Langmuir turbulence of a relativistic plasma in a strong magnetic field

D. G. Lominadze, A. B. Mikhaĭlovskiĭ, and R. Z. Sagdeev

Abastumani Astrophysical Observatory, Georgian SSR Academy of Sciences and Institute of Cosmic Studies, USSR Academy of Sciences
(Submitted 8 May 1979)
Zh. Eksp. Teor. Fiz. 77, 1951–1961 (November 1979)

We study the Langmuir turbulence of a relativistic plasma in a strong magnetic field. We obtain the equations which describe this turbulence and analyze them qualitatively in the strongly relativistic approximation. We assume that the Langmuir oscillations are excited by two-stream instability. We consider the cases of an electron-positron and an electron-ion plasma. We show that the main result of the nonlinear scattering of the waves excited due to a beam instability in an ultrarelativistic plasma is the absorption of oscillatory energy by the particles. The accumulation of long-wavelength Langmuir oscillations (the condensate) which is important in the theory of a nonrelativistic plasma is appreciably reduced. We conclude that, as in the nonrelativistic case, the dynamics of the beam instability is characterized by two consecutive stages: the quasilinear and the nonlinear ones. We show that these two stages last for times of the same order of magnitude.

PACS numbers: 52.60. + h, 52.35.Ra, 52.35.Py, 52.35.Mw

§1. INTRODUCTION

It is necessary to develop a theory of the turbulence of a relativistic plasma in connection with the ever increasing range of experiments with relativistic beams and the study of relativistic astrophysical objects. We analyze in the present paper the Langmuir turbulence of such a plasma. Kaplan and Tsytovich[1] have earlier studied the Langmuir turbulence of a relativistic plasma when there is no magnetic field present. In contrast to them we consider a plasma in a strong magnetic field. According to present-day ideas[2,3] there is a strong magnetic field (of the order of 10^{12} Oe) in the vicinity of neutron stars. Moreover,[4] the vicinity of neutron stars is occupied by a relativistic electron-positron plasma which is produced by effects connected with the magnetic field and the fast rotation of the star. We assume therefore that the problem of the Lnagmuir turbulence of a plasma in a strong magnetic field considered by us may, in particular, be of interest for the physics of the magnetosphere of neutron stars-of pulsars.

Our whole discussion is performed in a one-dimensional geometry, where we assume that the momenta of the particles and the wave vectors of the oscillations are oriented along the magnetic field. In that approximation the magnetic field strength does not occur in the problem of Langmuir turbulence. We perform our analysis in the simplifying assumption that the plasma is strongly relativistic in the frame of reference in which the average plasma velocity vanishes. According to present-day ideas[4] the pulsar plasma moves relative to the neutron star with a Lorentz factor of the same order as the Lorentz factor of the "thermal" spread of the plasma in the rest frame of the star. In the framework of those ideas one should consider the plasma in its own rest frame to be moderately relativistic (with a Lorentz factor of the order unity). One can obtain a qualitative picture of the turbulence of such an intermediate case by extrapolating the results of the analysis of the corresponding limiting cases of the plasma (nonrelativistic and ultra-relativistic cases).

We assume that the Langmuir oscillations are excited by a beam of high-energy particles. The problem of the excitation of Langmuir oscillations in a relativistic plasma by a beam of ultra-relativistic particles was considered earlier in Ref. 5. We studied there the quasi-linear beam relaxation. We neglected then the non-linear scattering of waves, i.e., we assumed that under conditions when the energy of the oscillations is comparable to the beam energy the linear growth rate is nevertheless larger than the non-linear damping rate. We make the same assumption also in the present paper. An alternative point of view was formulated by Tsytovich and Kaplan.[6] They stated that the non-linear damping rate is large compared to the linear growth rate and that the quasi-linear beam relaxation can not take place due to the fast non-linear removal of the Langmuir oscillations from the resonance region. We show that this point of view of Tsytovich and Kaplan[6] is erroneous and we explain the causes of this error.

We give the starting equations in §2. We summarize the results of the linear and the quasi-linear approximations which are necessary to the subsequent analysis in §3. The actual form of the equations of the Langmuir turbulence depend on whether we are dealing with an electron-positron or an electron-ion plasma. We therefore first consider an electron-positron plasma (§4) and then an electron-ion plasma (§5). We discuss the results in §6.

§2. STARTING EQUATIONS

We start from the relativistic Vlasov equation for each kind of particles, the Maxwell equations, and the expression for the current density

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial z}+eE\frac{\partial f}{\partial p}=0,$$
$$\frac{1}{4\pi}\frac{\partial E}{\partial t}+j=0,\quad j=\sum e\int vf\,dp. \tag{2.1}$$

Here $f(t,z,p)$ is the total particle distribution function, depending on the time t, the coordinate z, and the momentum p; $v=p/m\gamma$ is the particle velocity, e, and m its charge and mass; $\gamma=(1+p^2/m^2c^2)^{1/2}$ the Lorentz factor; c the velocity of light; E the electric field. The summation is over the kinds of particles.

