=-\pi/2+\arg V,\\ a_{\mathbf{k}_{1,2}}=|a_{\mathbf{k}_{1,2}}|\exp(i\phi_{1,2}-i\omega_{\mathbf{k}_{1,2}}t),\\ A_{\pm\mathbf{k}_0}=|A_{\pm\mathbf{k}_0}|\exp\{i(\tilde{\Psi}/2\pm\delta-{}^1/_2\omega_p t)\}.\end{aligned} \tag{4}$$

We use the following dimensionless variables and additional symbols:

$$\begin{aligned}x=2|S||A_{\mathbf{k}_0}|/\gamma, \quad z=|a_{\mathbf{k}_1}|^2=|a_{\mathbf{k}_2}|^2, \quad H=h/\gamma|V|,\\ F=T/2S, \quad x_0=(H^2-1)^{1/2}, \quad x_1=2|S|\gamma/|\Phi|^2.\end{aligned} \tag{5}$$

The coefficients S and T are assumed negative, as is the case in real antiferromagnets. The time is measured in units of the reciprocal damping γ^{-1}.

Starting from Eqs. (3) and taking the foregoing assumptions into account, we can obtain a system of four ordinary differential equations for the wave amplitudes and for the introduced combinations of their phases:

$$\begin{aligned}\dot{x}=(H\sin\Psi-1)x+(x/x_0)^{1/2}z\sin(\Psi/2-\chi),\\ \dot{\Psi}=-(1+F)x+Fx_0+H\cos\Psi+z(xx_1)^{-1/2}\cos(\Psi/2-\chi),\\ \dot{z}=-[1+(x/x_1)^{1/2}\sin(\Psi/2-\chi)]z,\\ \dot{\chi}=(x/x_1)^{1/2}\cos(\Psi/2-\chi).\end{aligned} \tag{6}$$

We investigate below the phase portrait and the sequence of bifurcations of the system (6), show that its phase trajectories are unstable to small changes of the initial conditions, and calculate the Kolmogorov entropy of this system for different values of the parameter x_0.

2. SINGULAR POINTS, LIMIT CYCLES, AND THEIR BIFURCATIONS

The phase trajectories of the system (6) do not go off to infinity. The evolution of the system is accompanied by compression of the phase volume, since the following condition is valid

$$\frac{\partial\dot{x}}{\partial x}+\frac{\partial\dot{\Psi}}{\partial\Psi}+\frac{\partial\dot{z}}{\partial z}+\frac{\partial\dot{\chi}}{\partial\chi}=-2<0. \tag{7}$$

When constructing the phase portrait it is convenient to use the "polar coordinates"

$$u=x\cos\Psi, \quad v=x\sin\Psi, \quad s=z\cos\chi, \quad q=z\sin\chi.$$

Then the condition $z=0$ specifies a plane in the indicated phase space.

We list now the stationary points of the system (6):

1. An unstable stationary point is the origin $O(x=z$

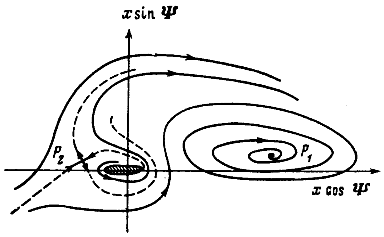

FIG. 1. Picture of the phase trajectories in the plane $z=0$.

$=0$). On the $z=0$ plane the point O is surrounded by an unstable limit cycle (see Fig. 1). The region of attraction of the point O is small, and all the trajectories of interest to us pass outside this region.

2. Two more stationary points lie on the $z=0$ plane, namely the point P_1:

$$x=x_0,\quad \Psi=\arcsin(1/H), \tag{8a}$$

and the point P_2:

$$x=\frac{F-1}{F+1}x_0,\quad \Psi=\pi-\arcsin\frac{1}{H}\quad (F>1). \tag{8b}$$

It follows from the third equation of the system (6) that none of the trajectories that start out from points on the plane $z=0$ leave this plane. In the plane $z=0$, the point P_1 is a stable focus, and the point P_2 is a saddle. The picture of the phase trajectories on the plane $z=0$ is shown in Fig. 1.

