paramagnetic ions is a characteristic feature of all van Vleck paramagnets. It must be borne in mind, however, that an estimate of the interaction energy on the basis of the measured (or calculated) values of $\alpha_{\parallel}$ and $\alpha_{\perp}$, in the case of ions with large ionic radii (such as Pr^{3+}) may be too low, since it does not take into account the additional enhancement of the interaction by the kinetic exchange of the $4f$ electrons.

The authors thank S. A. Al'tshuler for interest in the work and for advice in the course of its performance, and S. L. Korableva for preparing the sample.

[1)]In principle, the fact that the second moment of a paramagnetic-resonance line depends on the direction of the external magnetic field relative to a crystallographic axis has been known for a long time (see, e.g., Refs. 13 and 14). The difference in the present paper is that this angular dependence is used here to study nuclear spin-spin interactions.

[1]M. Chapellier, M. Goldman, V. H. Chan, and A. Abragam, J. Appl. Phys. **41**, 849 (1970).
[2]K. Andres and E. Bucher, Phys. Rev. Lett. **22**, 600 (1969).
[3]K. Andres, E. Bucher, J. P. Maita, and A. S. Cooper, Phys. Rev. Lett. **28**, 1652 (1972).
[4]K. Andres, E. Bucher, P. H. Schmidt, J. P. Maita, and S. Darack, Phys. Rev. B **11**, 4364 (1975).
[5]B. Bleaney, Twentieth AMPERE Congress, Abstracts, I23, Tallinn, 1978.
[6]R. E. Thoma, G. D. Brunton, R. A. Penneman, and T. K. Keenan, Inorg. Chem. **9**, 1096 (1970).
[7]I. A. Ivanova, A. M. Morozov, M. A. Petrova, I. G. Podkolzina, and P. P. Feofilov, Izv. Akad. Nauk SSSR Neorg. Mater. **11**, 2175 (1975).
[8]H. P. Jenssen, A. Linz, R. P. Leavitt, C. A. Morrison, and D. E. Wortman, Phys. Rev. B **11**, 92 (1975).
[9]I. S. Konov and M. A. Teplov, Fiz. Tverd. Tela (Leningrad) **18**, 1114 (1976) [Sov. Phys. Solid State **18**, 636 (1976)].
[10]M. V. Eremin, I. S. Konov, and M. A. Teplov, Zh. Eksp. Teor. Fiz. **73**, 569 (1977) [Sov. Phys. JETP **46**, 297 (1977)].
[11]A. A. Antipin, I. S. Konov, S. L. Korableva, R. M. Rakhmatullin, M. S. Tagirov, M. A. Teplov, and A. A. Fedii, Fiz. Tverd. Tela (Leningrad) **21**, 111 (1979) [Sov. Phys. Solid State **21**, 65 (1979)].
[12]P. L. Scott, H. J. Stapleton, and C. Wainstein, Phys. Rev. **137**, A71 (1965).
[13]U. Kh. Kopvillem, Fiz. Met. Metalloved. **8**, 8 (1959).
[14]D. E. O'Reilly and T. Tsang, Phys. Rev. **128**, 2639 (1972).
[15]A. Abragam, Principles of Nuclear Magnetism, Oxford, 1961. Russ. transl. IIL, 1963, p. 108.

Translated by J. G. Adashko

Parametric excitation of spin waves by noise pumping

A. S. Mikhaĭlov and I. V. Uporov

Moscow State University
(Submitted 30 May 1979)
Zh. Eksp. Teor. Fiz. **77**, 2383–2395 (December 1979)

A new method is proposed for parametric excitation of spin waves by noise pumping. The principal characteristics of the subthreshold regime are determined with the aid of Wyld's diagram technique and the above-threshold state is examined. The ineffectiveness of the "phase mechanism" of the above-threshold limitation is demonstrated and the role of various dissipative mechanisms is analyzed.

PACS numbers: 75.30.Ds

Besides ferromagnetic resonance, it is possible to excite parametrically spin waves in ferrodielectric crystals by applying an external alternating magnetic field parallel to the static magnetization of the crystal. The absorption of the energy in this pumping method is due to production of pairs of spin waves (SW) with opposite wave vectors and with a frequency close to half the pump frequency.[1] Measurements of the resonance threshold and investigations of the properties of the subthreshold state of the spin system is by now the subject of an extensive literature (see, e.g., Refs. 2–5).

Parametric excitation of waves is close to the known effect of parametric excitation of an oscillator, explained way back by Rayleigh.[6] This effect is the basis of the action of ordinary swings: if the length of the mathematical pendulum is periodically varied at the frequency equal to double the natural frequency of the pendulum, then an exponential buildup of oscillations sets in when the external action exceeds a certain threshold intensity.[1)] It is not quite obvious that parametric excitation of an oscillator can be produced also by a random (noise) variations of its parameters with time (see, e.g., Ref. 7).

We analyze in this paper the features of parametric excitation of spin waves by noise pumping in the subthreshold and in the (nonlinear) above-threshold regime.

1. NOISE PUMPING

Let $h(t)$ be the alternating magnetic pump field. To describe its properties it is convenient to consider the autocorrelation function[2)]

$$g(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_0^T h(t)h(t+\tau)\,dt \tag{1}$$

and its frequency spectrum

$$g(\omega)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} g(\tau)e^{i\omega\tau}\,d\tau. \tag{2}$$

 0038-5646/79/121149-07$02.40

The difference between random (noise) pumping and (quasi) periodic pumping manifests itself in the form of the spectrum of the autocorrelation function.[7] Noise pumping has a continuous spectrum $g(\omega)$, whereas the spectrum of (quasi) periodic pumping is discrete and consists of individual lines.

