# Stationary nonequilibrium ion distributions produced in interaction with an electron thermostat

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We discuss the stationary distribution of hot ions with energies $\varepsilon > T(M/m)^{1/3}$, at which interaction with the electron subsystem, which plays the role of a thermostat, prevails. The effective ion temperatures established in the interaction with a nonequilibrium electron gas are determined. For ion sources localized in energy (beams, nuclear reactions), the distribution contains an equilibrium part, which goes over into a power-law "tail" parametrized by the flux along the spectrum. It is shown that the obtained power-law distributions are stable under local isotropic perturbations. Formation of a power-law distribution of the ions under nonstationary and inhomogeneous conditions is considered, particularly in the case of excitation by a beam. It is shown that in the interiors of stars the power-law tails of the distributions influence the rates of the secondary nuclear reactions with large Coulomb barriers.

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## 1. INTRODUCTION

Stationary states of a plasma, produced in the presence of sources such as nuclear reaction or beams of high energy particles, can greatly deviate from equilibrium in definite energy intervals even if the source power is low. In particular, the plasma can become considerably enriched with high-energy ions, as is manifest in the simplest cases by the onset of power-law "tails" in the particle velocity distribution.[1–4] As already noted, the presence of such tails should greatly influence a number of properties of the plasma, including the dispersion properties, as well as the rates of the thermonuclear reactions in the plasma (see, e.g., Ref. 2). The flow of hot electrons and ions from sources plays an important role also in a low-temperature plasma.[5,6]

A considerable role in the formation of the distribution tails can be played by thermal particles. In particular, in the region of sufficiently high ion energies of interest for thermonuclear plasma

$$\varepsilon > T(M/m)^{1/3} \quad (T=T_e) \tag{1.1}$$

the ion–electron energy transfer predominates[7] and the energy relaxation of the ion, which is characterized by the reciprocal relaxation time

$$\nu_{ie}^{\varepsilon} \sim \begin{cases} \nu, & v \ll v_{Te} \\ \nu(v_{Te}/v)^3, & v \gg v_{Te} \end{cases}, \tag{1.2}$$

$$\nu=\sigma_{ie}n_e v_{Te} m/M, \quad \sigma_{ie}=\pi e^4 z^2 \lambda/T^2,$$

is determined from the interaction with the electrons:

$$\nu_{ie}^{\varepsilon} \gg \nu_{ii}=\sigma_{ii}n_i v, \quad \sigma_{ii}=\pi e^4 z^4 \lambda/\varepsilon^2.$$

For electrons, at the same time, only the interelectron interactions are of importance. Under these conditions, the kinetic equations for the ions become much simpler (if the electron distribution can be assumed given). Without allowance for diffusion in velocity space, such an equation was considered in Refs. 1 and 4, where it was used to analyze the relaxation of a high-energy ion beam in an equilibrium plasma at $v_{Ti} \ll v \ll v_{Te}$.

In this paper we discuss the differential kinetic equations for the distribution function of high-energy ions $v \gg v_{Ti}$; in addition to the electron-friction force, we take account in this equation also the diffusion (Sec. 2). This makes it possible to describe explicitly the transition from the equilibrium part of the distribution to the power-law tail, with transition region determined by the relation between the source power and the temperature (and the density) of the plasma (cf. Refs. 8 and 9); the transition occurs for a number of astrophysical and laboratory situations in the region (1.1) (Sec. 3).

We obtain below stationary solutions for an isotropic ion distribution function in the presence of a flux, over the spectrum, from sources localized at high energies. In the energy interval

$$T(M/m)^{1/3} < \varepsilon < TM/m \tag{1.3}$$

the distribution of the ions in the tails is $f \sim |J|\varepsilon^{-3/2}$ (in agreement with the result of Ref. 1), whereas at

$$\varepsilon > TM/m \tag{1.4}$$

the distribution flattens out, $f \sim |J|$ (Sec. 3), where $4\pi J$ is the ion flux along the spectrum and is equal to the source power. The proportionality of the ion density in the dietribution to the flux [and not to its square root as in Ref. 2, see (3.6)] is due to the fact that at the small fluxes considered in the present paper the tail is formed in collisions between ions and thermal electrons, while the ion–ion collisions of the particles in the tail are inessential.

If the electrons are not in equilibrium, then under stationary conditions the electron subsystem plays the role of a thermostat with an effective temperature that is a functional of the electron distribution. (A similar situation was investigated for a system of interacting electrons and radiation.[10,8]) In the absence of an ion source, the interaction with the electrons leads to the establishment of an "equilibrium" ion distribution with effective temperatures that are different, generally speaking, in the intervals (1.3) and (1.4) (Sec. 3). The power-law distributions of the ions in the tail turn out to be stable against isotropic local perturbations (Sec. 4). Sections 5 deals with the influence of the power-law tails on the rates of second-

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ary nuclear reactions in stars. It turns out that for reactions with a large Coulomb barrier it becomes important to take the disequilibrium into account. We discuss briefly the role of the spatial inhomogeneities (using an ion beam as an example), and of the nonstationary character of the sources, in the formation of the flux distributions (Sec. 6).

