rotation, a shift of the rays of the opposing wave. It follows also from (22) that the energy-momentum tensor of the neutrino field satisfies the energy-dominance condition[10]

$$\mathrm{Im}\,\rho \geqslant {}^1/_2 |\sigma| \tag{23}$$

only if $\mathrm{Im}\rho > 0$.

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[1)]A description of the method and several of its applications can be found in the paper by Frolov.[6]

[2)]The physical interpretation of the tetrad components of the Weyl tensor Ψ_i is the subject of a paper by Szekeres.[9]

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One-soliton cosmological waves

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Exact solutions of the gravitational equations which describe the evolution of gravitational solitons against the background of Friedmann cosmological models with the equation of state $\epsilon = p$ are derived and examined. The corresponding vacuum solutions are given.

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§1. INTRODUCTION

The method of inverse solution of the scattering problem has been used by Zakharov and the writer[1] to describe a procedure for integrating the gravitational equations for the case of a metric tensor depending on only two variables. The metric we used was written in the form[1)]

$$-ds^2 = f(-dt^2 + dz^2) + g_{ab}dx^a dx^b, \tag{1.1}$$

where the functions f and g_{ab} depend on the coordinates t and z. Our notation for the coordinates is $(x^0, x^1, x^2, x^3) = (t, x, y, z)$. The first Latin letters a and b always run through the values 1 and 2 and refer to the coordinates x and y. The Latin indices i and k, which occur later, refer to four-dimensional space and take the values 0, 1, 2, 3.

In the previous paper[1] we considered the Einstein equations corresponding to the interval (1.1) only in empty space. The application of a similar method to the integration of these equations in a space filled with matter is as yet an unsettled question. Meanwhile the solutions belonging to the class of metrics (1.1) include such fundamental exact solutions as the Friedmann cosmological models, for which the presence of matter is essential. It would certainly be interesting to construct new exact cosmological solutions describing the evolution of finite disturbances such as gravitational solitons, appearing against the background of a Friedmann space. For the reason we have noted, this cannot at present be done in general form.

There is, however, one special case in which the method already described[1] can still be applied even in a space with matter. This is the case of an ideal fluid with the "superrigid" equation of state $\varepsilon = p$, proposed by Zeld'ovich.[2] The specific form of this equation of state will not play any decisive part in our work, since we shall deal with soliton perturbations of the gravitational field itself, not of the matter, which remains unperturbed in our solutions. From this point of view the matter serves only for the provision and maintainance of the Friedmann background solution, and it can be hoped that the qualitative picture of the behavior of gravitational solitons on this background will remain approximately the same for other equations of state. Besides this, exact solutions of the Einstein equations, analogous (in the sense that the behavior of the metric coefficients g_{ab} remains the same in them) to those obtained here for a space with matter, exist also in vacuum. The way they are found in the general case is described in Sec. 2, and the actual construction is given in Sec. 4.

In this paper we shall consider one-soliton solutions on the background of Friedmann models of all three types. Let us point out their main qualitative peculiarities. These solutions are inhomogeneous cosmological models, in which the distribution of the gravitational field at the inital time shows a clearly expressed max-

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imum with respect to the spatial coordinates near some axis in three-dimensional space. During the expansion of the world this disturbance dies away, and after some finite interval of time it produces a gravitational wave moving away from the axis, with an amplitude decreasing with time. Accordingly, open models, during the final stages of the infinite expansion, go over into Friedmann models. In the closed model, this process of homogenization (and also of isotropization) continues only up to the moment of maximum expansion. During the stage of contraction of the world the fractional perturbation of the gravitational field increases again, and at the final moment of the evolution it is again concentrated on an axis in three-dimensional space. In the open models this axis is topologically equivalent to an infinite straight line, and the soliton disturbance possesses cylindrical symmetry relative to it. In the closed space this assertion retains its meaning only locally, since the axes on which the soliton is concentrated at the inital and final times are circumferences of great circles of the three-dimensional spherical space of the Universe. Furthermore, the initial and final circles do not coincide and nowhere have any common points, being disposed normal to each other.