An important particular case is noise pumping with a Lorentz spectrum

$$g(\omega)=\frac{I}{\pi}\frac{\Gamma}{(\omega-\omega_p)^2+\Gamma^2}. \quad (3)$$

The maximum of the spectrum occurs in this case at the frequency ω_p, the half-width of the maximum is given by Γ, and the parameter I characterizes the integral intensity of the noise:

$$I=\int_{-\infty}^{+\infty} g(\omega)\,d\omega=\lim_{T\to\infty}\frac{1}{T}\int_0^T h^2(t)\,dt. \quad (4)$$

In addition to the pair autocorrelation function we can define correlators of higher order. For example,

$$g(\tau_1,\tau_2,\tau_3)=\lim_{T\to\infty}\frac{1}{T}\int_0^T h(t)h(t+\tau_1)h(t+\tau_2)h(t+\tau_3)\,dt. \quad (5)$$

We assume the random function $h(t)$ to be Gaussian, i.e., that all the even correlators of higher order break up into partial correlators $g(\tau)$, while correlators of odd order are identically equal to zero. According to the central limit theorem for random processes,[8] a Gaussian noise can be obtained as a limit of an increasingly frequency sequence of random pulses of ever decreasing intensity. For a Gaussian noise it suffices to know the form of the pair correlation function $g(\tau)$ or (equivalently) its frequency spectrum $g(\omega)$.

In view of the instability of oscillators, there is no absolute monochromatic pump—each real source has a spectrum in the form of a line with small but finite width Γ. The reciprocal Γ^{-1} characterizes the temporal instability of the oscillation. However, if the relation $\gamma\gg\Gamma$ is satisfied, then the characteristic time of variation of the pump parameters, Γ^{-1} is much longer than the relaxation time of the spin system, the state of the latter manages to adjust itself to the instantaneous values of the pump parameters, and one can use the results obtained for a purely monochromatic external action.

In this paper we consider a different limiting case, when the pump parameters fluctuate at a frequency much higher than the SW relaxation frequency, i.e., when $\Gamma\gg\gamma$. We analyze also the situation that arises in pumping by a superposition of noise (which satisfies the condition $\Gamma\gg\gamma$ and a coherent monochromatic pump). In this case the spectrum $g(\omega)$ takes the form

$$g(\omega)={}^1/_4 h_0^2\delta(\omega-\omega_p)+g'(\omega), \quad (6)$$

where h_0 is the intensity of the alternating magnetic field of the coherent pump ($h(t)=h_0\cos\omega_p t$), while $g'(\omega)$ is the continuous spectrum of the noise component and reaches a maximum at the frequency of the coherent pump.

In both considered cases it is assumed that the width Γ of the noise-pump spectrum is small compared with the frequency ω_p at which the maximum of the spectrum is located.

2. INTERACTION OF LONGITUDINAL PUMP WITH THE SW SYSTEM IN A CRYSTAL

Since the deviation of the magnetization at one site remains exceedingly small ($\Delta M/M\lesssim 10^{-5}$) even at the largest experimentally obtainable excess above the threshold, it is natural to use the Holstein–Primakoff transformation. After diagonalization, the harmonic part of the Hamiltonian takes the usual form

$$H_0=\sum_{\mathbf{k}}\hbar\omega_{\mathbf{k}}a_{\mathbf{k}}^* a_{\mathbf{k}}. \quad (7)$$

We consider throughout the classical limit of large average occupation numbers, so that we can regard $a_{\mathbf{k}}^*$ and $a_{\mathbf{k}}$ as complex canonical amplitudes of the SW.

The interaction with the longitudinal pump is given by the Hamiltonian[3)]

$$H_p=\frac{\hbar}{2}\sum_{\mathbf{k}}\{h(t)V_{\mathbf{k}}a_{\mathbf{k}}^* a_{-\mathbf{k}}^*+\text{c.c.}\}. \quad (8)$$

The coupling coefficient $V_{\mathbf{k}}$ takes in this case the form

$$V_{\mathbf{k}}=\frac{g\omega_M}{2\omega_{\mathbf{k}}}\sin^2\theta_{\mathbf{k}}\exp(2i\varphi_{\mathbf{k}}), \quad (9)$$

where $\theta_{\mathbf{k}}$ and $\varphi_{\mathbf{k}}$ are the polar and azimuthal angles of the wave vector $\mathbf{k}$ in a spherical coordinate system whose axis z is directed along the vector of the static magnetization M_0 of the crystal; $\omega_M=4\pi g M_0$.

In addition to the cited terms (7) and (8) the Hamiltonian contains terms that describe the nonlinear interaction between the SW, and also between the SW and the phonon system. Recognizing that the SW produced in longitudinal pumping are concentrated in a narrow layer near the resonant surface $\omega_{\mathbf{k}}=\omega_p/2$, it is convenient to separate the subsystem of parametrically excited spin waves (PESW) with wave vectors in the resonant layer, and relegate all the remaining spin waves to the "thermostat." The energy from the external pump enters the PESW system and is then dissipated in the thermostat. It is important to note that although locally, in the resonant layer, the PESW density $n_{\mathbf{k}}$ can exceed by tens of thousands of times the equilibrium thermal density, the total number of PESW remains smaller, even at helium temperatures, than the total number of thermal SW in the crystal. To prevent overall heating, special efforts are made in the experiments—the crystals are placed in a reservoir with liquid helium, and a pulse procedure is used. It is therefore natural to assume that the thermostat is in thermal equilibrium at the ambient temperature.