## 2. KINETIC EQUATION FOR ISOTROPIC DISTRIBUTIONS OF HOT IONS

We consider the Landau kinetic equation for the ion distribution function $f^i = f$ in a plasma at sufficiently high ion energies (1.1) when, as is well known,[7] the ion-energy distribution is governed by the interaction with the electrons. It takes the form of the continuity equation in the momentum space of the ions:

$$\frac{\partial f_p}{\partial t} + \mathrm{div}_p \mathbf{j} = G_p, \tag{2.1}$$

where $G_p$ is the density of the particle sources. For an isotropic distribution of the ions $f(\varepsilon)$ and of the electrons $f^e(\varepsilon')$ the Landau expression[7] for $j_i$ can be transformed into $j_i = \pi_i j(p)$, where

$$j(p) = \alpha v \int d\mathbf{p}' \frac{v'^2[\pi\pi']^2}{|\mathbf{v}-\mathbf{v}'|^3}\left(f\frac{\partial f^e}{\partial \varepsilon'} - f^e\frac{\partial f}{\partial \varepsilon}\right), \quad \pi = \frac{\mathbf{p}}{p}. \tag{2.2}$$

Integrating over the angles, we obtain[11] for the radial component of the flux density:

$$j(p) = \frac{\alpha\pi}{3v^2}\int \frac{dp' p'^2}{v'}[v+v'-|v'-v|]^3\left(f\frac{\partial f^e}{\partial \varepsilon'} - f^e\frac{\partial f}{\partial \varepsilon}\right), \tag{2.2'}$$

here $\alpha = \pi e^4 z^2 \lambda$, $\lambda$ is the Coulomb logarithms, and the primed quantities pertain to electrons.

Introducing the diffusion coefficient $D(p)$ and the friction force $F(p)$:

$$\left\{\begin{matrix} D(p) \\ F(p) \end{matrix}\right\} = -\frac{\pi\alpha}{3v^3}\int dp' \frac{p'^2}{v'}[v'+v-|v'-v|]^3 \left\{\begin{matrix} f^e \\ v\partial f^e/\partial\varepsilon' \end{matrix}\right\}, \tag{2.3}$$

we write down the final equation for the isotropic distribution function of the ions under conditions when scattering by electrons prevail, in the form

$$\frac{\partial f}{\partial t} = -\frac{1}{p^2}\frac{\partial}{\partial p}J(p) + G(p), \quad J(p) = p^2 j(p) = -p^2\left[D(p)\frac{\partial f}{\partial p} - F(p) f\right]. \tag{2.4}$$

We consider first the stationary solution of (2.4) in the absence of an ion source ($G = 0$). The electron distribution is assumed stationary but, generally speaking, not in equilibrium. Equation (2.4) reduces to the condition $J(p) = \mathrm{const}$, and in view of the absence of a source the constant is equal to zero (cf. Sec. 3):

$$J(p) = p^2 F(p)\left[\frac{T_{\mathrm{eff}}}{v}\frac{\partial f}{\partial p} + f\right] = 0. \tag{2.5}$$

The effective temperature $T_{\mathrm{eff}}$ is defined by

$$T_{\mathrm{eff}}(v) = -vD(p)/F(p) \tag{2.6}$$

and coincides in the case of a Maxwellian distribution, as seem from (2.3), with the electron temperature. From (2.5) we obtain the stationary distribution of the ions:

$$f(\varepsilon) = C\exp\left(-\int\frac{d\varepsilon}{T_{\mathrm{eff}}(v)}\right). \tag{2.7}$$

It is seen from (2.3) that if the ion velocity $v$ is less than the characteristic electron velocity $\bar{v}_e$, then

$$D(p) \sim \mathrm{const}, \quad F(p) \sim -v, \quad v \ll \bar{v}_e. \tag{2.8}$$

For equilibrium electrons, as is well known,

$$D(p) = \frac{2}{3}\left(\frac{2}{\pi}\right)^{1/2}\frac{\alpha n_e m^{1/2}}{T^{1/2}}, \quad F(p) = -\frac{2}{3}\left(\frac{2}{\pi}\right)^{1/2}\frac{\alpha n_e m^{1/2} v}{T^{3/2}}, \tag{2.8'}$$

where $n_e = \int d\mathbf{p}' f^e$ is the particle density. It follows from (2.8) that in the interval (1.3) the distribution (2.7) takes the form of a uniform distribution with temperature $T^{(1)} = T_{\mathrm{eff}}(0)$:

$$T^{(1)} = -\int_0^\infty dv' v' f^e \Big/ \int_0^\infty dv' v' \frac{\partial f^e}{\partial \varepsilon'}, \quad v_{Ti}\left(\frac{M}{m}\right)^{1/4} \ll v \ll \bar{v}_e. \tag{2.9}$$

In the limiting case of high ion velocities $v \gg \bar{v}_e$ we have

$$D(p) \sim 1/v^3, \quad F(p) \sim -1/v^2, \tag{2.10}$$

including, for equilibrium electrons,

$$D(p) = 2\frac{\alpha n_e T}{mv^3}, \quad F(p) = -2\frac{\alpha n_e}{mv^2}. \tag{2.10'}$$

The distribution of the ions for these velocities also becomes Maxwellian with a temperature $T^{(2)} = T_{\mathrm{eff}}(\infty)$, more sensitive to the high-energy electrons than $T^{(1)}$:

$$T^{(2)} = -\int_0^\infty dv' v'^4 f^e \Big/ \int_0^\infty dv' v'^4 \frac{\partial f^e}{\partial \varepsilon'}, \quad \bar{v}_e \ll v. \tag{2.11}$$

In expressions (2.9) and (2.11) for $T^{(1,2)}$ it is assumed that $f^e(v')$ ensures convergence of the integrals.