In planes orthogonal to the plane $z=0$, the stationary point P_1 is stable at $x_0<x_1$ (complex node) and unstable at $x_0>x_1$ (complex saddle), while the stationary point P_2 is stable at $x_0<x_1(F+1)/(F-1)$ (complex node) and unstable at $x_0<x_1(F+1)/(F-1)$ (complex saddle).

3. At $x_1<x_0<x_1(F+1)/(F-1)$ we have an additional stationary point, the point P_3:

$$x=x_1,\quad \Psi=\Psi_1=\arccos\{H^{-1}[(1+F)x_1-Fx_0]\},$$
$$z=(H\sin\Psi_1-1)x_1,\quad \chi={}^1/_2(\pi+\Psi_1). \tag{9}$$

It is stable at small values of the excess of x_0 over x_1 (node-focus) and becomes stable at larger values of x_0 (see below).

When the parameter $x_0=(H^2-1)^{1/2}$, that characterizes the excess above the resonance threshold ($H=1$) for primary waves, is varied, the following sequence of bifurcations is observed.

At $0<x_0<x_1$ the stationary point P_1 is stable and all the trajectories that start out outside the region of attraction of the point O contract to the point P_1. This regime corresponds to excitation of primary SW and to the absence of secondary waves, with respect to which the primary waves serve as a pump source. The oscillations near P_1 (see Fig. 1) constitute (damped) collective oscillations about a stable stationary state of the system (see Ref. 6). The value $x_0=x_1$ determines the threshold of production of secondary waves. At $x_0=x_1$ the point P_1 loses stability and a new stationary stable point P_3, which no longer lies in the $z=0$ plane, becomes separated from P_1. In this regime, the amplitude of the primary waves is frozen on the second threshold ($x=x_1$), and the amplitudes z of the secondary waves remain stationary on account of the energy flux from the primary waves. With increasing x_0, the point P_3 moves away from P_1, and at a certain value x_0 it loses its stability.[3] The loss of stability takes place in that plane in which P_3 is a focus, and as a result there a small limit cycle containing the point P_3 (which has become unstable) is produced. The onset of instability is physically due to the fact that the primary waves through which energy is transferred from the external pump to the secondary waves are incapable of carrying a very large energy flux and at the same time their amplitude remains frozen at the threshold $x=x_1$. The limit cycle increases next and at a certain value x_0 "sticks" to the separatrix of the saddle point P_2. The result is a vanishing of this limit cycle and a topological change in the character of the phase trajectory: it now encompasses a cylinder (x, Ψ).

3. NUMERICAL SIMULATION OF THE DYNAMICS

To obtain the phase portrait of the system at different values of x_0, we have integrated Eqs. (6) numerically. The obtained trajectories were fed to a plotting unit with the aid of which the projections of the phase trajectories were constructed.[4] The initial coordinates were chosen near the point P_1. The parameters F and x_1 had values $F=x_1=2$. The results of the calculations are given in Figs. 2–5.

The character of a trajectory after the destruction of the initial limit cycle is the following. If the trajectory is "let out" from a certain point near the saddle-focus P_1, then it begins to twist along a spiral towards the point P_1 in the plane $z=0$, and simultaneously moves away from it in the direction, orthogonal to this plane, of the one-dimensional separatrix of the saddle-focus P_1 (Fig. 2). With further motion, an abrupt "spill" occurs near the unstable point P_3, and the trajectory returns to the plane $z=0$, after which the cycle of twisting and stretching returns. As seen from Fig. 2b, the phase Ψ increases by 2π after each such cycle.