Below the threshold and at sufficiently small excesses above the threshold, the main contribution to the PESW damping is made by processes in which a single PESW takes part. Since the total number of PESW is in this case much smaller than the number of SW and of the phonons in the thermostat, the probability of PESW scattering by a thermostat particle greatly exceeds the analogous probability for scattering by another PESW.

To describe the interaction of the PESW with the thermostat, we use the method of Langevin equations—we introduce the damping and the Gaussian random force in the equation of motion:

$$\frac{da_{\mathbf{k}}}{dt} = -\frac{i}{\hbar}\frac{\partial H}{\partial a_{\mathbf{k}}^*} - \gamma_{\mathbf{k}} a_{\mathbf{k}} + f_{\mathbf{k}}(t). \tag{10}$$

The method of Langevin equations is presently extensively used in quantum radiophysics problems.[9,10] The conditions for the applicability of this method to the problem of parametric excitation of SW with the aid of monochromatic pumping were discussed in Ref. 11 (see also the review[4]). Because of the proposed narrowness of the noise-pump spectrum ($\Gamma \ll \omega_p$), the results of Ref. 11 apply also to our case.

In a detailed derivation, the Langevin equation (10) appears after partially averaging the amplitudes $a_{\mathbf{k}}$ in the "interaction representation" ($a_{\mathbf{k}} \to a_{\mathbf{k}} \exp(i\omega_{\mathbf{k}} t)$) over time intervals Δt that are large compared with the characteristic "interaction time" $\tau_{\text{int}} \sim \omega_{\mathbf{k}}^{-1}$ of the considered system, but small compared with typical SW relaxation and with the reciprocal width of the noise-pump spectrum.

Since the main contribution to the PESW damping is made by processes in which only one PESW takes part, and the remaining waves belong the the thermostat that is in thermal equilibrium, the damping $\omega_{\mathbf{k}}$ is given by the same expression as in the absence of the pump. It is necessary to retain in the Hamiltonian H only the resonant terms that lead in Eq. (10) to terms with frequency close to that of the waves excited by the pump.

The Gaussian delta-correlated random force $f_{\mathbf{k}}(t)$ realizes the noise action of the thermostat (in the kinetic equation for the average occupation numbers it corresponds to the "arrival term"). The function $f_{\mathbf{k}}(t)$ has the correlators:

$$\langle f_{\mathbf{k}}(t)\rangle = 0, \quad \langle f_{\mathbf{k}}(t) f_{\mathbf{k}'}(t')\rangle = 0,$$
$$\langle f_{\mathbf{k}}(t) f_{\mathbf{k}'}^*(t')\rangle = 2\gamma_{\mathbf{k}} n_{\mathbf{k}}^0 \delta(t-t')\delta_{\mathbf{k},\mathbf{k}'}, \tag{11}$$

where $\delta_{\mathbf{k},\mathbf{k}'}$ is the Kronecker symbol and $n_{\mathbf{k}}^0 = k_B T/\hbar\omega_{\mathbf{k}}$ is the average thermal intensity of the SW with wave vector $\mathbf{k}$ at a temperature T. Averaging over the ensemble of the random forces in (11) and in the subsequent calculations reduces physically to taking the time average over the interval γ^{-1}, $\Gamma^{-1} \gg \Delta t \gg \tau_{\text{int}}$. We note that owing to the approximations used in its derivation, Eq. (10) can be used only to determine quantities that vary slowly in time compared with the "interaction time" τ_{int}.

Parametric excitation of SW is possible also in antiferromagnets of the easy-plane type. In this case the SW of the lower branch of the spectrum are excited. The interaction with the pump is described by the Hamiltonian (8). The expression for the coupling coefficient $V_{\mathbf{k}}$ is given for this case in Ref. 12.

3. INVESTIGATION OF THE SUBTHRESHOLD REGIME AND DETERMINATION OF THE NOISE-PUMP THRESHOLD

In the subthreshold regime there is no need to take into account the nonlinear interaction between the PESW, so that Eq. (10) takes the form

$$\left(\frac{d}{dt} + \gamma_{\mathbf{k}} + i\omega_{\mathbf{k}}\right) a_{\mathbf{k}} = -ih(t) V_{\mathbf{k}} a_{-\mathbf{k}}^* + f_{\mathbf{k}}(t). \tag{12}$$

It is convenient to use the equivalent integral equation for the Fourier components

$$a_{\mathbf{k},\omega} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega t} a_{\mathbf{k}}(t)\,dt, \tag{13}$$

which follows directly from Eq. (12):

$$a_{\mathbf{k},\omega} = G_{\mathbf{k},\omega}\left\{ f_{\mathbf{k},\omega} - iV_{\mathbf{k}} \int_{-\infty}^{+\infty} h(\omega+\omega') a_{\omega',-\mathbf{k}}^*\,d\omega'\right\}. \tag{14}$$

For brevity we have introduced here the notation

$$G_{\mathbf{k},\omega}^0 = [-i(\omega-\omega_{\mathbf{k}}) + \gamma_{\mathbf{k}}]^{-1}. \tag{15}$$

We shall sometimes use also the abbreviated form $q \equiv (\mathbf{k}, \omega)$.