Another peculiarity of these solutions is the very possibility of treating them as perturbation of Friedmann models, since these solutions reduce, by a continuous limiting procedure with respect to an arbitrary constant parameter, to Friedmann metrics. This property is not completely trivial, since one-soliton solutions do not admit limiting reduction with respect to parameters taking them directly to the metric on whose background they are constructed by the method expounded in the previous paper.[1] In the case studied here the one-soliton solution is close, not to the original background model, but to an exact copy of it, which can be obtained by a discrete symmetry transformation and can be regarded as a different specimen of the same solution on a different physical sheet. This is discussed in more detail in the Appendix. This interpretation means that after obtaining the final form of a one-soliton solution we forget about the method by which it was derived, and take as the background solution the one that is obtained by the appropriate passage to a limit.

The solutions obtained depend on two arbitrary constant parameters. Depending on the regions of variation of one of these parameters all solutions can be divided into two classes. Half of the solutions contain no singularities other than the usual cosmological singularities with respect to the time, which are already contained in the background solution itself. This fact, together with the existence of the limiting transition with respect to the parameters to the background Friedmann models makes this set of solutions extremely satisfactory from the physical point of view; they describe perturbations of the Friedmann models which are finite (but with an infinitesimal case) and everywhere regular. The other half of the solutions, in addition to the background cosmological singularities, have discontinuities of the energy density of the matter and of the first derivatives of the metric coefficients on the light cone. The existence of such discontinuities in one-soliton solutions was already pointed out in Ref. 1. We emphasize that everything we have said about limiting transitions to background solution, and about what is to be taken as being a background solution, relates only to the first set of regular solutions. We shall not consider the case of the discontinuous solutions in this paper.

Soultions with the indicated properties describe one possible mechanism for the production of gravitational waves of cosmological origin. Their sources are inhomogeneities of the gravitational field near the initial cosmological singularity and the dynamics of these inhomogeneities during the further expansion of space. In the course of time the inhomogeneities disappear (at least in open models), but they leave behind a trace in the form of decaying waves which still exist for some time in the universe at later stages of its evolution. This entire process, however–the appearance of an inhomogeneity, its prewave stage, and its final product, a gravitational wave, makes up a single whole, the evolution of a gravitational soliton. In the present case we are dealing with solitons that have cylindrical symmetry, and we cannot call them localized disturbances in the ordinary sense of the word. Nevertheless, the existence of this example allows us to suppose that analogous phenomena can occur with three-dimensional perturbations against the background of uniform cosmological models.

§2. THE GRAVITATIONAL EQUATIONS AND THE FRIEDMANN BACKGROUND MODELS

If the matter filling space is an ideal fluid with the equation of state $\varepsilon = p$, its energy-momentum tensor is

$$T_{ik} = 2\varepsilon u_i u_k + \varepsilon g_{ik}, \quad u^i u_i = -1, \tag{2.1}$$

and the Einstein equations take the form

$$R_{ik} = 2\varepsilon u_i u_k. \tag{2.2}$$

Since for the metric form (1.1) the components R_{0a} and R_{3a} of the Ricci tensor are identically equal to zero, it follows from Eq. (2.2) that the velocity components u_a must also be equal to zero. It can be seen from this that the main part of the Einstein equations, which determines the matrix components g_{ab}, has the form $R_{ab} = 0$, and is thus the same as in vacuum. For this reason the method developed in Ref. 1 can still be applied in the present case.