Thus, when ions interact with nonequilibrium electrons and there is no ion source, equilibrium distribution of the ions is established in each of the integrals (1.3) and (1.4), with effective temperatures $T^{(1,2)}$ that are functionals of the electron distribution. A similar situation is realized in the interaction of electrons with radiation (see the reviews[10,12]).

In the presence of given ion sources, the stationary solution of the kinetic equation (2.4) takes the form

$$f(p) = C\exp\left\{-\int_{p_{min}}^{p}\frac{v\,dp}{T_{\mathrm{eff}}(v)}\right\} - \int_{p_{min}}^{p}\frac{dp'}{p'^2 D(p')}\exp\left\{-\int_{p'}^{p}\frac{v''dp''}{T_{\mathrm{eff}}(v'')}\right\} \times\left[J(\infty) + \int_\infty^{p'} dp'''(p''')^2 G(p''')\right]. \tag{2.12}$$

The solution (2.12) presupposes satisfaction of the condition of the matching of the source and the sinks, which is an obvious consequence of (2.4):

$$J(\infty) - J(p_{min}) = \int_{p_{min}}^{\infty} dp p^2 G(p). \tag{2.13}$$

We assume that there is no flux at infinity: $J(\infty) = 0$.[1)] For the cases of physical interest, when the source is a beam of particles or a nuclear reaction, the source can be regarded as given and localized at energies $\varepsilon_0$ much higher than thermal, for example, using the representation

$$G(p) = \frac{I}{4\pi p_0^2}\delta(p - p_0), \tag{2.14}$$

where $I$ is the strength of the source. As to the sink, unless specially stipulated, we assume that it is outside the considered region at $T^{1,2}$, in contrast to the distributed sink in Ref. 4.

## 3. STATIONARY DISTRIBUTIONS WITH FLUX ALONG THE SPECTRUM

We now discuss the case when the sources of the ions are concentrated in an energy region much larger than that of the sinks. A flux of particle numbers from the sources to the sinks is then produced over the spectrum. This region can naturally be called the inertial interval, in analogy with the Kolmogorov turbulence in a liquid, or the Zakharov weak turbulence in a system of waves with dispersion.[12,13]

The solution in the inertial interval can be obtained as an intermediate asymptotic form of the general solution (2.12). It is more convenient, however, to obtain it in terms of the flux along the spectrum from Eq. (2.4). This equation makes it possible to take simultaneously into account also the role of the thermostat, which is quite significant. In the inertial interval $G=0$ and the stationary solution corresponds to the vanishing of the collision integral or, equivalently, to constancy of the ion flux along the spectrum, $J(p)=J$. In the energy region (1.3) (region I) the equation takes the form

$$T\frac{\partial f}{\partial \varepsilon}+f=-\frac{2J\varepsilon^{-3/2}}{\nu_1(2M)^{3/2}},\qquad T(M/m)^{1/3}<\varepsilon<\min(\varepsilon_0, TM/m),\quad \nu_1=\frac{4}{3\pi^{1/2}}\nu. \tag{3.1}$$

The solution of this equation is a sum of the Maxwellian term $f^M$ (the solution of the homogeneous equation) and the power-law "tail" $f^{\mathrm{pl}}$, which describes the flux of the ions towards the lower energies (Fig. 1):

$$f=f^M+f^{\mathrm{pl}}=\frac{n}{(2\pi MT)^{3/2}}\exp(-\varepsilon/T)+\frac{2|J|\varepsilon^{-3/2}}{\nu_1(2M)^{3/2}},\qquad J<0. \tag{3.2}$$

It is recognized here that our analysis, including the initial equation, is itself valid only at $\varepsilon\gg T$.

The conditions under which the power-law term in (3.2) predominates include limitations on the flux (if $\varepsilon_0>TM/m$, then it must be replaced by $TM/m$):

$$\varepsilon_0^{3/2}\exp(-\varepsilon_0/T)/2(\pi T)^{3/2}\ll|J|/\nu_1 n\ll 1.$$

The left-hand side of the inequality corresponds to the fact that the source is located at higher energies than the "joining point" $\varepsilon_*$:

$$\varepsilon_0>\varepsilon_*\approx T\ln\frac{\nu_1 n}{2\pi^{3/2}|J|},$$

at which the thermal and flux terms of (3.2) become equalized. The right-hand side of the inequality corresponds to the condition that the number of particles in

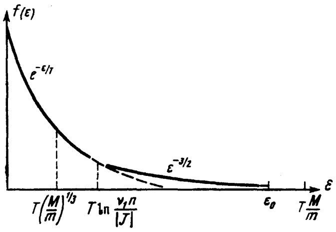


FIG. 1. Distribution function $f(\varepsilon)$ of the ions in region I ($v \ll v_{Te}$) under the condition $n\nu_1\gg|J|\gg n\nu_1(\varepsilon_0/T)^{3/2}\exp(-\varepsilon_0/T)$. The power-law tail $f^{pl}\sim|J|^{-3/2}$ is formed in the presence of a flux $J$ along the spectrum from a particle source localized at $\varepsilon=\varepsilon_0<TM/m$.

the tail be small compared with the total number of particles. The number of particles in the tail

$$n_t^{(1)}\approx 8\pi|J|\tau_1\ln p_0/p_T \tag{3.3}$$

is determined by the strength of the source (the flux along the spectrum), multiplied by the relaxation time $\tau_1\equiv\nu_1^{-1}$. The lower limit of the region of the coordinate of the sink is replaced in (3.3), with logarithmic accuracy, by the thermal momentum.