Using (2.1) we get the equations corresponding to the weak turbulence approximation.[7-9] We put $f=f_0+\tilde{f}$, where f_0 is the distribution function averaged over the oscillations and $\tilde{f}$ its oscillatory part. We write $\tilde{f}$ in the form

$$\tilde{f}=\int f_{k\omega}(t,p)\exp(-i\omega t+ikz)\,dk\,d\omega.$$

The time-dependence of $f_{k\omega}(t)$ takes into account the slow variation of the amplitude of the electric field oscillations and of the quasi-stationary distribution function f_0. Using an iteration method we get from the first of Eqs. (2.1) $f_{k\omega}=f_{k\omega}^{(1)}+f_{k\omega}^{(2)}+f_{k\omega}^{(3)}+\ldots$, where

$$f_{k\omega}^{(1)}=-\frac{ieE_{k\omega}}{\omega-kv}\frac{\partial f_0}{\partial p}+\frac{e\partial E_{k\omega}/\partial t}{(\omega-kv)^2}\frac{\partial f_0}{\partial p},\tag{2.2}$$

$$f_{k\omega}^{(2)}=-\frac{e^2}{\omega-kv}\frac{\partial}{\partial p}\int dk'\,d\omega'\,\frac{\partial f_0/\partial p}{\omega'-k'v}\left(E_{k'\omega'}E_{k-k',\omega-\omega'}-\langle E_{k'\omega'}E_{k-k',\omega-\omega'}\rangle\right),\tag{2.3}$$

$$f_{k\omega}^{(3)}=\frac{ie^3E_{k\omega}}{\omega-kv}\frac{\partial}{\partial p}\int dk'\,d\omega'\,\frac{I(k',\omega')}{\omega-\omega'-(k-k')v}\times\frac{\partial}{\partial p}\left[\left(\frac{1}{\omega-kv}-\frac{1}{\omega'-k'v}\right)\frac{\partial f_0}{\partial p}\right].\tag{2.4}$$

The quantities $E_{k\omega}(t)$ are defined similarly to $f_{k\omega}(t)$. The symbol $\langle\ldots\rangle$ indicates an average over the random phases of the oscillations. This average is connected with the function $I(k,\omega)$ through the relation

$$\langle E_{k\omega}E_{k'\omega'}\rangle=I(k,\omega)\delta(k+k')\delta(\omega+\omega').$$

Hence it follows that the function $I(k,\omega)$ characterizes the average spectral density of the energy of the electric field oscillations

$$\int I(k,\omega)\,dk\,d\omega=\langle E^2(z,t)\rangle.$$

Using the expression (2.1) for the current density and Eqs. (2.2) to (2.4) we change the Maxwell Eq. (2.1) to the form

$$\frac{\partial W(k,t)}{\partial t}=-\operatorname{Re}[\sigma^{(1)}(k)+\sigma^{(3)}(k)]I(k)-\operatorname{Re}\int dk'\,d\omega'\langle E_{k'\omega'}j_{k\omega}^{(2)}\rangle,\quad W(k)=\omega_k\frac{\partial\operatorname{Re}\varepsilon}{\partial\omega_k}\frac{I_k}{8\pi}.\tag{2.5}$$

Here $W(k)$ is the spectral density of the oscillation energy; the function $I(k)$ is related to $I(k,\omega)$ through the formula $I(k,\omega)=I(k)\delta(\omega-\omega_k)$; ω_k is the solution of the dispersion equation

$$\operatorname{Re}\varepsilon(k,\omega_k)=0;\tag{2.6}$$

$\varepsilon(k,\omega)$ is the permittivity in the linear approximation:

$$\varepsilon(k,\omega)=1-\sum\frac{4\pi e^2}{m}\int\frac{f_0\,dp}{\gamma^3(\omega-kv)^2};\tag{2.7}$$

$\operatorname{Re}\sigma^{(1)}$ and $\operatorname{Re}\sigma^{(3)}$ are the real parts of the conductivities connected with the functions $f_{k\omega}^{(1)}$ and $f_{k\omega}^{(3)}$:

$$\operatorname{Re}\sigma^{(1)}(k)=-\sum\pi e^2\int v\delta(\omega_k-kv)\frac{\partial f_0}{\partial p}dp,\tag{2.8}$$

$$\operatorname{Re}\sigma^{(3)}(k)=\sum\frac{\pi e^4}{m^2}\omega_k\int(k-k')I(k')\,dk'\times\int\frac{dp}{\gamma^6(\omega_k-kv)^4}\delta(\omega_k-\omega_{k'}-(k-k')v)\frac{\partial f_0}{\partial p};\tag{2.9}$$

$j_{k\omega}^{(2)}$ is the current density connected with the function $f_{k\omega}^{(2)}$:

$$j_{k\omega}^{(2)}=\sum e\int vf_{k\omega}^{(2)}dp.$$

The current $j_{k\omega}^{(2)}$ gives rise to a correlation between the phases of the electric field oscillations with different k and ω. This effect is important for the calculation of the last term on the right-hand side of (2.5). It is taken into account through the substitution

$$E_{k\omega}\to E_{k\omega}-4\pi ij_{k\omega}^{(2)}/\omega\varepsilon(k,\omega).\tag{2.10}$$

