continues to exist at all time when $p>1$, but narrows down rapidly.

As $t\to\infty$, $\varphi\approx -kV_0t\to\infty$ we get $y_r\approx(\varphi - r\pi)/p$. Since $-1<y<1$, we get for integer r the restriction $r_{\min}<r<r_{\max}$, where $r_{\min}=p(1-a)/\pi a$, $r_{\max}=p(1+a)/\pi a$. The number of mutually penetrating streams $N=r_{\max}-r_{\min}=2p/\pi$ increases linearly with time.

The concentration of each stream is

$$n_r=|1+p(-1)^r(1-y_r^2)^{1/2}|^{-1}\approx p^{-1}(1-y_r^2)^{-1/2}.$$

The density is calculated by summing over r, which can be replaced by integration over the equidistant spectrum y_r, with $\Delta y_r=y_{r+1}-y_r=\pi\Delta r/p$:

$$n=\sum_{r_{\min}}^{r_{\max}} n_r=\int_{r_{\min}}^{r_{\max}} n_r\,dr=\int_{-1}^{1} n(y)\frac{p}{\pi}\,dy=\frac{1}{\pi}\int_{-1}^{1}(1-y^2)^{-1/2}\,dy=1. \qquad (17)$$

At $t\to\infty$ it is natural to regard the quantity

$$f(y)=\begin{cases}\pi^{-1}(1-y^2)^{-1/2}, & |y|<1\\ 0, & |y|>1\end{cases} \qquad (18)$$

as the velocity distribution function with normalization (17). The average velocity is

$$\langle y\rangle=\int yf(y)\,dy=0,$$

The mean squared velocity of the "thermal" motion is

$$\langle y^2\rangle=\int y^2f(y)\,dy=\frac{1}{2}.$$

Thus, with the exception o narrow regions containing density singularities, the system evolves into a homogenous state with a distribution function (18), characterized by a density $n=1$, an average stream velocity $u=V_0$, and an effective temperature

$$T_{\text{eff}}=\frac{m}{2}\langle(u-V_0)^2\rangle=\frac{m}{4}(aV_0)^2.$$

Allowance for the thermal motion in the initial distribution (1) leads to a spreading of the inhomogeneities and to a finite amplitude of the density peaks. The equilibrium velocity distribution is established when account is taken of the collisions. Motions of the considered type are possible in a plasma consisting of cold ions and thermal electrons (see, e. g., Refs. 3 and 4), in which case the potential is

$$e\varphi=T_e\ln(n/n_0).$$

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[1]B. B. Kadomtsev, Kollektivnye yavleniya v plazme (Collective Phenomena in Plasma), Nauka, 1976, Ch. 3.
[2]B. B. Kadomtsev and V. I. Karpman, Usp. Fiz. Nauk **103**, 193 (1971) [Sov. Phys. Usp. **14**, 40 (1971)].
[3]I. A. Akhiezer and A. E. Borovik, Zh. Eksp. Teor. Fiz. **51**, 1227 (1966) [Sov. Phys. JETP **24**, 823 (1966)].
[4]A. V. Gurevich and L. P. Pitaevskiĭ, Zh. Eksp. Teor. Fiz. **56**, 1778 (1969) [Sov. Phys. JETP **29**, 954 (1969)].

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Self-action of electromagnetic waves in a plasma subject to modulational instability

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We study the nonlinear penetration of an electromagnetic wave into a plasma upon development of modulational instability. We use an averaging method which employs the substantial scale difference between the plasma and the electromagnetic waves to find the change in the refractive index of the incident wave when the plasma is layered on a fine scale. We consider in detail the stationary self-action of an s-wave in an initially uniform layer of an overdense plasma with sharp boundaries. We find the field distribution and the dependence of the transmission coefficient on the amplitude of the field of the incident wave.