With increasing intensity of the external pump, i.e., with increasing value of x_0, the phase portrait of the

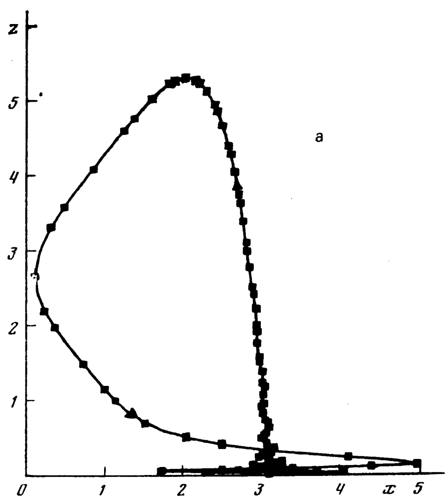

FIG. 2a. Projection of the phase trajectory on the plane (x,z) at $x_0=1.55x_1$ with $x_1=F=2$. The arrows show the direction of motion.

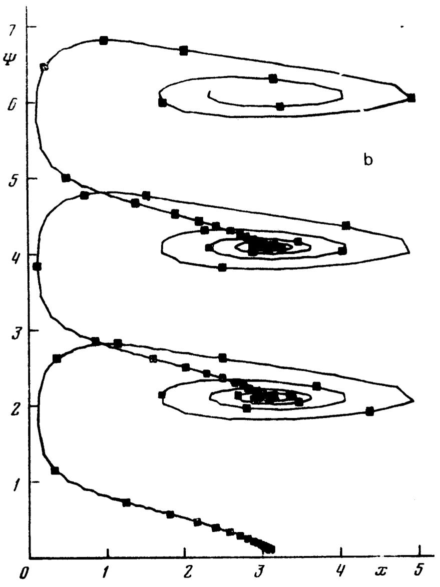

FIG. 2b. Projection of the phase trajectory on the (x, Ψ) plane for the same values of the parameters. The values of Ψ measured in units of π are marked on the ordinate axis. The squares on the curves are separated by equal time intervals.

system becomes more and more complicated (Fig. 3), and it is natural to verify whether the behavior of the system at these values of x_0 is stochastic.

4. PHASE-TRAJECTORY INSTABILITY AND KOLMOGOROV ENTROPY

To check on the assumption that the obtained self-oscillations are stochastic, we have numerically calculated the divergences of the phase trajectories. We calculated the quantity

$$k(t)=\frac{1}{t}\ln\frac{D(t)}{D(0)}, \tag{10}$$

where

$$D(t)=\{(x^{(1)}-x^{(2)})^2+(\Psi^{(1)}-\Psi^{(2)})^2+(z^{(1)}-z^{(2)})^2+(\chi^{(1)}-\chi^{(2)})^2\}^{1/2}$$

is the distance, at the instant of time t, between two phase trajectories 1 and 2 that are separated at the initial instant of time $t=0$ by a small distance $D(0)$.

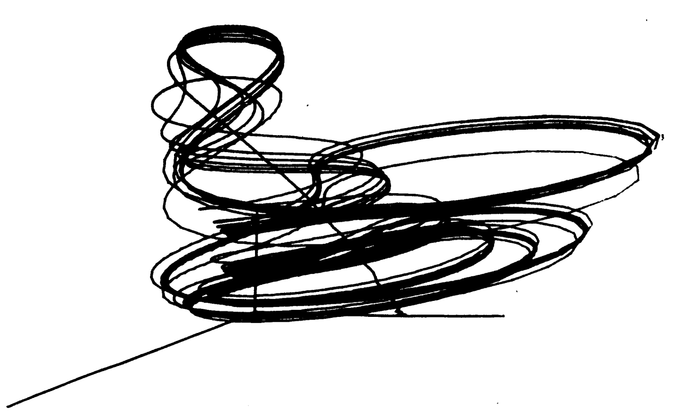

FIG. 3. Three-dimensional projection of phase trajectory on the space $(x\cos\Psi,\ x\sin\Psi,\ z)$ at parameter values $x_0=4x_1$, $x_1=F=2$.