It is necessary to supplement Eqs. (2.5) to (2.10) by relations which characterize the change with time of the function f_0. For this we use the first Eq. (2.1) from which we get, using the two formulae given above

$$\frac{\partial f_0}{\partial t}=\pi e^2\frac{\partial}{\partial p}\int I(k)\,dk\,\delta(\omega_k-kv)\frac{\partial f_0}{\partial p}+S^{(2)}+S^{(3)},\tag{2.11}$$

$$S^{(2)}=-e\frac{\partial}{\partial p}\int dk\,d\omega\,dk'\,d\omega'\langle E_{k'\omega'}f_{k\omega}^{(2)}\rangle,\tag{2.12}$$

$$S^{(3)}=-\frac{\pi e^4}{m}\frac{\partial}{\partial p}\int(k-k')\,dk\,dk'\,I(k)I(k')\frac{1}{\omega_k-kv}\times\frac{\partial}{\partial p}\left\{\frac{\delta(\omega_k-\omega_{k'}-(k-k')v)}{\gamma^3(\omega_k-kv)^2}\frac{\partial f_0}{\partial p}\right\}.\tag{2.13}$$

Equations (2.5) and (2.11) and the relations given above which explain them are the starting point for the analysis of the Langmuir turbulence of a relativistic plasma which follows. These equations are the relativistic generalization of the analogous non-relativistic equations.[7-9] We note, however, that such a generalization is not complete. We take into account only nonlinear resonances of the kind $v=(\omega_k-\omega_{k'})/(k-k')$ but neglect resonances of the kind $v=(\omega_k+\omega_{k'})/(k+k')$ (cf. Ref. 7) which are unimportant for what follows. Moreover, we assume that apart from ω_k and $\omega_{k'}$ also k and k' are positive, i.e., we neglect also resonances of the kind $v=(\omega_k-\omega_{k'})/(|k|+|k'|)$ corresponding to particles with non-relativistic velocities. Let us explain that resonances of the kind $v=(\omega_k+\omega_{k'})/(k+k')$ do not occur in our case, since we use the relation $I(k,\omega)=I(k)\delta(\omega-\omega_k)$. In the general case one has instead of that relation the formula $I(k,\omega)=\sum I(k)\delta(\omega-\omega_k)$ where the summation is over all roots of the dispersion equation (2.6), among which there are roots with $\omega_k<0$.

§3. RESULTS OF THE LINEAR AND THE QUASI-LINEAR APPROXIMATIONS

The dispersion equation (2.6) has a simple solution in two limiting cases: when $\omega/k\gg c$ and when $\omega/k\approx c$. In the first case

$$\omega_k^2=\overline{\gamma^{-3}}\omega_p^2+3k^2c^2(1-\overline{\gamma^{-5}}/\overline{\gamma^{-3}}),\tag{3.1}$$

and in the second one

$$\omega_k=c[k-\alpha(k-k_0)+\beta(k-k_0)^2/2k_0].\tag{3.2}$$

Here $\omega_p^2=\sum 4\pi e^2n_0/m$ is the sum of the squares of the non-relativistic plasma frequencies of all kinds of particles; $n_0=\int f_0dp$ is the equilibrium density of each kind of particle. The bar on top indicates averaging over momenta with the square of the plasma frequency:

$$\bar{X}=\frac{1}{\omega_p^2}\sum\frac{4\pi e^2}{m}\int Xf_0\,dp.$$

The quantities k_0, α and β are defined by the relations

$$k_0^2=\frac{2\gamma_0\omega_p^2}{c^2},\quad \alpha=\frac{\gamma_0}{2\overline{\gamma^3}},\qquad \beta=\frac{3}{\overline{\gamma^3}}\overline{\gamma^5\left(\frac{\gamma_0}{\overline{\gamma^3}}-\frac{1}{\gamma^2}\right)^2},\tag{3.3}$$

and the quantity γ_0 indicates the same as $\bar{\gamma}$, i.e., $\gamma_0\equiv\bar{\gamma}$. Tsytovich and Kaplan[6] were the first to obtain Eqs. (3.1), (3.2) for a particular form of f_0 (as $\beta\to 0$); Eq. (3.2) as $\beta\to 0$ for a general form of f_0 was derived earlier in Ref. 5.

We note that as to order of magnitude $\overline{\gamma^n}\sim\gamma_0^n$ and $\overline{\gamma^{-n}}\sim\gamma_0^{-1}$, where n is a positve integer. In particular, therefore, $\overline{\gamma^{-3}}\sim\gamma_0^{-1}$ so that $\omega(0)\equiv\omega_{k=0}\sim\omega_p/\gamma_0^{1/2}$. Using this remark and matching formulae (3.1) and (3.2) at the limits of their applicability we get a qualitative dispersion curve ω_k which is given in the figure.