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INTRODUCTION

It is well known that the processes of the modulational instability of Langmuir waves, which lead to the formation of Langmuir solitons and of the corresponding inhomogeneities in the plasma density (cavitons), play an essential role when strong electromagnetic waves interact with a dense collisionless plasma. One usually studies the modulational instability in order to determine the magnitude of the effective collision frequency ν_{eff} which characterizes the additional electromagnetic-energy loss connected with the excitation of fine-scale electric fields. In that case one does not take into account the fact that appearance of cavitons (plasma stratification) also leads to a change in the real part of the refractive index of the electromagnetic wave and, thereby, to a change in the distribution of the large-scale electric fields in the plasma. It is clear that taking such "reactive" non-linear effects into account is important for the determination of the characteristics of the absorption of the wave in the plasma and that without it the interpretation of experimental results when strong electromagnetic waves interact with a plasma is impossible.

The problem of the effect of the stratification of an overdense plasma on the disperison characteristics of an electromagnetic wave was first stated in Ref. 1. The present paper is a direct extension of Ref. 1 and is devoted to a study of the non-linear penetration of an electromagnetic wave into an overdense plasma. When solving the problem we apply a variant of the averaging method which employs the substantial difference in the scales of the plasma and electromagnetic waves. We first (Sec. 1) determine the stationary distribution of plasma waves which are excited by the given pump wave field and after that we obtain, as the result of averaging over the characteristic scale of the instability, an equation which describes the self-action of the average field in a medium with some effective value of the dielectric permittivity. As an example we consider in Sec. 2 the stationary self-action of a TEM-wave in an initially uniform layer of an overdense plasma with sharp boundaries. We find the field distribution in the layer and the way the transmission coefficient depends on the field amplitude of the incident wave.

1. EFFECTIVE PERMITTIVITY OF A STRATIFIED PLASMA

We consider the normal incidence of a plane monochromatic wave with a frequency ω close to the plasma frequency onto a plasma layer with a permittivity $\varepsilon_0(z) < 0$ while in the non-linear regime there arises the possibility of illumination of the plasma.[1] The mechanism of the non-linear penetration is connected with the modulational instability of plasma oscillations which leads to a stratification of the plasma and the formation of waveguide channels in the skin layer of the incident wave. The initial set of equations for the slow (in the scale $2\pi/\omega$) field amplitude $\mathbf{E}=E(t,x,z)\mathbf{x}^0$, when we take into account the small density perturbations arising from the action of the electromagnetic pressure as well as the linear spatial dispersion, has the form[1]

$$-\frac{2i}{\omega}\frac{\partial E}{\partial t}+3r_d^2\frac{\partial^2 E}{\partial x^2}+[\varepsilon_0(z)-n]E=-k_0^{-2}\frac{\partial^2 E}{\partial z^2}, \qquad (1)$$

$$\frac{\partial^2 n}{\partial t^2}-v_s^2\frac{\partial^2 n}{\partial x^2}=v_s^2\frac{\partial^2}{\partial x^2}\frac{|E|^2}{16\pi N_0 T}, \qquad (2)$$

where r_d is the electron Debye radius, $k_0=\omega/c$, n is the relative density perturbation (N_0 is its unperturbed value), and $v_s=(T/M)^{1/2}$ is the ion-sound speed.

In the case considered the electric field is in fact a superposition of the fields of the electromagnetic and plasma waves which have essentially different spatial scales. This enables us to use an averaging method and to get the solution of the set (1), (2) in two stages. We first of all find the plasma wave distribution in the plasma $z=$const which are excited (in the quasi-static approximation) by the given magnitude of the electric induction

$$D=-k_0^{-2}\frac{\partial^2 E}{\partial z^2}=\text{const}.$$

After that by averaging (1) over the characteristic stratification scale we are led to an equation which describes the self-action of the pump wave (averaged over the field stratification scale) in a medium with some value of the effective permittivity:

$$-\frac{2i}{\omega}\frac{\partial \bar{E}}{\partial t}+\varepsilon_{\text{eff}}\bar{E}+k_0^{-2}\frac{\partial^2 \bar{E}}{\partial z^2}=0. \qquad (3)$$

The exact calculation of $\varepsilon_{\text{eff}}=D/\bar{E}$ is connected with the first-stage solution of the non-linear Eqs. (1) and (2) and with a consideration of the development of the modulation instability.