The Fourier components $h(\omega)$ and f_q of the random Gaussian functions $h(t)$ and $f_{\mathbf{k}}(t)$ should be formally regarded as random Gaussian functions of the frequency (see Ref. 7), and their correlators are

$$\langle f_q\rangle = \langle f_q f_{q'}\rangle = 0, \quad \langle f_q f_{q'}^*\rangle = \gamma_{\mathbf{k}}\pi^{-1} n_{\mathbf{k}}^0 \delta_{\mathbf{k},\mathbf{k}'}\delta(\omega-\omega'), \tag{16}$$

and analogously

$$\langle h(\omega)\rangle = 0, \quad \langle h(\omega) h^*(\omega')\rangle = g(\omega)\delta(\omega-\omega'). \tag{17}$$

Since $h(t)$ is a real function, its Fourier component satisfies the relation $h(\omega) = h^*(-\omega)$. The random functions $f_{\mathbf{k}}(t)$ and $h(t)$ are statistically independent.

The main principal interest attaches to the calculation of the average wave intensities $n_{\mathbf{k}} = \langle a_{\mathbf{k}}^*(t) a_{\mathbf{k}}(t)\rangle$ (we recall that we have in mind averaging over a time interval $\Delta t \gg \Gamma^{-1} \gg \omega_{\mathbf{k}}^{-1}$). In addition, since $a_{\mathbf{k}}(t)$ is a certain classical random process, we can take its spectral density n_q, defined in analogy with the spectral density $g(\omega)$ [see (1)], or equivalently by the relation

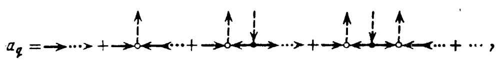
$$n_q \delta_{\mathbf{k},\mathbf{k}'}\delta(\omega-\omega') = \langle a_q a_{q'}^*\rangle. \tag{18}$$

To find these quantities we have used Wyld's diagram technique,[13] which was initially developed for hydroproblems involving SW interactions (see, e.g., Ref. 14).

We note first that the integral equation (14) can be solved by successive iterations, choosing as the first approximation $a_q^{(0)} = G_q^0 f_q$, which is then substituted in the integral term, and finding the first correction

$$a_q^{(1)} = -iG_q^0 V_{\mathbf{k}} \int_{-\infty}^{\infty} h(\omega+\omega') G_{-\mathbf{k},\omega'}^{0*} f_{-\mathbf{k},\omega'}^*\,d\omega', \tag{19}$$

substituting then this correction again in the integral term, and finding the second correction, etc. The resultant solution for a_q is a series in powers of $h(\omega)$, which can be conveniently represented in graphic form:


(20)

using the symbols

$$\longrightarrow\ = G_q^0, \quad \longleftarrow\ = G_q^{0*},$$
$$\cdots\cdots\!\!\rightarrow\ = f_q, \quad \leftarrow\!\!\cdots\cdots\ = f_q^*, \tag{21}$$
$$---\!\!\rightarrow\ = h(\omega), \quad \leftarrow\!\!---\ = h^*(\omega).$$

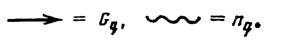
The white and black circles stand for the vertices $-iV_{\mathbf{k}}$

and $iV_{\mathbf{k}}^{*}$, respectively. The first diagram in (20) corresponds to the term $a_q^{(0)}$, the second to $a_q^{(1)}$ [see (19)], and the next ones to higher iterations.

To calculate the quantities of interest to us we need also the values of the correlators $G_{\mathbf{k}}$, defined by

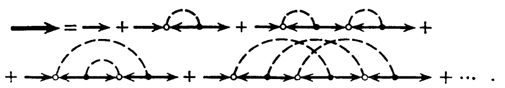
$$G_q = \langle a_q f_{q'}^* \rangle / \langle f_q f_{q'}^* \rangle \tag{22}$$

and called Green's functions. We assign to the functions n_q and G_q the graphic elements



(23)

If we now take the formal solution for a_q in the form of the iteration series (20) and substitute it in the definition (22) of the Green's function, after averaging over the random pump $h(\omega)$ and the thermostat noise f_q, we obtain for G_q a solution in the form of an infinite series:



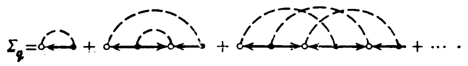
(24)

The dashed line corresponds to $g(\omega)$. There is no diagram with an odd number of vertices, since the corresponding terms yield zero after averaging.

Summing in the usual form the weakly coupled diagrams, we obtain Dyson's integral equation for G_q:

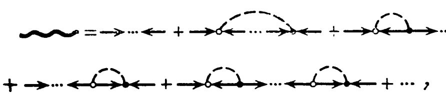
$$G_q = G_q^0 \{1 + \Sigma_q G_q\}, \tag{25}$$

in which the self-energy part Σ_q is given by the series



(26)

If we substitute the formal solution for a_q and $a_{q'}^*$ in the definition (18) of n_q and average, we get the formal solution for n_q:

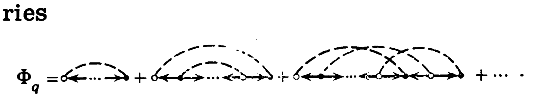


(27)

where the dots stand for the quantity $\varphi_{\mathbf{k}} = (\gamma_{\mathbf{k}}/\pi) n_{\mathbf{k}}^0$. The series (27) likewise admits of summation of weakly coupled diagrams, leading to an integral equation of Wyld's type