We recall that both this dispersion curve and Eqs. (3.1) to (3.3) refer to an ultra-relativistic plasma with a one-dimensional momentum distribution of the particles. The dispersion law for longitudinal oscillations for the case of an isotropic ultra-relativistic distribution considered by Silin and Rukhadze[10,11] and by Tsytovich[12] differs both quantitatively and qualitatively from the one considered by us. The most important qualitative difference is connected with the value of k_0 and the corresponding frequency $\omega_{k_0}\equiv\omega_0\equiv ck_0$: according to Ref. 12 in the isotropic case $k_0\sim(\omega_p\gamma_0^{-1/2}/c)\ln^{1/2}\gamma_0$ and correspondingly $\omega_0\sim\omega_p\gamma_0^{-1/2}\ln^{1/2}\gamma_0$ whereas in our case $k_0\sim\omega_p\gamma_0^{1/2}/c$, $\omega_0\sim\omega_p\gamma_0^{1/2}$. As regards the frequency of the oscillations with the longest wavelength $\omega(0)$, according to Refs. 10 and 11 it remains in the three-dimensional case the same as in the one-dimensional one, $\omega(0)\sim\omega_p\gamma_0^{-1/2}$.

It follows from (3.2) that when $k>k_0$ the phase velocity of the oscillations does not exceed the velocity of light, $\omega/k<c$, so that in that range of wave numbers a linear Cerenkov interaction between the particles and the oscillations is possible. Such an interaction is according to (2.5) described by the equation

$$\partial W(k)/\partial t=-\operatorname{Re}\sigma^{(1)}(k)I(k).\tag{3.4}$$

For an equilibrium particle momentum distribution

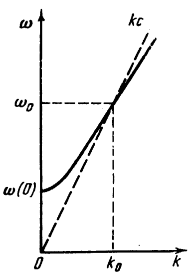

FIG. 1. Wave-number dependence of the frequency of the Langmuir oscillations in the case of an ultra-relativistic plasma.

($\partial f_0/\partial p<0$) the conductivity is postive ($\operatorname{Re}\sigma^{(1)}>0$) and the oscillations are damped. This damping is important approximately when $(k-k_0)\sim k_0$.[5,6] The presence of a particle beam with a Lorentz factor $\gamma_b\gg\gamma_0$ leads to a pumping of the oscillations, i.e., to a beam instability, for wave numbers very close to k_0, $(k-k_0)/k_0\sim(\gamma_0/\gamma_b)^2$.[5] These oscillations lead to a quasi-linear relaxation of the beam distribution function, described by the "contracted" Eq. (2.11)

$$\frac{\partial f_0}{\partial t}=\pi e^2\frac{\partial}{\partial p}\int I(k)\,dk\,\delta(\omega_k-kv)\frac{\partial f_0}{\partial p},\tag{3.5}$$

and a corresponding change in the conductivity $\operatorname{Re}\sigma^{(1)}$, see Eq. (2.8). As a result the beam distribution function acquires, as in the non-relativistic case, the shape of a plateau stretching up to the characteristic plasma particle momentum. The noise arising due to this occupies a range of wave numbers which are characterized by the relation $(\gamma_0/\gamma_b)^2\lesssim(k-k_0)/k_0\lesssim 1$. According to Ref. 5 the energy included in these oscillations is equal to half the initial beam energy

$$\int W(k)\,dk\sim n_b mc^2\gamma_b/2.\tag{3.6}$$

The duration of the quasi-linear relaxation process stage is of the order of[5]

$$\tau_{QL}\sim\Lambda\frac{n_0\gamma_0}{n_b\gamma_b}\frac{\gamma_0^{1/2}}{\omega_p},\tag{3.7}$$

where Λ is a quantity of the order of the Coulomb logarithm.

One can write the result (3.7) also in the form $\tau_{QL}\sim\Lambda/\Gamma_L$, where Γ_L is a characteristic linear growth rate of the beam instability in the stage of a strongly smeared-out beam:

$$\Gamma_L\sim\frac{n_b\gamma_b}{n_0\gamma_0}\frac{\omega_p}{\gamma_0^{1/2}}.\tag{3.8}$$

We assume now that the quasi-linear relaxation has basically ended and that the behavior of our system is determined by the non-linear terms in Eqs. (2.5), (2.11). We turn now to an explanation of effects described by these terms.

§4. ELECTRON-POSITRON PLASMA

We assume that the plasma consists of electrons and positrons with the same distribution functions f_0. As the masses of the particles of the different kinds are the same and the charges the same in magnitude, but opposite in sign, the second-order electric current vanishes: $j_{k\omega}^{(2)}=0$. The non-linear relaxation equations (2.5), (2.11) reduce thus for an electron-positron plasma to the following:

$$\partial W(k)/\partial t=-\operatorname{Re}\sigma^{(3)}(k)I(k),\tag{4.1}$$

$$\partial f_0/\partial t=S^{(3)}.\tag{4.2}$$

The non-linear terms in these equations are caused according to (2.9), (2.13) by resonance of waves with particles which have velocities $v_{res}=(\omega_k-\omega_{k'})/(k-k')$. Using (3.2) we find that in the case of oscillations with wave numbers close to k_0 this resonance condition corresponds to a Lorentz factor γ_{res} which satisfies the relation

$$\gamma_{res}^2=\gamma_0/\overline{\gamma^3}-\lambda(k+k'-2k_0)/k_0\overline{\gamma^3},\tag{4.3}$$

where λ is a coefficient of the order unity. It is clear that as to order of magnitude $\gamma_{res} \sim \gamma_0$, i.e., particles with average momentum values are at resonance, rather than the low-energy particles, as would be the case for a non-relativistic plasma.