The construction of the microstructure of turbulence (the non-linear stage of the modulational instability) is as yet far from solved. There are several approaches to a solution of that problem[2-7] and of the different models corresponding to them for the strongly turbulent plasma state. It is important that the effective permittivity is a rough characteristic of that state since it is essentially determined by the average dipole moment per unit volume of the strongly non-uniform (over a wavelength of the electromagnetic wave) plasma. Its calculation does not encounter the difficulties due to the actual details of the structure of the stratification, so that one can use any model for the non-linear regime and expect at least agreement between the main qualitative features. The quantitative estimate of the effects considered may, of course, depend on whether we use a model of collapsing cavitons or a model of quasi-stationary interacting solitons.

The set of Eqs. (1)–(3) takes only the stratification of the plasma in the x-direction into account although the modulational instability can, in general, lead to the formation of three-dimensional solitons and density cavitons. We assume that in the case of linear polarization of the field of the incident electromagnetic wave the characteristic size of the solitons and cavitons in the direction of propagation of the wave L_z: $L_\perp \ll L_z$. This kind of assumption is corroborated by the well known results of a numerical study of the non-uniform modulational instability (see, e. g., Ref. 5).

Taking into account what we have said above we shall for the evaluation of ε_{eff} base ourselves upon the one-dimensional dynamical model of the non-linear stage of the modulational instability[6] in which the averaging method is applied most simply. The calculations given for this model[6,7] show that as a result of the development of the modulational instability a succession of solitons (along the x-axis) is formed which differ somewhat in amplitude. As we noted already the effective dielectric permittivity is a rough characteristic of the system as it is sufficient to know for its determination solely the way the average distance between the solitons and their average amplitude depend on the magnitude of the field of the electromagnetic wave which excites the modulational instability. It is natural to expect that the distance between the solitons in the stationary case is the same as the scale of the perturbation in the linearized problem which grows fastest. For instance, in Ref. 1 it was assumed that the scale of the stratification is everywhere in the instability region the same and determined by the maximum value of the field at the initial moment ($t=0$). The estimates which are obtained in that case for the illumination parameters (penetration times and threshold field intensities) describe rather well the observational data and some experimental relations.[8,9] We find in the present paper the stationary solution of the set of Eqs. (1) and (2) in another limiting case where we assume a local connection between the stratification scale and the field of the electromagnetic wave. Such an approach, in contrast to the one used in Ref. 1, enables us to solve much more simply the self-consistent problem of the penetration of the field into a dense plasma, taking reflection into account. The structure of the fine-scale stratification is then determined by the following equation in dimensionless variables

$$\partial^2 a/\partial\zeta^2+(\varepsilon+a^2-\overline{a^2})a=d, \tag{4}$$

while the self-action of the average field (3) is described by the set of equations

$$\frac{\partial^2\bar{a}}{\partial\eta^2}-\bar{a}\left(\frac{\partial\varphi}{\partial\eta}\right)^2=-d=-\varepsilon_{\text{eff}}(\bar{a})\bar{a}, \tag{5}$$

$$\frac{\partial\varphi}{\partial\eta}\overline{a^2}=C=\text{const}; \tag{6}$$

$$a(\zeta,\eta)=\left(\frac{3M}{4m}\right)^{1/2}\frac{\omega}{\omega_p}\frac{|E|}{(16\pi N_0T)^{1/2}},$$

where $a(\zeta,\nu)$ and $\varphi(\nu)$ are the amplitude and the phase of the wave;

$$d=\frac{\omega^3}{\omega_p^3}\left(\frac{3M}{4m}\right)^{3/2}\frac{D}{(16\pi N_0T)^{1/2}},\qquad \varepsilon=\frac{3M}{4m}\frac{\omega^2}{\omega_p^2}\varepsilon_0,$$

$$\zeta=\frac{2}{3}\left(\frac{m}{M}\right)^{1/2}\frac{\omega_p}{\omega}\frac{x}{r_d},\qquad \eta=\left(\frac{4m}{3M}\right)^{1/2}\frac{\omega_p z}{c}.$$

The bar in (4) to (6) indicates averaging over the period of the stationary field distribution in the planes η = const.