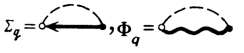
$$n_q = |G_q|^2 \{\varphi_{\mathbf{k}} + \Phi_q\}, \tag{28}$$

in which the "self-noise" part is given by the infinite series



(29)

We use now the small parameter γ/Γ which is at our disposal (see Sec. 1). By direct calculation we can establish that any diagram in the series (26) or (29), containing an intersection of dashed lines, is less than the diagram of the same order without intersections by a factor $(\gamma/\Gamma)^{m-1}$, where m is the number of the corresponding intersections. Taking this into account, we can discard all the diagrams that contain intersections, and sum the remaining diagrams. For Σ_q and Φ_q this yields



(30)

or in analytic form

$$\Sigma_q = |V_{\mathbf{k}}|^2 \int_{-\infty}^{+\infty} g(\omega+\omega') G_{-\mathbf{k},\omega'}^{*} \, d\omega', \tag{31}$$

$$\Phi_q = |V_{\mathbf{k}}|^2 \int_{-\infty}^{+\infty} g(\omega+\omega') n_{-\mathbf{k},\omega'} \, d\omega'. \tag{32}$$

Equations (25) and (28), with account taken of expressions (31) and (32), form a system of integral equations for n_q and G_q. Using once more the presence of the small parameter γ/Γ, these equations can be easily solved and we get ultimately

$$G_q^{-1} = -i(\omega-\omega_{\mathbf{k}}) + \gamma_{\mathbf{k}} - \pi |V_{\mathbf{k}}|^2 g(\omega+\omega_{\mathbf{k}}), \tag{33}$$

$$n_q = |G_q|^2 \frac{\gamma_{\mathbf{k}}}{\pi} n_{\mathbf{k}}^0 \frac{1-\pi|V_{\mathbf{k}}|^2 g(2\omega_{\mathbf{k}})/\gamma_{\mathbf{k}}}{1-2\pi|V_{\mathbf{k}}|^2 g(2\omega_{\mathbf{k}})/\gamma_{\mathbf{k}}}. \tag{34}$$

By integrating (34) with respect to the frequency ω we obtain also the average intensity of the SW:

$$n_{\mathbf{k}} = \frac{n_{\mathbf{k}}^0}{1-2\pi|V_{\mathbf{k}}|^2 g(2\omega_{\mathbf{k}})/\gamma_{\mathbf{k}}}. \tag{35}$$

Thus, in the presence of noise pump, the SW becomes heated, and this heating is a most strongly pronounced near the resonant surface $2\omega_{\mathbf{k}} = \omega_p$, since the function $g(\omega)$, by assumption, reaches its maximum at the frequency $\omega = \omega_p$. On the resonant surface the heating is most intense at points where the ratio $|V_{\mathbf{k}}|^2/\gamma_{\mathbf{k}}$ is maximal. The threshold of the parametric resonance is determined by the same value of the maximum of the spectral pump intensity $g(\omega_p)$, at which the average intensity of the SW becomes infinite at least at one point on the resonant surface. According to (35), this yields $g(\omega_p) = g_c$, where

$$g_c = \min \{\gamma_{\mathbf{k}}/2\pi|V_{\mathbf{k}}|^2\}, \tag{36}$$

and the minimum is sought on the resonant surface.

It is useful to write down an expression for the threshold in the case of a noise pump with a Lorentz spectrum. The critical intensity I_c at a fixed halfwidth Γ is then given by the expression

$$I_c = \min \{\gamma_{\mathbf{k}}\Gamma/2|V_{\mathbf{k}}|^2\}. \tag{37}$$

Comparing (35) and (33), we note that in contrast to the case of a monochromatic pump, in this case the renormalized damping of the PESW $\nu_k = \gamma_{\mathbf{k}} - \pi|V_{\mathbf{k}}|^2 g(2\omega_{\mathbf{k}})$, which enters in the Green's function G_q, does not vanish at the threshold at those k-space points where the average intensity of the waves is infinite, but remains finite and equal to $\nu_{\mathbf{k}} = \frac{1}{2}\gamma_{\mathbf{k}}$ at the threshold.

In the derivation of (35) we have neglected the nonlinear interaction between the PESW and the nonlinear-damping effects. It is clear, however, that in the immediate vicinity of the threshold and particularly above the threshold [where (35) yields formally negative values of $n_{\mathbf{k}}$ for certain $\mathbf{k}$] we must take the nonlinear effects into account.

4. INEFFECTIVENESS OF THE PHASE MECHANISM

We consider first the role of the "phase mechanism,"[4] which leads frequently to an above-threshold limitation in the case of monochromatic pumping. It comes into being because the quadrupole nonlinear interaction between the PESW, with account taken of the existing anomalous correlation of the PESW ($\sigma_{\mathbf{k}} = \langle a_{\mathbf{k}} a_{\mathbf{k}} \rangle$), the amplitude of this wave becomes renormalized under monochromatic pumping and "freezes" at the threshold level, i.e., the PESW system "pushes out" the coherent pump from the sample. The anomalous correlators do not vanish in this case because the external coherent pump predetermines the predominant value of the time phase $\Psi_{\mathbf{k}}$ of the PESW pairs, defined as $\Psi_{\mathbf{k}}=\varphi_{\mathbf{k}}+\varphi_{-\mathbf{k}}$, where $a_{\mathbf{k}}\equiv\rho_{\mathbf{k}}^{1/2}\exp(i\varphi_{\mathbf{k}})$.