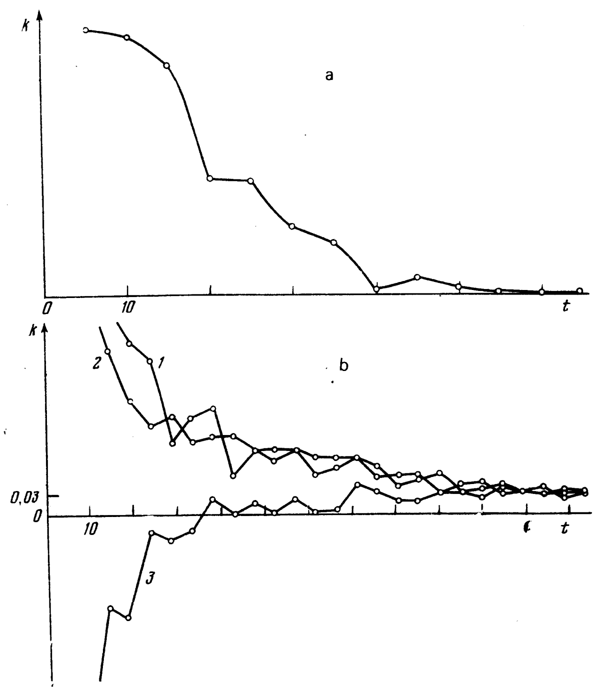

FIG. 4. Dependence of the quantity k on the time: *a*) $x_0=1{,}4x_1$, $x_1=F=2$; *b*) $x_0=4x_1$, $x_1=F=2$. Curves 1–3 correspond to different choices of the initial points of the two trajectories.

If in the course of time the trajectories are attracted to a certain stable limit cycle, the quantity $k(t)$ should tend to zero at large t. On the other hand if the motion is stochastic, then the distance between the trajectories increases exponentially with time and there should exist a nonzero positive limit[5)]

$$K=\lim_{t\to\infty}k(t). \tag{11}$$

The quantity K is customarily called in the physical literature the Kolmogorov entropy[7] [strictly speaking, K is the maximum characteristic Lyapunov number; it gives the upper bound of the true Kolmogorov entropy (see Ref. 8)]. We note that K is a dimensional quantity; it characterizes the average rate of separation of the phase trajectories.

Figure 4b shows a plot of $k(t)$ for the value $x_0=4x_1$ at various choices of the initial points of the two phase trajectories. It is seen that in the course of time $k(t)$ tends to a certain positive limit that does not depend on the initial conditions. As follows from the definition (10) of $k(t)$, this means that the distance $D(t)$ between two initially closed phase trajectories increases on the average exponentially in the course of time. Thus, in the investigated system we observe indeed instability of the phase trajectories relative to a small perturbation of the initial conditions. For comparison, Fig. 4a shows a plot of $k(t)$ for $x_0=1.4x_1$, a value for which a stable limit cycle is reached when the phase portrait is numerically constructed.

We have calculated the functions $k(t)$ and their limit values K for a number of values of the parameter x_0. The obtained values of the Kolmogorov entropy K are shown in Fig. 5. The value of K reaches a maximum in the vicinity of the point $x_0=4x_1$. At $x_0>6x_1$, alternation of stable limit cycles and of regions of stochastic behav-

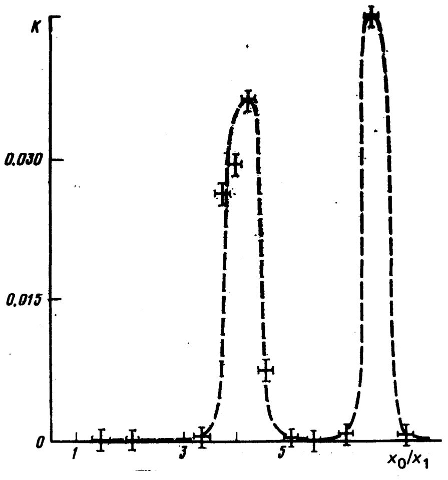

FIG. 5. Dependence of the Kolmogorov entropy K on the value of the parameter x_0/x_1.

ior are observed, in analogy with what was observed in Ref. 9 for the Lorenz model. In the vicinity of $1.4<x_0<3$ the behavior of the system is apparently stochastic ($K>0$), but the accuracy of the calculations does not make this conclusion unequivocal.