It is also clear from (4.3) that at the limit of applicability of Eq. (3.2), when k and k' are appreciably different from k_0, the qualitative relation $\gamma_{res} \sim \gamma_0$ still remains valid. Moreover, when k and k' decrease, γ_{res} also decreases according to (4.3). A similar analysis of the part of the spectrum corresponding to longer wavelengths, which is described by Eq. (3.1), shows that such oscillations already correspond to non-relativistic resonance particles. Taking the qualitative curve of ω_k of our figure into account we can thus conclude that one can consider the resonance particles to be relativistic up to $(k, k') \sim \omega_p / c\gamma_0^{1/2}$.

It is clear from (2.9) that the contribution to the integral for the conductivity $\mathrm{Re}\,\sigma^{(3)}(k)$ is negative for $k' > k$ and positive for $k' < k$. This means that, as in the case of a non-relativistic plasma, non-linear scattering of waves by particles leads to the pumping of noise energy from the short-wavelength to the long-wavelength part of the spectrum. Similar to the case of a non-relativistic plasma, this transfer occurs with conservation of the "number of quanta"

$$\frac{\partial}{\partial t} \int \frac{dk\, W(k)}{\omega_k} = 0. \tag{4.4}$$

One can check this by turning to Eq. (4.1) and using the fact that according to (29)

$$\int dk\, I(k) \sigma^{(3)}(k)/\omega_k = 0.$$

Since $\partial\omega/\partial k > 0$ the transfer of noise to the long-wavelength region means the same as a transfer to the low-frequency region. In the case of a non-relativistic plasma the frequency of the oscillations depends weakly on the wave number and is approximately equal to the Langmuir frequency. In that case condition (4.4) means that the whole energy of the noise is approximately conserved in the process of the non-linear scattering by particles. However, in the case of the relativistic dispersion law, shown in the figure, the difference in frequency of the short-wavelength and the long-wavelength parts of the spectrum is appreciable, $\omega_0/\omega(0) \sim \gamma_0$. The ("final") noise energy after it has been transferred to the long-wavelength region is small compared to the energy of the initial noise:

$$\left[\int W(k)\,dk\right]_{\text{fin}} \Big/ \left[\int W(k)\,dk\right]_{\text{in}} \sim \gamma_0^{-1}. \tag{4.5}$$

The main part, however, of the initial energy of the noise is transferred to the resonance particles, i.e., goes into plasma heating. This effect is described by Eqs. (4.2), (2.13). Kaplan and Tsytovich[1] noted a similar effect of an appreciable heating of the plasma when discussing the Langmuir turbulence of a relativistic plasma when there is no magnetic field. However, they assumed that the order of magnitude of the frequencies of all oscillations (both the initial and the final ones) is the same. In the case considered by them the order of magnitude of the oscillation energy remained therefore also the same. In our case, on the other hand, there is a considerable change in the order of magnitude of the frequency and, as a consequence, in the order of magnitude of the oscillation energy.

We now find an estimate of the non-linear damping rate caused by the presence of the non-zero right-hand side of (4.1). Defining the non-linear damping rate by the relation $\Gamma_{NL} = \frac{1}{2}\partial \ln W(k)/\partial t$ and dropping the numerical factor $\frac{1}{2}$ we can write (4.1) as an order of magnitude relation

$$\Gamma_{NL} \sim \mathrm{Re}\,\sigma^{(3)}(k) I(k)/W(k). \tag{4.6}$$

According to Ref. 5

$$I(k)/W(k) \sim \gamma_0^{-2}. \tag{4.7}$$

We find an estimate for $\mathrm{Re}\,\sigma^{(3)}(k)$ by putting in (2.9)

$$\omega_k \sim \omega_0,\ k-k' \sim k_0,\ \delta(\omega_k - \omega_{k'} - (k-k')v) \sim (\omega_k - kv)^{-1}.$$

Moreover, we use the fact that for a one-dimensional momentum distribution of the particles we have for waves with the dispersion law (3.2) the following order-of-magnitude relation

$$\omega_k - kv \sim k_0 c \gamma_0^{-2}. \tag{4.8}$$

Finally, using (4.7) and introducing the quantity $W^l \equiv \int W(k)dk$ which denotes the total energy of the Langmuir oscillations we find the estimate

$$\mathrm{Re}\,\sigma^{(3)}(k) \sim \omega_0 W^l / n_0 \gamma_0 m c^2. \tag{4.9}$$