We first of all turn to the determination of the transverse field structure. It is described by the solution of the integro-differential Eq. (4) which is a linear-fractional combination of several constants and an elliptical function. However, it is in the general case in practice impossible to use this expression for the evaluation of averages. It is therefore important to find a simple approximate solution. To do this we note that for a value of the induction equal to the threshold value for the modulational instability $d_M=2^{-1/2}|\varepsilon|^{3/2}$ the field distribution is localized and has the form of a Lorentzian soliton on a constant pedestal. We therefore approximate the solution in the form of a sequence of such solitons on a common pedestal with a distance L between one another:

$$a(\zeta)=-\left(\frac{d}{2}\right)^{1/3}+\sum_{n=-\infty}^{\infty}\frac{4(d/2)^{1/3}}{1+2(d/2)^{4/3}(\zeta-nL)^2}. \tag{7}$$

It is clear that the parameter L occurring here is completely determined as a function of d because as d tends to $d_M=2^{-1/2}|\varepsilon|^{3/2}$ the solution (7) must change to the exact solution ($L\to\infty$). To determine this behavior we use the averaged Eq. (4):

$$\overline{a^2}-\varepsilon=(\overline{a^3}-d)/\bar{a}. \tag{8}$$

Hence, up to terms of order[1)] of $1/L^2$ we find the expression

$$L=4/3^{1/2}[(2d^2)^{1/3}-|\varepsilon|]^{1/2}, \tag{9}$$

which as $d\to d_M$ is the same as the expression for the optimum scale of the modulational instability. We can now, by using (6) and (8) evaluate $\bar{a}$, $\overline{a^2}$, and $\overline{a^3}$ and obtain the following expression for the effective dielectric permittivity:

$$\varepsilon_{\text{eff}}(d\geqslant d_M)=2^{1/2}d/[\pi 3^{1/2}((2d^2)^{1/3}-|\varepsilon|)^{1/2}\operatorname{sign}d-(2^{1/2}d)^{1/3}]. \tag{10}$$

In the opposite case, $d<d_M$ there is no stratification and ε_{eff} is equal to the unperturbed linear value

$$\varepsilon_{\text{eff}}(d<d_M)=\varepsilon. \tag{11}$$

It follows from (10) that in the approximation considered the threshold field for illumination of an initially uniform overdense plasma differs somewhat from the one obtained earlier[1] and is equal to

$$d_{\text{thr}}=\left(\frac{3\pi^2}{3\pi^2-1}\right)^{3/2}\frac{|\varepsilon|^{3/2}}{2^{1/2}}, \tag{12}$$

that is, it exceeds somewhat (by a factor 1.17) the threshold for the modulational instability. At smaller values of $d(d<d_{\text{thr}})$ the stratification of the plasma is insufficient for its illumination and the plasma remains overdense. When d decreases the absolute magnitude of ε_{eff} increases and reaches its unperturbed values at the threshold value (11) for the modulational instability.

2. SELF-ACTION OF AN ELECTROMAGNETIC WAVE IN A DENSE PLASMA LAYER

Once we have found the function ε_{eff} we have the usual non-linear electrodynamic problem. To get some idea about the structure of the solution and in order to find the electrodynamic characteristics of a dense plasma layer we turn to the simplest stationary case. We consider normal incidence of a plane wave from the vacuum onto an initially uniform plasma layer which is bounded by the planes $z=l$ and $z=-l$. Using the emission condition ($z\to-\infty$) the field in the vacuum has the form

$$E=E_0\exp(ik_0z)+E_1\exp(-ik_0z),\ z>l, \qquad E=E_2\exp(ik_0z),\ z<(-l), \tag{13}$$

where E_0, E_1, and E_2 are, respectively, the amplitudes of the incident, the reflected, and the transmitted waves.