It is not hard to show that with the use of gravitational hydrodynamics we can, without limiting the generality of our solution, express the matter field in terms of a single scalar function φ, which we call the fluid potential:

$$\varepsilon = p = -{}^1/_2 \varphi_{;k} \varphi^{;k}, \quad u_i = (2\varepsilon)^{-1/2} \varphi_{;i}. \tag{2.3}$$

The Einstein equations and the equations of hydrodynamics now become the following system:

$$R_{ik} = \varphi_{;i} \varphi_{;k}, \quad \varphi_{;k}^{;k} = 0. \tag{2.4}$$

The possibility of this representation of an ideal liquid with the equation of state $\varepsilon = p$ was noted in Ref. 3. The fact that the components u_a of the velocity are zero means that the potential is a function of only two variables, t and z. Denoting differentiation with respect to t with a dot, and that with respect to z with a prime, we get from Eqs. (2.3) and (1.1):

$$\varepsilon = p = (2f)^{-1}(\dot\varphi^2 - \varphi'^2), \quad (2.5)$$

$$u_0 = (2\varepsilon)^{-1/2}\dot\varphi, \quad u_3 = (2\varepsilon)^{-1/2}\varphi', \quad u_a = 0. \quad (2.6)$$

As in our earlier paper,[1] we shall denote by g a two-rowed matrix g_{ab}, and for its determinant and derivatives we introduce the notations

$$\det g = \alpha^2, \quad A = -\alpha g_{,\zeta} g^{-1}, \quad B = \alpha g_{,\eta} g^{-1}, \quad (2.7)$$

where the comma indicates ordinary differentiation and instead of t and z we have introduced the light variables ζ and η:

$$t = \zeta - \eta, \quad z = \zeta + \eta. \quad (2.8)$$

If we now write the metric coefficient F as a product

$$f = f_V F, \quad (2.9)$$

it is easy to show that the equations (2.4) can be divided into four groups. The first and second of them exactly repeat the Einstein equations in vacuum for the metric

$$-ds^2 = f_V(-dt^2 + dz^2) + g_{ab}dx^a dx^b. \quad (2.10)$$

These equations can be written (cf. Ref. 1) in the form

$$(\alpha g_{,\zeta} g^{-1})_{,\eta} + (\alpha g_{,\eta} g^{-1})_{,\zeta} = 0, \quad (2.11)$$

$$(\ln f_V)_{,\zeta} = (\ln\alpha)_{,\zeta\zeta}/(\ln\alpha)_{,\zeta} + (\mathrm{Sp}\, A^2)/4\alpha\alpha_{,\zeta}, \quad (2.12)$$

$$(\ln f_V)_{,\eta} = (\ln\alpha)_{,\eta\eta}/(\ln\alpha)_{,\eta} + (\mathrm{Sp}\, B^2)/4\alpha\alpha_{,\eta}. \quad (2.13)$$

The third group is just a wave equation for the potential φ:

$$(\alpha\varphi_{,\zeta})_{,\eta} + (\alpha\varphi_{,\eta})_{,\zeta} = 0 \quad (2.14)$$

and the fourth group determines the factor F which corrects for the matter

$$(\ln F)_{,\zeta} = \varphi_{,\zeta}^2/(\ln\alpha)_{,\zeta}, \quad (\ln F)_{,\eta} = \varphi_{,\eta}^2/(\ln\alpha)_{,\eta}. \quad (2.15)$$

It follows from Eqs. (2.10) and (2.7) that the function α satisfies the usual wave equation as before:

$$\alpha_{,\eta\zeta} = 0. \quad (2.16)$$

With this condition the equations (2.15) are automatically compatible if φ satisfies Eq. (2.14).

Accordingly, we see that to solve the problem we must first integrate Eqs. (2.11)–(2.15), thus constructing some exact solution of the Einstein equations with the metric (2.10). This part of the problem has already been studied in Ref. 1. After this we must determine the fluid potential φ from Eq. (2.14) and with it find from Eq. (2.15) the coefficient F. Substituting this in Eq. (2.9), we get the desired metric (1.1), and the potential φ determines the energy density and the components of the velocity of the matter in accordance with the relations (2.5) and (2.6).