Our analysis, wherein the relaxation is effected by ion-electron collisions, is valid only if the number of ions in the tail is much less than the number of thermal electrons (and accordingly, ions). In fact, let us compare the frequencies of the relaxation of the tail ions on electrons and on ions belonging to the same tail. The condition

$$\nu_{ie}^{\varepsilon}\gg\nu_{ii}^{t}\sim e^4z^4\lambda n_t/\varepsilon^{3/2}M^{1/2} \tag{3.4}$$

imposes on the flux a limitation equivalent to smallness of the number of particles in the tail, $n_t\ll n$:

$$|J|\tau_1\ll n. \tag{3.5}$$

If the flux is so large that $|J|\gg\nu_1 n$, then the picture changes substantially. For quasi-equilibrium electrons, ion-ion collisions predominate, and the transport is essentially integral. The problem was solved under these conditions in Ref. 2. For a high-energy source, the distribution of the ions did correspond to a constant energy flux $J_{\mathrm{en}}$ along the spectrum (the contribution of the particle flux in this region can be neglected), and according to Ref. 2 we have

$$f\sim|J_{\mathrm{en}}|^{1/2}\varepsilon^{-5/4},\qquad |J|>\nu_1 n. \tag{3.6}$$

The square-root dependence on the flux is due to the fact that the spectrum is caused by collisions of the tail particles with one another (for details see Ref. 2, as well as Ref. 12) and, as we have seen above, it is replaced by a linear dependence for small fluxes.

In the energy region (1.4) (region II) the stationary distribution of the ions, corresponding to a constant flux, is obtained from the equation

$$T\frac{\partial f}{\partial\varepsilon}+f=-\frac{2J}{\nu_2(2MT)^{3/2}},\qquad \nu_2=\nu\left(\frac{M}{m}\right)^{3/2} \tag{3.7}$$

and takes the form (Fig. 2)

$$f=\frac{n}{(2\pi MT)^{3/2}}\exp\left(-\frac{\varepsilon}{T}\right)+\frac{2|J|}{\nu_2(2MT)^{3/2}},\qquad T\frac{M}{m}<\varepsilon<\varepsilon_0. \tag{3.8}$$

The flux term in (3.8) is significant if $|J|>\nu_2 n\exp-\varepsilon_0/T$). For fluxes such that

$$\exp(-M/m)>|J|\nu_2 n>\exp(-\varepsilon_0/T),$$

the flux distribution joins up with the equilibrium distribution in the region II [see (3.8)]. On the other hand if

$$|J|\nu_2 n>\exp(-M/m),$$

then the plateau of (3.8) goes over at $\varepsilon\sim TM/m$ into the power-law tail of region I (Fig. 3). The appearance of high-energy power-law tails in the distribution functions is thus a rather general consequence of the large distance between the high-energy sources and the sinks at low energies. Under these conditions, a constant flux is produced (by virtue of the conservation law) along the spectrum of the hot particles.

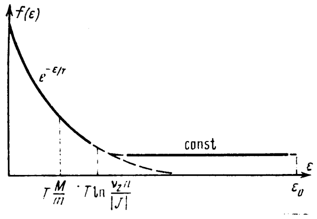


FIG. 2. Ion distribution function $f(\varepsilon)$ in the region II ($v \gg v_{Te}$) at fluxes $n\nu_2 \exp(-M/m) \gg |J| \gg n\nu_2 \exp(-\varepsilon_0/T)$. The tail of this distribution takes the form of a plateau at $f^{pl} \sim |J|$.

The appearance of standard tails is also quite typical of the electron distribution.[5,6] These tails should arise if an electron source exists (for example, an electron of neutral beam) at energies $\varepsilon_0^e \gg T$. We consider in this connection the influence of the tails of the electron distribution on the ion distribution. If the number of particles in the electron tail is small, $n_t^e/n_e \ll 1$, then the distribution of the electrons in the region $T \ll \varepsilon \ll \varepsilon_0^e$, where $\varepsilon_0^e$ is the energy of the electron source, is formed on account of the interaction of the electrons from the region of the tail with the thermal electrons. It is described by an equation similar to (2.4) and (2.10), where $f$ must be replaced by the distribution of the electrons of the tail, and $f^e$ by the Maxwellian distribution. The solution of this equation is analogous to (3.8):

$$f^e=\frac{n_e}{(2\pi mT)^{3/2}}\exp(-\varepsilon/T)+\frac{2|J_e|}{\nu_{ee}(2mT)^{1/2}}\qquad \nu=\frac{2^{1/2}\pi e^4\lambda n_e}{m^{1/2}T^{3/2}}. \qquad (3.9)$$

The power-law tail of the electron distribution leads to a dependence of the effective temperature of the ions on the velocity in accordance with (2.6). This influence is small in region I and can become substantial in region II, where the maximum value of the temperature is reached at $v > v_0^e$:

$$T_{max}=T\left(1+\frac{2}{5}\frac{n_t^e}{n}\frac{\varepsilon_0^e}{T}\right),\quad n_t^e=\frac{8\pi}{3}|J_e|\tau_{ee}\left(\frac{\varepsilon_0^e}{T}\right)^{1/2}.$$

Under the condition

$$n_eT/\varepsilon_0^e \ll n_t^e \ll n_e$$

the effective temperature of the ion is much higher than that of the electrons.