Using (4.6) to (4.8) we get finally

$$\Gamma_{NL} \sim \frac{W^l}{n_0 \gamma_0 m c^2} \frac{\omega_p}{\gamma_0^{1/2}}. \tag{4.10}$$

We compare (4.10) with (3.8). We note that as the energy of the Langmuir oscillations reaches the order of magnitude of the beam energy only after the quasi-linear relaxation is finished (see (3.6)), in the quasi-linear relaxation stage $W^l < n_b m c^2 \gamma_b$, and hence

$$\Gamma_{NL} < \Gamma_L. \tag{4.11}$$

This justifies the neglect of the non-linear processes in the study of the quasi-linear relaxation, as was done in Ref. 5 and in the present paper.

Tsytovich and Kaplan[6] used in their paper, as an estimate for the linear growth rate, a correct expression of the type (3.8), but has instead of (4.10) an incorrect estimate for the non-linear damping rate $\Gamma_{NL} \sim \omega_0 W^l / n_0 \gamma_0 m c^2$, which is larger than (4.10) by a factor γ_0^2. This led those authors to conclude incorrectly that the level of saturation of the oscillation energy is too low and, as a consequence, that the non-linear effects are unimportant.

It is interesting to note that after the quasi-linear stage is finished the non-linear damping rate is of the same order of magnitude as the linear one:

$$\Gamma_{NL} \sim \Gamma_L. \tag{4.12}$$

Hence the duration of the process of the transfer of the oscillation energy and of the absorption of it by the particles in the plasma must be of the same order as the duration of the quasi-linear stage so that $\tau_{NL} \sim \tau_{QL}$, where τ_{QL} is characterized by the estimate (3.7).

§5. ELECTRON-ION PLASMA

In the case of an electron-ion plasma $j^{(2)}_{k\omega} \neq 0$. Using (2.15) we then get from (2.5) and (2.12) instead of (4.1), (4.2) the non-linear relaxation equations

$$\partial W/\partial t = [\mathrm{Re}\,\sigma^{(2)}(k) + \mathrm{Re}\,\sigma^{(3)}(k)]I(k), \tag{5.1}$$

$$\partial f_0/\partial t = S^{(2)} + S^{(3)}. \tag{5.2}$$

Here

$$\mathrm{Re}\,\sigma^{(2)}(k) = \frac{4\pi e^6\omega_k}{m^2}\int dk'\,I(k')\,\mathrm{Im}\frac{(V_{k,k'})^2}{\varepsilon(k-k',\omega_k-\omega_{k'})}, \tag{5.3}$$

$$S^{(2)} = \frac{4\pi e^6\omega_k}{m^2}\int dk\,dk'\,I(k)I(k')\,\mathrm{Im}\frac{(V_{k,k'})^2}{\varepsilon(k-k',\omega_k-\omega_{k'})}, \tag{5.4}$$

$$V_{k,k'} = \int\frac{\partial f_0}{\partial p}\frac{dp}{\gamma^3(\omega_k-kv)(\omega_{k'}-k'v)[\omega_k-\omega_{k'}-(k-k')v]}, \tag{5.5}$$

$$v_{k,k'} = \frac{\partial}{\partial p}\frac{1}{\omega-kv}\frac{\partial}{\partial p}\frac{\omega_k-kv}{(\omega_{k'}-k'v)[\omega_k-\omega_{k'}-(k-k')v]}. \tag{5.6}$$

$V_{k,k'}$ and $v_{k,k'}$ are connected through the relation

$$\int mc^2\gamma v_{kk'}\,dp = \frac{\omega_k}{m}V_{k,k'}. \tag{5.7}$$

Just as §4 we assume that $\mathrm{Re}\sigma^{(2)}(k)$ and $S^{(2)}$ are caused by resonances of the kind $\omega_k - \omega_{k'} = v(k-k')$. As the resonance velocity lies in the range of thermal velocities the imaginary terms in the integrals over the momenta on the right-hand side of (5.3) and the analogous imaginary terms in (5.6) are not small compared to the real ones. To simplify the calculations we can make a rough approximation of the problem by assuming those imaginary terms to be large compared to the real ones. We then get, in particular,

$$\begin{aligned}\mathrm{Re}\,\sigma^{(2)}(k) &= \pi\frac{e^4}{m^2}\omega_k\int(k-k')I(k')\,dk'\\ &\times\left\{\int dp\frac{\delta(\omega_k-\omega_{k'}-(k-k')v)}{\gamma^3(\omega_k-kv)^2}\frac{\partial f_0}{\partial p}\right\}^2\\ &\times\left\{\int\frac{\partial f_0}{\partial p}\delta(\omega_k-\omega_{k'}-(k-k')v)\,dp\right]\right\}^{-1}.\end{aligned} \tag{5.8}$$