5. CONCLUSION

The onset of self-oscillations in the presence of an energy flux over the spectrum can be illustrated with the following simple example.

We consider a situation wherein an external energy source feeds only one normal mode of the excitation of the medium, while the remaining normal modes "draw" energy from this first mode. The first mode becomes unstable at a certain critical value of the power p of the external source, and the next instability—instability of one of the modes not directly coupled to the pump—sets in when the intensity N_1 of the first mode exceeds a certain critical value N_{cr}. If in the entire region of interest to us the intensities of the two modes N_1 and N_2 remain small enough, then by analogy with the mechanism of the onset of Landau turbulence we can use the model

$$\dot{N}_1=\alpha(p-p_{cr})N_1-\beta N_1^2-\eta N_2, \qquad \dot{N}_2=\nu(N_1-N_{cr})N_2. \tag{12}$$

Elementary analysis shows that at small values of the excess above the threshold of the first instability, a stable stationary state is established in the model (12), wherein the second mode is not excited ($N_2=0$), and the intensity of the first mode is limited because of the effect of the positive nonlinear damping and is equal to $\overline{N}_1=(\alpha/\beta)(p-p_{cr})$.

The new stationary state is established when, with increasing power p of the external source, the intensity $\overline{N}_1$ of the first mode exceeds the critical value N_{cr}. In this new stationary state, the intensity of the first mode is quenched at the threshold value N_{cr}, and the intensity of the second mode differs from zero and increases with increasing power p fed into the medium:

$$N_1=N_{cr}, \quad N_2=[\alpha(p-p_{cr})-\beta N_{cr}]N_{cr}/\eta. \tag{13}$$

An investigation of the stability of the stationary state (13) relative to small deviations of the intensity N_1 and N_2 shows that it is stable only at small excesses above the threshold of generation of the second mode $p_2=p_{cr}+(\beta/\alpha)N_{cr}$, when the power p of the external source satisfies the condition $p_2<p<p_2+(\beta/\alpha)N_{cr}$. At higher values of p, the system (12) does not have even one stationary stable state, and since the phase trajectories in this system do not go off to infinity, the system has at $p>p_{cr}+2(\beta/\alpha)N_{cr}$ a stable limit cycle—periodic self-oscillations of the intensities of the two modes.[6)]

Compared with the simple model (12), the situation in parametric excitation of SW turns out to be more complicated in that the limitation of the growth of the first unstable state—a pair of primary parametrically excited SW—is insured not by the positive nonlinear damping, but by a phase mechanism[6] of dynamic origin and connected with the partial forcing out of the external pump from the sample. It is important, in addition, that the first unstable mode has in this case two degrees of freedom and is characterized by two slow variables—amplitude x and time phase Ψ.

Nonetheless, the sequence of the instabilities remains qualitatively the same as in the simple example (12). At relatively small excess above the threshold of parametric resonance, no secondary waves are excited and a stable stationary state of the phase theory $x=(H^2-1)^{1/2}$, $\sin\Psi=1/H$ is realized. If the amplitude of the primary waves x in this stationary state exceeds the critical value x_1, it becomes unstable and a new stationary state appears, in which the amplitude x is "frozen" at the threshold value $x=x_1$. Being stable when produced, the new stationary state loses, however, its stability at a certain higher external-pump power. Just as in the case considered above, the loss of stability is connected with the fact that, in analogy with the mechanism of the positive nonlinear damping in the model (12), the phase mechanism of the post-threshold limitation is incapable of maintaining the system stable under conditions when an ever more intense energy flux to the secondary modes must flow through the primary mode with a "frozen" amplitude.

The resultant self-oscillations of the amplitudes x and z and of the phases Ψ and x are at first regular. The randomization of the self-oscillation takes place when the swing of the oscillations of the temporal phase Ψ reaches a value 2π. The phase portrait of the system then acquires the form of an "uncoiling" spiral and turns out to be close in character to the "spiral chaos" previously observed[10] for a number of models with chemical reactions. We note also that the results obtained in the present paper confirm the conclusion stated in the review[3] that the decay processes play an important role in the onset of turbulence.