A noise pump does not have a defined constant and is by itself incapable of establishing an anomalous correlation. Bearing in mind the possibility of a nonequilibrium phase transition, it is necessary, however to exercise caution in the analysis of the above-threshold state in the case of noise pumping, since spontaneous breaking of the symmetry and the onset of anomalous correlation can occur above the threshold. We shall therefore proceed in the spirit of the theory of Bogolyubov's "quasi-averages,"[15] namely, we consider the above-threshold regime for the superposition of noise and weak coherent pumping $h_0(t)$, so that the coherent component will play the role of a sort of "external field" that breaks the symmetry. If a phase transition occurs under noise pumping, this should manifest itself in the fact that the anomalous correlators remain in the limit as $h_0\to 0$.

To analyze the "phase mechanism" we use the reduced Hamiltonian[4]

$$H_{int}=\hbar\sum_{\mathbf{k},\mathbf{k}'}\left\{T|a_{\mathbf{k}}|^2|a_{\mathbf{k}'}|^2+\frac{S}{2}a_{\mathbf{k}}^*a_{-\mathbf{k}}^*a_{\mathbf{k}'}a_{-\mathbf{k}'}\right\}, \tag{38}$$

where the summation is confined to vectors $\mathbf{k}$ from the resonant layer.

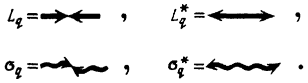
In view of the presence of a coherent component in the pump, we must now introduce, besides the Green's functions G_q and the correlators n_q introduced in Sec. 3, also the anomalous Green's function

$$L_q=\langle a_q f_{-q}'\rangle/\langle f_q f_{q'}^*\rangle \tag{39}$$

and the anomalous correlator

$$\sigma_q=\frac{\langle a_q a_{q'}\rangle}{\delta_{\mathbf{k},-\mathbf{k}'}\delta(\omega+\omega'-\omega_p)}. \tag{40}$$

They correspond to new graphic elements:


(41)

The vertices T and S will be denoted respectively by

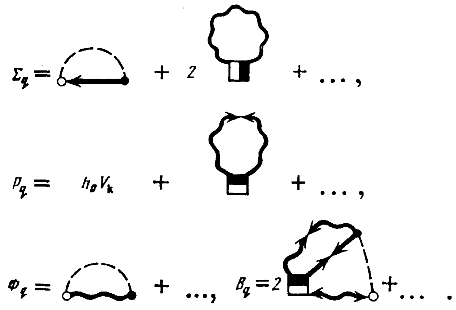

(42)

Proceeding in analogy with Sec. 3 and summing the weakly coupled diagrams, we obtain now a system of four integral equations

$$G_q=G_q^0\{1+\Sigma_q G_q+P_q L^*_{-\mathbf{k},\omega_p-\omega}\},\quad L_q=G_q^0\{P_q G^*_{-\mathbf{k},\omega_p-\omega}+\Sigma_q L_q\},$$
$$n_q=|G_q|^2\{\varphi_{\mathbf{k}}+\Phi_q\}+|L_q|^2\{\varphi_{-\mathbf{k}}+\Phi^*_{-\mathbf{k},\omega_p-\omega}\}+G_q L_q^* B_q+L_q B_q^* G_q^*, \tag{43}$$
$$\sigma_q=G_q\{\varphi_{\mathbf{k}}+\Phi_q\}L_{-\mathbf{k},\omega_p-\omega}+L_q\{\varphi_{-\mathbf{k}}+\Phi_{-\mathbf{k},\omega_p-\omega}\}G_{-\mathbf{k},\omega_p-\omega}+L_q^2 B_q^*+G_q B_{-\mathbf{k},\omega_p-\omega}G_{-\mathbf{k},\omega_p-\omega}.$$

Here $\varphi_{\mathbf{k}}=\pi^{-1}\gamma_{\mathbf{k}}n_{\mathbf{k}}^0$, $q\equiv(\mathbf{k},\omega)$.

The first diagrams for Σ_q, P_q, Φ_q, and B_q are given below.



We note that P_q gives the renormal value of the coherent pump. The diagrams left out are small in the parameter γ/Γ and can be neglected. The series for B_q begins with diagrams of the order of γ/Γ, and therefore in the same approximation we must put $B_q=0$.

The integral equations (43) can then be solved in two important limiting cases: a) weak coherent component ($h_0\ll h_c$, where $h_c=\gamma/|V|$) and b) small noise disturbance (with intensity $I\ll I_c$, where $I_c=\gamma\Gamma/2|V|^2$). We analyze initially the first case.

a) Addition of the weak coherent component lowers somewhat the noise-pump intensity threshold:

$$I_c^*=I_c\{1-2(h_0/h_c)^2\}. \tag{44}$$

The total PESW intensity

$$N=\sum_{\mathbf{k}}(n_{\mathbf{k}}-n_{\mathbf{k}}^0)$$

above the threshold $I=I_c^*$ is given by the expression

$$N=\frac{1}{2}\frac{\gamma}{|S|}\left(\frac{I-I_c^*}{I_c-I}\right)^{1/2}. \tag{45}$$

It becomes infinite as $I\to I_c$. The anomalous PESW correlators $\sigma_{\mathbf{k}}$ take the form

$$\sigma_{\mathbf{k}}=\frac{h_0V_{\mathbf{k}}n_{\mathbf{k}}}{\gamma_{\mathbf{k}}-2iSN}\exp(i\omega_p t). \tag{46}$$

Thus, in the limit as $h_0\to 0$ the anomalous correlators vanish.[4)] Since we have $I_c^*\to I_c$ as $h_0\to 0$ it is seen from (45) that the "phase mechanism" does not impose an above-threshold limitation in the case of noise pumping.