We note that there is also another region, that of hot ions (region III):

$$T\ll\varepsilon\ll T(M/m)^{1/3}, \qquad (3.10)$$

where a flux distribution can be formed and is essential for joining the tail to the equilibrium part of the distribution. In this energy region the main energy-transfer processes are ion-ion collisions. If we assume the region (3.10) to be broad enough (this is the case for heavy ions), then we can again separate the thermal and hot ions, and the flux in the kinetic equation for $f$ coincides in form with (2.4) and (2.8). The stationary solution is a sum of a Maxwellian and a power-law part

$$f=\frac{n_i\exp(-\varepsilon/T_i)}{(2\pi MT_i)^{3/2}}+\frac{2|J|\tau_{ii}}{(2MT_i)^{1/2}},\qquad \tau_{ii}^{-1}=\frac{2^{1/2}\pi e^4z^4\lambda n_i}{M^{1/2}T_i^{3/2}}$$

and the nonequilibrium part of the distribution function becomes noticeable above the equilibrium background at $|J| > (n_i/\tau_{ii})\exp[-(M/m)^{1/3}]$.

## 4. STABILITY OF FLUX DISTRIBUTIONS

We examine now the stability of the stationary solutions with respect to local perturbations. To this end, we turn to the nonstationary equation (2.4), which takes in the integral (1.3) the form

$$\frac{\partial f}{\partial t}=x^{-1/2}\frac{\partial}{\partial x}x^{3/2}\left(\frac{\partial f}{\partial x}+f\right),\qquad x=\frac{\varepsilon}{T}, \qquad (4.1)$$

where $t$ is measured in units of $\tau_1 \equiv \nu_1^{-1}$. Putting

$$f(x,t)=f_0(x)+g(x)e^{\alpha t}, \qquad (4.2)$$

we obtain for the perturbation $g(x)$ the equation

$$x\frac{d^2g}{dx^2}+\left(x+\frac{3}{2}\right)\frac{dg}{dx}+\left(\frac{3}{2}-\alpha\right)g=0. \qquad (4.3)$$

The linearly-independent solutions of this equation are hypergeometric functions,[14] which satisfy the condition of locality of the perturbations

$$g(x)/f_0(x)=o(1),\quad x\to\infty,\quad x\to 0, \qquad (4.4)$$

where $f_0(x)$ is the stationary solution of (4.1), but at $\alpha < 0$. This result can be made clearer by considered the asymptotic region of large $x$. At a power-law decrease of $g(x)$ at infinity, the second derivative in (4.3) can be neglected compared with the first, and the decreasing solution actually has a power-law form (at $\alpha < 3/2$)

$$g(x)\sim x^{\alpha-3/2}.$$

The requirement (4.4) that the initial perturbation be local, i.e., that $g(x)$ decrease faster than the solution $f_0(x) \sim x^{-3/2}$ tested for stability, means that $\alpha$ should be negative and, thus, the solution (3.2) is stable. The conclusion that the power-law tail is stable in region I agrees with the result of Ref. 15 that the power-law solution[9,15] of the Kompaneets equation for the photon distribution is stable.

For the perturbations $g(x)$ of the solution (3.8) in region II we obtain the equation

$$\frac{d^2g}{dx^2}+\frac{dg}{dx}-\alpha x^{1/2}g=0. \qquad (4.5)$$

Making the substitution $g(x)=e^{-x/2}u(x)$, we reduce (4.5) to a Schrödinger equation

$$\frac{d^2u}{dx^2}-\left(\frac{1}{4}+\alpha x^{1/2}\right)u=0. \qquad (4.6)$$

In the case $\alpha > 0$, this equation does not have bounded solutions.[16] Consequently, the solution (3.8) is also stable.

We note that the investigated solutions belong to the class of the so-called intermediate asymptotics, and general problems in their theory, including stability, are considered in Ref. 17.

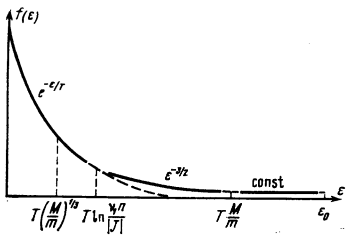


FIG. 3. Ion distribution function $f(\varepsilon)$ at fluxes $n\nu_1 \gg |J| \gg n\nu_1(M/m)^{3/2}\exp(-M/m)$, $\varepsilon_0 > TM/m$. The power-law asymtotics are formed in regions I and II.

## 5. INFLUENCE OF DEVIATION FROM EQUILIBRIUM IN THE INTERIOR OF A STAR ON THE RATE OF SECONDARY NUCLEAR REACTIONS

In connection with the problem of solar neutrinos,[18] deviations from equilibrium distributions in the interior of the sun have been discussed in recent years.[19] We examine the influence of the excess of high-energy particles (3.2) on the rate of nuclear reactions. Since the energy $\varepsilon$ of the particles produced in nuclear reactions is of the order of several MeV and the thermal energy $T$ the order of several keV, there exists an energy interval (1.3) in which the ion distribution is the result of their interaction with the thermal electrons and is described by Eqs. (2.4) and (2.8). The source can be regarded as localized [Eq. (2.14)] at $\varepsilon_0 \sim TM/m$, and the sink should be located at energies of the order of $\varepsilon_s \sim T(M/m)^{1/3}$, i.e., of the lower limit of the interval (1.3). Thus, for $Li^7(H,\gamma)Be^7$, $C^{12}(H,\gamma)N^{13}$, $O^{16}(H,\gamma)F^{17}$, and others the region of the sink is at $\varepsilon \sim 10$–$20$ keV, and for $He^3(He^4,\gamma(Be^7$, $N^{14}(He^4,\gamma)F^{18}$, $O^{16}(He^4,\gamma)Ne^{20}$, and others at ~20–40 keV.[20] Just as above, we take the presence of such a sink into account by the boundary condition and consider the stationary solution (3.2).