It is clear from a comparison of (5.8) and (2.9) that $\mathrm{Re}\sigma^{(2)}$ has the opposite sign of $\mathrm{Re}\sigma^{(3)}$. This means that the correlations between the phases of the oscillations lead to a weakening of the process of the transfer of the oscillation energy to the long-wavelength part of the spectrum considered in §4. A similar effect also occurs in the case of a non-relativistic plasma. In that case the terms $\mathrm{Re}\sigma^{(2)}$ and $\mathrm{Re}\sigma^{(3)}$ almost-up to terms of the order of the small parameter $(kv/\omega_k)^2$-completely cancel each other. The direction of the transfer (from short to long wavelengths) remains the same as before (i.e., as when $\mathrm{Re}\sigma^{(2)}$ is neglected), but the rate of the transfer is considerably weakened (by a factor $(kv/\omega_k)^2$). However, in the case of a relativistic plasma there is no small parameter of the kind $(kv/\omega_k)^2$. The sum $\mathrm{Re}\sigma^{(3)} + \mathrm{Re}\sigma^{(2)}$ is thus of the same order of magnitude as $\mathrm{Re}\sigma^{(3)}$ so that there is no appreciable weakening of the transfer rate. As to the direction of the transfer, one can verify that it is conserved. To do this one must show that the sum $\mathrm{Re}[\sigma^{(3)} + \sigma^{(2)}]$ has the sign of $\mathrm{Re}\sigma^{(3)}$. For this purpose we write that sum in the form

$$\begin{aligned}\mathrm{Re}\,\sigma^{(3)} + \mathrm{Re}\,\sigma^{(2)} &= -\pi\frac{e^4}{m^2}\omega\int(k-k')I(k')\,dk'\\ &\left\{\int a^2(p)\,dp - \left[\int a(p)b(p)\,dp\right]^2\Big/\int b^2(p)\,dp\right\},\end{aligned} \tag{5.9}$$

where

$$a(p) = \left[-\frac{\delta(\omega_k-\omega_{k'}-(k-k')v)}{\gamma^6(\omega_k-kv)^4}\frac{\partial f_0}{\partial p}\right]^{1/2}, \tag{5.10}$$

$$b(p) = [-\delta(\omega_k-\omega_{k'}-(k-k')v)\,\partial f_0/\partial p]^{1/2}.$$

Next, taking into account Schwartz's inequality

$$\int a^2\,dp\int b^2\,dp \geqslant \left(\int ab\,dp\right)^2,$$

we verify the foregoing.

All qualitative estimates obtained in §4 for an electron-positron plasma thus remain valid also for an electron-ion plasma.

§6. DISCUSSION OF THE RESULTS

We have obtained equations for the Langmuir turbulence of a relativistic plasma with a one-dimensional particle momentum distribution and we have analyzed them qualitatively in the approximation of the strong relativistic case, $\gamma_0 \gg 1$. As in the case of a non-relativistic plasma these equations describe non-linear scattering of waves by particles. The main result of such a scattering in an ultra-relativistic plasma is the absorption of the oscillation energy by the plasma particles. As to the accumulation of the long-wavelength Langmuir oscillations (condensate), which is important in the theory of a non-relativistic plasma, this effect is considerably weaker in an ultra-relativistic plasma. Only a small fraction of the initial oscillational energy-of the order of γ_0^{-1}-is transformed into the condensate.

We have considered an electron-positron and an electron-ion plasma. In the case of an electron-positron plasma with the same distribution functions for the two kinds of particles there is no current in second order of the wave amplitude. (This peculiarity of an electron-positron plasma was also noted in Ref. 13.) This means in terms of Ref. 7 that the process in which an eigen-oscillation decays into another eigenoscillation, and no strongly damped density perturbation is present in an electron-positron plasma. However, in the relativistic case this difference in the structure of the equations describing an electron-positron and an electron-ion plasma does not lead to important consequences as would be the case in the non-relativistic situation. The reason is that in our problem there is no small parameter such as kv/ω with the corresponding cancellation of contributions from different turbulent processes.

It follows from our analysis that, as in the non-relativistic case, the dynamics of two-stream instability in a relativistic plasma is characterized by two consecutive stages, the quasi-linear and non-linear. These two stages have durations of the same order of magnitude, given by Eq. (3.7).

We noted in §1 that the problems of the dynamics of a relativistic plasma with a one-dimensional distribution function is of interest in connection with the problem of pulsar radiation. One considers it to be generally accepted[4] that the polar region of the magnetosphere of a pulsar is a relativistic plasma penetrated by an ultra-relativistic beam. The importance of the two-stream instability in the physics of the pulsar magnetosphere has often been emphasized (see Ref. 5 and the literature cited there). However, there is as yet no definite point of view about the actual mechanism connecting this instability with the observed pulsar emission. This is connected with the complexity of the problem and the fact that the theory of two-stream instability has not been worked out sufficiently for the particular case of the pulsars. In particular, the results given above indicate that the dynamics of the Langmuir turbulence excited by that instability in the pulsar plasma must evidence itself appreciably differently from what would happen in a non-relativistic plasma. It is also clear that one needs further studies for an actual use of the ideas of Langmuir turbulence in the problem of the interpretation of pulsar emission.