Different self-oscillations are quite frequently observed in parametric excitation of spin waves. The observed effect of stochastic self-oscillations can take place in ferromagnets and antiferromagnets when the conservation laws allow the decay of parametrically excited spin waves into two spin waves, into a pair of spin waves and a phonon, or into two phonons. It must be emphasized that the results obtained in this paper are applicable directly only to experiments with very small samples, in which only one pair of parametrically excited spin waves appears above the threshold. If we use the estimates given in Ref. 11 for the nonlinear self-action amplitudes S and Φ for ferromagnets in the case of decay into two spin waves ($S \sim \mu_0 M_0/\mathcal{N}$, $\Phi \sim \mu_0 M_0/\mathcal{N}^{1/2}$, where $\mathcal{N}$ is the number of unit cells in the crystal, and M_0 is the magnetic moment per unit volume), then we can obtain for the threshold of the onset of the self-oscillations ($x_0 \sim x_1$) the estimate $\Delta h/h_c \sim \hbar\gamma/\mu_0 M_0$, and consequently the self-oscillations should occur immediately after the threshold of the parametric resonance is exceeded. Actually, however, in experiments on parametric excitation of the spin waves above threshold, there is always excited a rather wide packet of waves.[12] If the individual waves in the packet are strongly enough correlated with one another, one can expect the qualitative picture of the phenomenon to remain the same (but the threshold of the stochastic self-oscillations increases compared with the estimate given above, by a factor equal to the number of individual waves in the packet). On the whole we hope the results of the present paper to attract stronger attention to the study of self-oscillations in parametric excitation of spin waves in magnetically ordered crystals.

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1)We assume that the dimensions of the crystal are small enough and that modes with other (discrete) wave vectors do not enter in the resonance region.

2)As a check, we simulated numerically the dynamics of the system without assuming equality of the amplitudes and of the damping of the secondary waves; this simulation revealed no qualitative deviations from the results presented below.

3)The value of x' is given by a root of a transcendental equation; at $F=2$ and $x_1=2$ it lies in the interval.

4)This part of the work was performed at the Computation Center of the Institute of Physics Problems of the USSR Academy of Sciences. The authors thank the staff of the Computation Center for help with the calculations.

5)Usually, in the determination of K, one must not use too large values of t, for if the phase trajectories do not go off to infinity, the distance between them cannot increase without limit. Since in the definition of $k(t)$ we do not regard the phases Ψ or χ that differ by an integer multiple of 2π as identical, no such difficulties arise here.

6)The system (12) cannot have a stochastic behavior, since it contains only two variables.

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ERRATA

Erratum: Spatial dispersion of spin susceptibility of conduction electrons in a superconductor [Sov. Phys. JETP 49, 291–295 (1979)]

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Equation (5.2) on p. 294 should read:

$$\chi(r) = -\frac{g^2\mu_B^2}{2}\left(\frac{N(0)\pi}{p_0 r}\right)^2 T\sum_\omega \left\{\left[\frac{u^2}{1+u^2}\cos(2p_0 r + 2\Phi) + \frac{1}{1+u^2}\right]\right.$$
$$\times\exp\left[-\frac{r}{l_p}\left(1-\frac{l_p}{l_s}+\frac{\Delta(1+u^2)^{1/2}}{\varepsilon_0}p_0 l_p\right)\right] + \frac{3r}{l_p}\frac{1}{1+u^2}\left(1-\frac{l_p}{l_s}+\frac{\Delta(1+u^2)^{1/2}}{\varepsilon_0}p_0 l_p\right)$$
$$\left.\times\exp\left\{-\frac{r}{l_p}\left[3\left(1-\frac{l_p}{l_s}+\frac{\Delta(1+u^2)^{1/2}}{\varepsilon_0}p_0 l_p\right)\left(\frac{\Delta(1+u^2)^{1/2}}{\varepsilon_0}p_0 l_p - \frac{2l_p}{3l_s}\frac{2u^2+1}{1+u^2}\right)\right]^{1/2}\right\}\right\}.$$