b) Another situation of practical interest is one when a weak noise component is superimposed on an intense coherent pump. Our analysis has shown that in this situation the coherent-pump threshold is lowered compared with the value h_c for fully monochromatic pumping:

$$h_c^*=h_c\{1-{}^3/_4 I/I_c\}, \tag{47}$$

where I_c is given by (37). The total intensity N of the PESW above the threshold is then

$$N=\left(1+2\frac{I}{I_c}\right)\frac{\gamma}{|S|}[(h_0/h_c^*)^2-1]^{1/2}. \tag{48}$$

5. DISSIPATIVE MECHANISMS

The dissipative mechanisms of the above-threshold limitation are connected with the "renormalization" of the PESW damping. If the damping γ_k increases with increasing number N of the PESW, the above-threshold limitation is due to the fact that a value $N=N_0$ is established at which the given intensity of the external pump is the threshold value for the corresponding damping $\gamma_{\mathbf{k}}(N_0)$.

When the damping $\gamma_{\mathbf{k}}$ is a linear function of N [i.e., $\gamma_{\mathbf{k}}(N)=\gamma_{\mathbf{k}}^0+\eta N$, with $\eta>0$], the stationary value of N_0 established above the threshold is given by

$$N_0=\frac{\gamma}{\eta}\left(\frac{I}{I_c}-1\right). \tag{49}$$

The strongest mechanisms of nonlinear positive damping are due to three-wave processes with participation of PESW. It turns out, however, that owing to the stringent requirements imposed by the conservation laws for these processes, the mechanisms are allowed for not all the experimental conditions.

1. *Merging of two PESW (the Suhl mechanism*[16]*).* In a ferromagnet (FM) this process is allowed at $gH_0 < \frac{2}{3}(\omega_p/2)$, where H_0 is the value of the static magnetic field. For antiferromagnets of the easy plane type (AFM), the amplitude of this process is zero.

2. *Decay of PESW into two SW (the Le Gall mechanism*[17]*).* Nonlinear positive damping is produced in this case because of the heating of the small SW system to which the energy flux is directed from the PESW (the "bottleneck" situation). For FM the process is allowed at $gH_0<\frac{1}{3}(\omega_p/2)$, and for AFM the amplitude of the process is equal to zero.

3. *Coalescence of two PESW into a phonon.* The process is allowed only in the presence of an intersection of the spectrum of the phonons and the SW, under the condition that half the pump frequency be less than the frequency ω^* at which this crossing takes place, i.e., $\omega_p/2<\omega^*$.

4. *Decay of a PESW into an SW and a phonon.* The process is allowed if the group velocity $v=|\partial\omega_{\mathbf{k}}/\partial\mathbf{k}|$ of the PESW exceeds the speed of sound s. For ferromagnets the condition $v>s$ can be satisfied only for PESW wave vectors $\mathbf{k}$ much larger than the typical value $|\mathbf{k}|\sim 10^5$ cm^{-1} attained in experiment. For AFM the process is forbidden (if the SW and phonon spectra have a crossing point).

When three-wave processes are allowed by the conservation laws, they give a strong nonlinear damping ($\eta/|S|\sim 1$) and dominate in the case of monochromatic pumping over the "phase" mechanism of the above-threshold limitation.

We see, however, that there is an interval of static magnetic field H_0 and of the average pump frequency ω_p in which the foregoing three-wave processes are forbidden. In the case of monochromatic pumping in this interval the above-threshold limitation is due to the "phase" mechanism. In noise pumping, the "phase" mechanism is not effective and the above-threshold limitation in the indicated interval should be due to the much weaker mechanism of positive nonlinear damping due to the four-wave process of coalescence of two PESW to produce two secondary SW. We have calculated the coefficient of positive nonlinear damping η for these processes:

$$\eta=\frac{40}{3}\pi\frac{|\Psi|^2N_T}{v k_0}, \tag{50}$$

where Ψ is the average amplitude of the four-magnon interaction and N_T is given by the integral

$$N_T=\frac{\Omega}{(2\pi)^3}\int\left(n_{\mathbf{k}}^0+\frac{1}{2}\right)d\mathbf{k}, \tag{51}$$

in which the integration is carried out over the $\mathbf{k}$-space region defined by the condition $\omega_0<\omega_{\mathbf{k}}<\omega_p-\omega_0$ ($\omega_0=\omega(\mathbf{k}=0)$); Ω is the volume of the crystal and $\mathbf{k}_0$ is the wave vector of the PESW.

It is useful to compare the efficiencies of the above-threshold limitation due to the "phase" mechanism and to the considered four-wave process. Assume that we have exceeded by a factor of 2 (in power) the threshold of parametric resonance under monochromatic and noise pumping, i.e., we choose the values of h_0 and I such that

$$(h_0/h_c)^2=I/I_c=2. \tag{52}$$

Then, recognizing that in the dissipative mechanism the number of PESW above the threshold is given by (49), while in the "phase" mechanism in the monochromatic pumping the following relation is valid

$$N=\frac{\gamma}{|S|}[(h_0/h_c)^2-1]^{1/2}, \tag{53}$$

we see that at the chosen values of h_0 and I the ratio of the corresponding values of $N(h_0)$ and $N(I)$ is

$$\frac{N(h_0)}{N(I)}=\frac{\eta}{|S|}\sim 40\frac{|S|N_T}{\omega_p}, \tag{54}$$

which can range, depending on the experimental conditions, from tenths of one percent to several percent ($\eta/|S|\sim 10^{-1}-10^{-2}$).