Inside the layer the distribution of the amplitudes and phases of the field are described by the set of Eqs. (5), (6). The parameter C in them corresponds to the energy flux density in the wave which is transmitted (through the layer), written in dimensionless variables. The self-consistent dependence of the induction d on the average field $\bar{a}$ which need in what follows is determined by the equation $d=\varepsilon_{\text{eff}}(d)\bar{a}$, where $\varepsilon_{\text{eff}}(d)$ is described by Eqs. (10), (11). One easily establishes that it is not single valued. However, it is clear from an analysis of the non-stationary problem of the stratification and the subsequent illumination that the transition from ε_{eff} determined by Eq. (10) to ε occurs for an average field corresponding to the threshold field for the modulational instability ($\bar{a}_M=d_M/|\varepsilon|$).

Thus we are led to the following set of equations which describe the self-action of the average field:

$$\frac{d^2\bar{a}}{d\eta^2}-C^2\bar{a}\left[\frac{3\pi^2-1}{\bar{a}+(3\pi^2[\bar{a}^2+(3\pi^2-1)|\varepsilon|/2])^{1/2}}\right]^4+2\left[\frac{\bar{a}+(3\pi^2[\bar{a}^2+(3\pi^2-1)|\varepsilon|/2])^{1/2}}{3\pi^2-1}\right]^3=0,\quad \bar{a}_M\leqslant\bar{a}<\infty, \tag{14}$$

$$d^2\bar{a}/d\eta^2-C^2/\bar{a}^3-|\varepsilon|\bar{a}=0,\quad -\bar{a}_M<\bar{a}<\bar{a}_M, \tag{15}$$

$$\frac{d^2\bar{a}}{d\eta^2}-C^2\bar{a}\left[\frac{3\pi^2-1}{\bar{a}-(3\pi^2[\bar{a}^2+(3\pi^2-1)|\varepsilon|/2])^{1/2}}\right]^4+2\left[\frac{\bar{a}-(3\pi^2[\bar{a}^2+(3\pi^2-1)|\varepsilon|/2])^{1/2}}{3\pi^2-1}\right]^3=0,\quad -\infty<\bar{a}\leqslant-\bar{a}_M. \tag{16}$$

It must be supplemented by the condition that the field $\bar{a}$ and the derivative $d\bar{a}/d\eta$ are continuous both at the boundaries of the layer $z=\pm l$ and in those points inside the layer where the average field is comparable to the threshold.

The solutions of the set of Eqs. (14) to (16) were studied qualitatively through looking at the behavior of the integral curves in the $(\bar{a},\ d\bar{a}/d\eta)$ phase plane of the system and numerically for the purpose of determining the way the transmission coefficient $T=|E_2|/|E_0|$ depends on the amplitude $b_0=a_0/a_M$ of the incident wave. For a layer of length $k_0L=5$, $|\varepsilon_0|=0.1$ this dependence is shown in the figure and has a hysteresis character which is rather normal in such cases.[10-12] It is clear that for some values of the field of the incident wave total transmission ($T=1$) becomes possible. These resonance states are characterized in the figure by a number which is determined as the ratio of the length of the layer to the period of the non-linear wave. We note that the dependence (of T on b_0) which we have described is in the resonance region considerably smoother than in the case of a cubic non-linearity.

Depending on the magnitude of the energy flux density C there are two forms of solutions of the set (14) to (16). When $C<C^*\approx 8.1$ the layer splits up into a sequence of regions with overdense (unperturbed) and transparent (stratified) plasma. In the case $C>C^*$ the whole layer turns out to be transparent (stratified). The transition from one kind to the other kind of solution for $C=C^*$ is explained by the existence of two resonance states with the same number.

In this paper we have restricted ourselves to an analysis of only reactive non-linear effects giving as an example the determination of the effective permittivity and the finding of the non-linear field structure in the plasma. After solving that problem the change in the power of the incident wave dissipated in the plasma due to a change in the real part of the refractive index can be found by standard methods:

$$P=\int \nu \bar{E}^2(z)\,dz.$$

In the case considered the size of the region occupied by the field increases by approximately a factor 2 as compared to its value evaluated using the linear formula, while the maximum value of the field increases by a factor 2 to 4. It is clear that this fact fundamentally changes the quantity P even in the simplest case of purely collisional losses.