[1] S. A. Kaplan and V. N. Tsytovich, Plazmennaya astrofizika (Plasma astrophysics), Nauka, 1972 §17 [English translation published by Pergamon Press, Oxford].
[2] Ya. B. Zel'dovich and I. D. Novikov, Teoriya tyagoteniya i évolyutsiya zvezd (Theory of gravitation and stellar evolution) Nauka, 1971, Ch. 13.
[3] V. L. Ginzburg and V. V. Zheleznyakov, Ann. Rev. Astron. Astroph. **13**, 511 (1975).
[4] M. A. Ruderman and P. G. Sutherland, Astroph. J. **196**, 51 (1975).
[5] D. G. Lominadze and A. B. Mikhaĭlovskiĭ, Zh. Eksp. Teor. Fiz. **76**, 959 (1979) [Sov. Phys. JETP **49**, 483 (1979)].
[6] V. N. Tsytovich and A. S. Kaplan, Astrofiz. **8**, 441 (1972) [Astroph. **8**, 260 (1974)].
[7] A. A. Galeev and R. Z. Sagdeev, Vopr. Teor. Plazmy **7**, 3 (1973) [Rev. Plasma Phys. **7**, 1 (1979)].
[8] B. B. Kadomtsev, Vopr. Teor. Plazmy **4**, 188 (1964) [English translation published by Academic Press under the title: Plasma Turbulence].
[9] V. N. Tsytovich, Teoriya turbulentnoĭ plazmy (Theory of a turbulent plasma) Atomizdat, 1971, Ch. 2 [English translation published by Consultants Bureau, New York].
[10] V. P. Silin, Zh. Eksp. Teor. Fiz. **38**, 1577 (1960) [Sov. Phys. JETP **11**, 1136 (1960)].
[11] V. P. Silin and A. A. Rukhadze, Elektromagnitnye svoĭstva plazmy i plazmopodobnykh sred (Electromagnetic properties of plasmas and plasma-like media) Gosatomizdat, 1961, § 13.
[12] V. N. Tsytovich, Zh. Eksp. Teor. Fiz. **40**, 1775 (1961) [Sov. Phys. JETP **13**, 1249 (1961)].
[13] J. Arons. C. A. Norman, and C. E. Max, Phys. Fluids **20**, 1302 (1977).

Translated by D. ter Haar

Possibility of diagnostics of magnetic fields in a laser plasma using the spectral composition of the scattered radiation

V. A. Roslyakov and A. N. Starostin

(Submitted 6 June 1979)
Zh. Eksp. Teor. Fiz. 77, 1962–1964 (November 1979)

The possibility of diagnostics of the spontaneous magnetic fields in a laser plasma using the spectral composition of the scattered radiation near $\omega_L/2$ is investigated. It is shown that at $\Omega\tau > 1$ it is possible to determine, from the shift of the spectral components, the intensity of the magnetic field and at sufficient spatial resolution also its orientation.

PACS numbers: 52.70.Kz, 52.50.Jm, 52.25.Ps

Calculations and experiment[1,2] show that magnetic fields of the order of or larger than 1 MOe can be produced in a laser plasma because of the presence of spatial inhomogeneity of the density and of the temperature. This circumstance can influence the symmetrical compression and the heating of the laser target, so that it is important to have reliable information on the intensities and force-line configurations of the magnetic fields. Measurements with the aid of magnetic probes or by rotation of the plane of polarization encounter great difficulties because of the small dimensions and inhomogeneities of the laser plasma. It is of interest in this connection to be able to effect the diagnostics of the magnetic fields by means of the spectral composition of the scattered radiation near $\omega_L/2$ (ω_L is the frequency of the laser radiation). Diagnostics using the spectral composition of other harmonics ($3\omega_L/2, 2\omega_L$, etc.) is less direct because the interpretation of the $3\omega_L/2$ and $2\omega_L$ spectra calls for analysis of the mechanism of the coalescence of the waves that result from the decay.[3,4]

Magnetic fields generated in a laser plasma influence the dispersion of the electromagnetic waves and the processes of transformation of the incident radiation. Parametric instability of the pump wave near a density $n_{cr}/4$ can lead in a magnetoactive plasma to frequency shifts of the scattered waves relative to $\omega_L/2$ by an amount $\sim\Omega$ (the cyclotron frequency of the electrons),[5] if one of the scattered waves propagates collinearly with the laser radiation in a cone with apex angle $\lesssim k_0/k_L$ ($k_0 \equiv (\Omega\omega_p)^{\frac{1}{2}}/c$, $k_L = 3^{\frac{1}{2}}\omega_p/c$, $k_2 \sim k_L$), while the other has a wave vector $k_3 < k_0$. In this case $\omega(k_3) = \omega_p \pm \Omega/2$ or $\omega(k_3) = \omega_p$ at any orientation of the magnetic field relative to the vector $\mathbf{k}_3$.

To calculate the thresholds and the increments of the paramagnetic instability of the incident wave, we consider the system of equations for slow amplitudes of

0038-5646/79/110932-02$02.40