We shall be interested henceforth in the steady-state regime. The rate of any reaction of the cycle, and consequently also the strength I of the source, is determined by the slowest reaction. Thus, for the proton cycle the slowest reaction is $H(H,e^+\nu)D^2$, and for the carbon cycle $N^{14}(H,\gamma)O^{15}$. We do not consider at present reactions for which the sink is substantially distributed or located near the source. We then obtain rather general relations that depend little on a concrete reaction of a given class.

As shown above, the stationary solution of Eqs. (2.4) and (2.8) contains, besides the Maxwellian part $f^M$, also, at epithermal energies, a power-law tail $f^{pl}$ [see (3.2)]. Under the considered conditions, the number of tail particles is certainly low and the criterion (3.5) of the applicability of (3.2) is satisfied. Since the rates of the nuclear reactions are quite sensitive to the number of high-energy particles, it follows, as will be seen, that an important role in the rate of a number of reactions is assumed by the contribution of the power-law tails in the distributions of H and $He^4$. Let us estimate their role in the hydrogen and carbon cycles. The rate of the reaction $1+2\rightarrow\ldots$, where $f_1$ and $f_2$ are the distribution functions of particles 1 and 2,

$$I_{12}=\int d\mathbf{p}_1\, d\mathbf{p}_2 f_1(\mathbf{p}_1) f_2(\mathbf{p}_2)\sigma_{12}(\varepsilon)v, \tag{5.1}$$

$$\sigma_{12}(\varepsilon)=\frac{S_{12}}{\varepsilon}\exp(-[\varepsilon_G/\varepsilon]^{1/2}), \tag{5.2}$$

contains the product of the probabilities $(f_1 f_2)$ of the particle collisions and of the penetration through the Coulomb barrier (characterized by the Gamow energy $\varepsilon_G$:

$$G_{12}=\varepsilon_G^{1/2}=31{,}3z_1z_2A^{1/2}[\mathrm{keV}^{1/2}], \tag{5.3}$$

where $z_1$ and $z_2$ are the charge numbers, and $A$ is the reduced mass number), $S_{1,2}$ is the astrophysical factor.[20] We consider first nonresonant reactions, where $S_{12}$ depends little on the energy; these reactions include $C^{12}(H,\gamma)N^{13}$, $N^{14}(He^4,\gamma)F^{18}$, $O^{16}(H,\gamma)F^{17}$, and others. Owing to the presence of the power-law section in the particle distribution (3.2), the reaction rate (5.1) contains, besides the usual Maxwellian terms $I_{12}^M$ (Ref. 20), a term that represents the contribution made to the reaction rate by the interaction of the fast particles of the power-law tail with the equilibrium particles of the other sort.

For $f_1=f_1^{pl}$ and $f_2=f_2^M$ we obtain from (3.2) and (5.1)

$$I_{12}^{pl}=\frac{2I\tau n_2}{(2M_1)^{1/2}}S_{12}\frac{1}{G_{12}}\exp(-G_{12}\varepsilon_0^{-1/2}), \tag{5.4}$$

as well as an analogous term with the contribution of the tail particles of species 2 and the equilibrium particles of species 1. The ratio of the power-law and the Maxwellian contributions is

$$\frac{I_{12}^{pl}}{I_{12}^M}=\pi^{1/2}\frac{I\tau T^{1/2}}{2n_2G_{12}\Delta\varepsilon_{12}}\exp\left\{3\left(\frac{G_{12}}{2}\right)^{2/3}T^{-1/3}-G_{12}\varepsilon_0^{-1/2}\right\}, \tag{5.5}$$

where $\Delta\varepsilon_{12}\sim G_{12}^{1/3}T^{5/6}$ is the width of the Gamow peak.[20] We note that the strength of the source $I$, which enters in (5.4), is defined for the hydrogen cycle in terms of the temperature and density by the expression

$$I=I_{HH}^M=\left(\frac{2}{\pi}\right)^{1/2}\frac{n_H^2S_{HH}\Delta\varepsilon_{HH}}{T^{3/2}M^{1/2}}\exp\left\{-3\left(\frac{G_{HH}}{2}\right)^{2/3}T^{-1/3}\right\}. \tag{5.6}$$

Let us estimate the reactions at which the condition

$$I_{12}^{pl}/I_{12}^M\geqslant 1 \tag{5.7}$$

is satisfied. Substituting in (5.7) and (5.5) the expression for the source power (5.6) and the relaxation time $\tau=2\nu_1^{-1}$, we get

$$3\left(\frac{G_{12}}{2}\right)^{2/3}-15\geqslant(G_{12}\varepsilon_0^{-1/2}-\ln\xi_H-\ln\xi_{12})T^{1/3}, \tag{5.8}$$

where

$$\xi_H=\frac{3n_HS_{HH}G_{HH}^{1/3}T^{5/6}}{\pi^{1/2}e^4\lambda n_e}\left(\frac{M}{m}\right)^{1/2},\qquad \xi_{12}=\frac{T^{4/3}}{G_{12}\Delta\varepsilon_{12}}.$$