Thus, in the absence of the Suhl and Le Gall three-wave processes, the only limitation mechanism given by the four-wave processes is very weak, and the total number of PESW increases rapidly with increasing excess above threshold.

The absorbed power in the above-threshold regime under noise pumping is given by

$$W=\frac{\hbar\omega_p}{2}\frac{\gamma^2}{\eta}\frac{I}{I_c}\left(\frac{I}{I_c}-1\right) \tag{55}$$

and at large excesses above the threshold it increases like the square of the pump power.

6. CONCLUSION

In ferromagnets and antiferromagnets of the easy plane type it is possible to excite parametrically SW with the aid of noise microwave pumping. The threshold of the parametric resonance is given by Eqs. (36) and (37) and is attainable with presently available noise generators. In contrast to the case of monochromatic pumping, the "phase" mechanism of above-threshold limitation turns out to be ineffective in this case: the PESW is unable to "push out" of the crystal the incoherent noise pump. The limitation on the growth of the number of PESW above threshold is due exclusively to the dissipative mechanisms. In the absence of three-wave Suhl and Le Gall processes the above-threshold limitation is the result of the much weaker four-wave mechanism. Experiments on noise pumping can be useful for the study of various dissipative mechanisms and for distinguishing between "phase" and dissipative effects.

The authors are deeply grateful to A. S. Borovik-Romanov and I. M. Lifshitz for interest in the work and to A. G. Gurevich and M. I. Kaganov for helpful discussions.

Note added in proof (20 October 1979). An additional positive nonlinear damping mechanism is due to the fact that the expansion for Σ_q contains diagrams that contain one vertex of four-wave interaction S and two vertices $V_{\mathbf{k}}$ of external noise pumping. Estimates show, however, that just as in the case of positive nonlinear damping due to coalescence of two PESW with production of two SW, the corresponding nonlinear damping coefficient η turns out to be quite small ($\eta \sim \gamma\Gamma^{-1}|S|$, $\eta/|S| \ll 1$), so that the conclusions of Sec. 5 remain in force. The authors thank V. S. L'vov for a helpful remark.

[1)] When waves are parametrically excited above threshold an exponential growth takes place of the two oscillators (waves) coupled via the pump.

[2)] We assume throughout the paper that the mean value of the pump

$$\langle h(t)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T h(t+\tau)\,d\tau$$

is equal to zero.

[3)] Taking the condition $\Gamma \ll \omega_p$ into account, we can leave out nonresonant terms of the form $h(t)a_{\mathbf{k}}^* a_{\mathbf{k}}$, which in this case, just as in monochromatic pumping, lead to negligibly small corrections to the results.

[4)] This result is valid also below threshold.

[1] M. I. Kaganov and V. M. Tsukernik, Zh. Eksp. Teor. Fiz. **37**, 823 (1959) [Sov. Phys. JETP **10**, 587 (1960)].
[2] Ya. A. Monosov, Nelineĭnyĭ ferromagnitnyĭ rezonans (Nonlinear Ferromagnetic Resonance), Nauka, 1971.
[3] A. G. Gurevich, Magnitnyi rezonans v ferritakh i antiferromagnetikakh (Magnetic Resonance in Ferrites and Antiferromagnets), Nauka, 1973.
[4] V. E. Zakharov, V. S. L'vov, and S. S. Starobinets, Usp. Fiz. Nauk **114**, 609 (1974) [Sov. Phys. Usp. **17**, 896 (1975)].
[5] W. Haubenreisser, Th. Klupsch, and F. Voight, Phys. Status Solidi B **66**, 9 (1974).
[6] J. W. S. Rayleigh, The Theory of Sound, Dover, 1945. Vol. 1.
[7] R. L. Stratonovich, Izbrannye voprosy teorii fluktuatsii v radiotekhnike (Selected Problems in the Theory of Fluctuations in Radio Engineering), Sov. radio, 1977.
[8] V. I. Tikhonov and M. A. Mironov, Markovskie protsessy (Markov Processes), Sov. radio, 1977.
[9] M. Lax, Fluctuations and Coherent Phenomena, (Russ. transl.), Mir, 1974.
[10] H. Haken and W. Waidelich, Quantum Theory of Lasers, transl. in: Kvantovye fluktuatsii izlucheniya lazera (Quantum Fluctuations of Laser Radiation), Mir, 1974, p. 143.
[11] A. S. Mikhaĭlov, Fiz. Tverd. Tela (Leningrad) **18**, 494 (1976) [Sov. Phys. Solid State **18**, 285 (1976)].
[12] V. I. Ozhogin, Zh. Eksp. Teor. Fiz. **58**, 2079 (1970) [Sov. Phys. JETP **31**, 1121 (1970)].
[13] H. W. Wyld, Ann. Phys. **14**, 193 (1961).
[14] V. S. L'vov, Zh. Eksp. Teor. Fiz. **69**, 2079 (1975) [Sov. Phys. JETP **42**, 1057 (1975)].
[15] N. N. Bogolyubov, Izbrannye trudy (Selected Works), Vol. 3, Kiev, Naukova dumka, 1971, p. 17.
[16] P. Gottlieb and H. Suhl, J. Appl. Phys. **33**, 1508 (1962).
[17] H. Le Gall, B. Lemaire, and D. Sere, Solid State Commun. **5**, 919 (1967).

Translated by J. G. Adashko