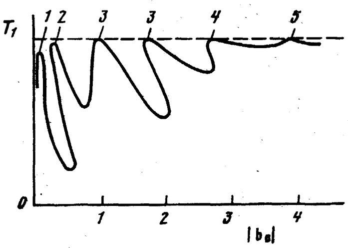

FIG. 1.

We note in conclusion that the effects of the change in the dielectric permittivity of the plasma as a result of the development of the modulational instability of the Langmuir oscillations can also play an essential part in problems about the interaction between a plasma and high-current electron beams.

1) The expansion up to terms of order L^{-1} corresponds to the exact solution of (4) and because of this turns (8) into an identity.

1. A. G. Litvak, V. A. Mironov, and G. M. Fraĭman, Pis'ma Zh. Eksp. Teor. Fiz. 20, 365 (1975) [JETP Lett. 20, 164 (1975)].
2. A. S. Kingsep, L. I. Rudakov, and R. N. Sudan, Phys. Rev. Lett. 31, 1482 (1973); V. V. Gorev, A. S. Kingsep, and L. I. Rudakov, Radiofiz. 19, 691 (1976) [Radiophys. Qu. Electron. 19, 486 (1977)]; L. I. Rudakov and V. N. Tsytovich, Phys. Repts. 40, 1 (1978).
3. A. A. Galeev, R. Z. Sagdeev, Yu. S. Sigov, V. D. Shapiro, and V. N. Shevchenko, Fiz. Plazmy 1, 10 (1975) [Sov. J. Plasma Phys. 1, 5 (1975)]; A. A. Galeev, R. Z. Sagdeev, V. D. Shapiro, and V. N. Shevchenko, Zh. Eksp. Teor. Fiz. 73, 1352 (1977) [Sov. Phys. JETP 46, 711 (1977)].
4. F. Kh. Khakimov and V. N. Tsytovich, Zh. Eksp. Teor. Fiz. 68, 95 (1975) [Sov. Phys. JETP 41, 47 (1975)].
5. V. E. Zakharov, Zh. Eksp. Teor. Fiz. 62, 1745 (1972) [Sov. Phys. JETP 35, 908 (1972)]; L. M. Degtyarev, V. E. Zakharov, and L. I. Rudakov, Zh. Eksp. Teor. Fiz. 68, 115 (1975) [Sov. Phys. JETP 41, 57 (1975)].
6. A. G. Litvak, V. Yu. Trakhtengerts, T. N. Fedoseeva, and G. M. Fraĭman, Pis'ma Zh. Eksp. Teor. Fiz. 20, 544 (1974) [JETP Lett. 20, 248 (1974)]; B. A. Al'terkop, A. S. Volokitin, and V. P. Tarakanov, Pis'ma Zh. Tekh. Fiz. 1, 534 (1975) [Sov. Tech. Phys. Lett. 1, 245 (1975)].
7. N. R. Pereira, R. N. Sudan, and J. Denavit, Phys. Fluids 20, 271 (1977).
8. Yu. Ya. Brodskiĭ, V. G. Eremin, A. G. Litvak, and Yu. A. Sakhonchik, Pis'ma Zh. Eksp. Teor. Fiz. 13, 136 (1971) [JETP Lett. 13, 95 (1971)]; Yu. Ya. Brodskiĭ, V. L. Gol'tsman, V. A. Mironov, and S. I. Nechuen, Zh. Eksp. Teor. Fiz. 74, 1636 (1978) [Sov. Phys. JETP 47, 853 (1978)].
9. G. M. Batanov and V. P. Silin, Pis'ma Zh. Eksp. Teor. Fiz. 14, 445 (1971) [JETP Lett. 14, 303 (1971)]; Proc. Lebedev Phys. Inst. 73, 87 (1974); 92, 3 (1977).
10. V. A. Mironov, Radiofiz. 14, 1450 (1971) [Radiophys. Qu. Electron. 14, 1138 (1974)].
11. V. B. Gil'denburg, Preprint No 57, NIRFI (Gor'kii) 1974.
12. L. M. Gorbunov and K. Zauer, Fiz. Plazmy 3, 1302 (1977) [Sov. J. Plasma Phys. 3, 724 (1977)].

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