For the hydrogen cycle ($T\sim 1$ keV, $n\sim 10^{26}\,\mathrm{cm}^{-3}$) we can assume $\log\xi_H\approx -25$ and $\log\xi_{12}\approx -2$. Recognizing also that the energy of the produced protons and $\alpha$ particles is $\varepsilon_0\sim 4$ MeV [the reaction $He^3(He^3, 2H)He^4$], an estimate for the round bracket in (5.8) yields 65. Then the condition (5.7), (5.8) for the predominance of the contribution of the tail nuclei is satisfied at

$$z_1z_2\left(\frac{A_1A_2}{A_1+A_2}\right)^{1/2}>8.8,\qquad A=M/M_H. \tag{5.9}$$

If the particle of species 2 is hydrogen, then (5.9) is satisfied at $z_1>8$ ($O^{16}(H,\gamma)F^{17}$, $Ne^{20}(H,\gamma)Na^{21}$), and if it is helium, then at $z_1>3$ ($N^{14}(He^4,\gamma)F^{18}$, $O^{16}(He^4,\gamma)He^{20}$, and others). In the carbon cycle at $T\sim 2$ keV, high-energy $He^4$ nuclei are produced (the region of the source is again of the order of several MeV), and an inertial interval can again exist, in which the flux distribution (3.2) with a source $I_{NH}^M$ should be established [cf. (5.6)]. The condition under which the contributions of the $He^4$ tail particles to the rates of the secondary nuclear reactions predominate for the carbon cycle is of the form

$$3(G_{12}/2)^{2/3}-67>37T^{1/3}. \tag{5.10}$$

Thus, for example, for the reaction $C^{12}(He^4,\gamma)O^{16}$ it is satisfied at $T<8$ keV.

The enrichment of the plasma with fast particles increases the rates of those resonant reactions at which the resonance energy $\varepsilon_r$ lies in the interval (1.3), on account of the resonant behavior of the cross section of the interacting particles[21] and for narrow resonances ($\Gamma \ll \Delta\varepsilon$), when

$$S(\varepsilon)\sim\delta(\varepsilon-\varepsilon_r)$$

the ratio analogous to (5.5) takes the form

$$\left(\frac{I_{12}^{\mathrm{pl}}}{I_{12}^{M}}\right)_r \sim\frac{n_t}{n_2}\left(\frac{T}{\varepsilon_r}\right)^{3/2}\exp\left(\frac{\varepsilon_r}{T}\right). \quad (5.11)$$

If follows therefore that the main contribution to the rates of the resonant reactions is determined by the power-law parts of the distribution functions, provided the following conditions are satisfied:

a) for the hydrogen cycle [the source $I=I_{\mathrm{HH}}{}^{M}$ (5.6) is determined by the rate of the reaction $\mathrm{H}(\mathrm{H},e^{+}\nu)\mathrm{D}^2$]

$$\varepsilon_r>78\,\mathrm{keV}; \quad (5.12)$$

b) for the hydrogen cycle [the source $I=I_{\mathrm{NH}}{}^{M}$ is determined by the rate of the reaction $\mathrm{N}^{14}(\mathrm{H},\gamma)\mathrm{O}^{15}$]

$$\varepsilon_r/T-67/T^{1/3}>25. \quad (5.13)$$

One of the examples of resonant reactions is the combustion of helium $3\mathrm{He}^4\to\mathrm{C}^{12}$, which proceeds via formation of the unstable nucleus $\mathrm{Be}^{8*}$. Recognizing that $\varepsilon_r=278$ keV for the reaction $\mathrm{Be}^{8*}(\mathrm{He}^4,\gamma)\mathrm{C}^{12}$, we find from (5.13) that at $T\leqslant 4.1$ keV the production of $\mathrm{C}^{12}$ proceeds mainly on account of the interaction of the $\mathrm{Be}^{8*}$ nuclei with the particles of the power-law tail of the distribution function.

Thus, it is seen that although allowance for the disequilibrium of the distribution functions does not influence the present-day parameters of the sun (neither the $H$ cycle nor the neutrino yield[20]), this influence fluence is quite substantial on a number of secondary reactions (with sufficiently high Coulomb barrier). The mechanisms discussed should "include" reactions occuring in the course of the evolution at lower temperatures than would be the case at complete thermodynamic equilibrium. This circumatance possibly influences the chemical composition and the parameters of stars in the evolution process.

## 6. INFLUENCE OF THE NONSTATIONARITY AND INHOMOGENEITY OF THE SOURCES ON THE POWER-LAW SPECTRA OF THE IONS

Distributions with constant flux can be formed also under nonstationary and inhomogeneous conditions. For example, when a source $G(p)$ is turned on at $t=0$ in the region of large momenta, where the energy diffusion is negligibly small: $|D(p)|\ll p|F(p)|$, the ions begin to shift to the region of small momenta under the influence of the friction force, and form a flux distribution. For a power-law friction $F(p)=-Ap^{-\xi+1}$, $A>0$ it follows from (2.4) that

$$f(p,t)=A^{-1}p^{\xi-3}\int_p^a dp'p'^2G(p'),\qquad a=(p^\xi+A\xi t)^{1/\xi}, \quad (6.1)$$

for a zero initial condition $f(p,0)=0$. If the source $G(p)$ is concentrated near $p=p_0$ in a region of width $\Delta p\ll p_0$, then the leading front of the flux distribution is located at

$$p(t)=(p_0{}^\xi-A\xi t)^{1/\xi}$$

and its width is [2]

$$\Delta p(t)=(p(t)/p_0)^{1-\xi}\Delta p.$$

For a linear friction law $F(p)=-p/\tau$ (region I) Eq. (6.1) goes over into

$$f(p,t)=\tau p^{-3}\int_p^b dp'p'^2G(p'),\qquad b=p\exp(t/\tau), \quad (6.2)$$

and this leads to $f(p)=\tau|J|/p^3$ behind the front far from the source. Thus, the flux distribution can be also quasistationary in the absence of a sink, and the nonstationary character is due to the motion of the leading front (seeRefs. 4 and 22). The time of establishment of the spectrum (6.2) in the region $p_1<p<p_0$ is

$$t=\tau\ln(p_0/p_1).$$

We consider now the formation of the flux distribution in an inhomogeneous plasma using using an example of physical interest, when the role of the source is played by a beam of high-energy ions incident on the plasma[3] with a distribution $f_0(\mathbf{p})$ on the boundary. For an anisotropic distribution of the ions in the beam it is necessary to take into account the diffusion over the transverse momenta, which leads to a broadening of the distribution. For an ion beam, generally speaking, an important role is played by diffusion due to scattering by thermal ions. However, confining ourselves to the region I (1.3), we can verify that the reciprocal transverse-relaxation time at not too small transverse momenta

$$\Theta^2=\frac{\Delta p_\perp{}^2}{p^2}\gg\Theta_0{}^2=\frac{m}{M}\left(\frac{v_{Te}}{v}\right)^3$$

is much smaller than the reciprocal energy relaxation time $\nu_1$. Thus, if the initial angular width of the beam $\Theta^2\gg(m/M)(v_{Te}/v)^3$, then, neglecting the small angle broadening of the beam, we obtain a stationary kinetic equation in the form

$$\mathbf{v}\frac{\partial f(\mathbf{p},\mathbf{r})}{\partial\mathbf{r}}=\frac{M}{\tau}\frac{\partial}{\partial\mathbf{p}}\mathbf{v}f(\mathbf{p},\mathbf{r}). \quad (6.3)$$

Let the beam propagate in the positive $x$ direction and enter the plasma at $x=0$. For a momentum distribution that has azimuthal symmetry and depends only on $x$, the solution of (6.3) is

$$f(p,\theta,x)=\left(\frac{v+x/\tau\cos\theta}{v}\right)^3 f_0(v+x/\tau\cos\theta,\theta), \quad (6.4)$$

where $\theta$ is the angle between the momentum $\mathbf{p}$ and the $x$ axis. If $f_0(p,\theta)$ is concentrated at $p\approx p_0$, then with increasing depth of penetration into the plasma the maximum of the beam distribution shifts to the region of low energies at a rate $|dp/dx|=M/\tau$, and the energy loss takes place over a mean free path $L=p_0\tau/M$. If we average over distances of the order of the energy length $L$, then it follows from (6.4) that

$$\langle f\rangle=\frac{1}{L}\int f(p,x)\,dx=\frac{n_b}{4\pi p^3},$$

where $n_b=\int d\mathbf{p}f_0(\mathbf{p})$ is the concentration of the particles in the beam. Thus, the obtained distribution is equivalent on the average to a spectrum with a constant flux $4\pi|J|=n_b/\tau$.

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*Note added in proof (24 January 1980).* We note that the tail of the distribution in region II [(3.8), Figs. 2 and 3] should decrease slowly with increasing energy, because of the dependence of the Coulomb logarithm on the ion velocity $v$, which exceeds $v_{Te}$ in this region.

The authors have learned that the asymptotic forms of the ion distributions, given in Sec. 3 and in the preprint[24], were obtained also in Ref. 25.

[1)]We exclude in this case a solution with positive flux, for which $J(\infty)>0$ would assume the role of a sink. A flux of this kind can be formed by diffusion only at $\varepsilon \leqslant T$, and manifests itself in loss of particles from the corresponding section of the "thermal" distribution.

[2)]Neglect of diffusion is valid, of course, so long as $\Delta p \gg |D(p)/F(p)|$. If $\Delta p < |D(p_0)/F(p_0)|$, then the width and the structure of the front are determined by diffusion.

[3)]In the presence of a magnetic field, an important role is played by quasilinear relaxation of the fast ions (see the review[23] and the references therein).


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# The magnetic field in a conducting fluid in two-dimensional motion


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The evolution of a magnetic field in a fluid moving so that one of the velocity components vanishes is considered. A dependence of the other velocity components, of the electric conductivity, as well as of the magnetic field on the corresponding coordinate is assumed. Thus a significant generalization of the antidynamo theorem proved by one of the authors in 1956 is obtained. The result allows one to classify the dynamo solutions in terms of the magnitude of the magnetic Reynolds number.



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## 1. INTRODUCTION

The classical problem of the magnetic dynamo, which solves the problem whether the amplifcation or maintenance of a magnetic field is possible in a moving conducting fluid has, as is well known, recieved an affirmative solution. Many concrete examples of magnetic dynamos have been constructed and this has stimulated