In the framework of the metric (1.1) we must now determine the Friedmann solutions. The standard forms for these, when four-dimensional spherical coordinates are used, contain a dependence on two space coordinates, while the interval (1.1) assumes a dependence on only one space variable. However, there exists a transformation of the three-dimensional coordinates which allows us to reduce the Friedmann solution to the form (1.1). This transformation (found for a different reason) is given in Appendix D of Ref. 4, and here we need only a special case of the result. The element of length in three-dimensional space in the closed model is given by the expression

$$dl^2 = a^2(d\chi^2 + \sin^2\chi \sin^2\theta d\varphi^2 + \sin^2\chi d\theta^2), \quad (2.17)$$

where the variables χ, θ, φ range over the limits $0 \le \chi \le \pi$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. The transformation

$$\sin z = \sin\chi \sin\theta, \quad \cos z \sin y = \sin\chi \cos\theta, \quad x = \varphi \quad (2.18)$$

reduces (2.17) to the following form:

$$dl^2 = a^2(dz^2 + \sin^2 z dx^2 + \cos^2 z dy^2), \quad (2.19)$$

in which the ranges of variation of the coordinates are $0 \le z \le \pi/2$, $0 \le x \le 2\pi$, $-\pi \le y \le \pi$. The three-dimensional line element of the open space is described by the expression

$$dl^2 = a^2(d\chi^2 + \mathrm{sh}^2\chi \sin^2\theta d\varphi^2 + \mathrm{sh}^2\chi d\theta^2) \quad (2.20)$$

with the following ranges for the coordinates: $0 \le x \le \infty$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. The transformation analogous to Eqs. (2.18) is

$$\mathrm{sh}\, z = \mathrm{sh}\,\chi \sin\theta, \quad \mathrm{ch}\, z\, \mathrm{sh}\, y = \mathrm{sh}\,\chi \cos\theta, \quad x = \varphi, \quad (2.21)$$

and this reduces Eq. (2.20) to the form

$$dl^2 = a^2(dz^2 + \mathrm{sh}^2 z dx^2 + \mathrm{ch}^2 z dy^2), \quad (2.22)$$

where the coordinates vary in the range $0 \le z < \infty$, $0 \le x \le 2\pi$, $-\infty < y < +\infty$. We choose the three-dimensional length element of the planar space in the form

$$dl^2 = a^2(dz^2 + z^2 dx^2 + dy^2) \quad (2.23)$$

and assume that in this expression $0 \le z \le \infty$, $0 \le x \le 2\pi$, $-\infty \le y \le +\infty$, so that the variables z, x, y form an ordinary cylindrical coordinate system.

With the use of Eqs. (2.23), (2.22) and (2.19) it is now easy to establish the form of the Friedmann solutions in the framework of the metric (1.1). For the flat model we have

$$-ds^2 = t(-dt^2 + dz^2 + z^2 dx^2 + dy^2), \quad \varphi = (3/2)^{1/2}\ln t, \quad \varepsilon = 3/4t^3, \quad t \geq 0. \quad (2.24)$$

For the open model

$$-ds^2 = a_0^2 \mathrm{sh}\, 2t(-dt^2 + dz^2 + \mathrm{sh}^2 z dx^2 + \mathrm{ch}^2 z dy^2), \quad \varphi = (3/2)^{1/2}\ln \mathrm{th}\, t, \quad \varepsilon = 3a_0^{-2}\mathrm{sh}^{-3} 2t, \quad t \geq 0. \quad (2.25)$$

And, finally, for the closed model we have

$$-ds^2 = a_0^2 \sin 2t(-dt^2 + dz^2 + \sin^2 z dx^2 + \cos^2 z dy^2), \quad \varphi = (3/2)^{1/2}\ln \mathrm{tg}\, t, \quad \varepsilon = 3a_0^{-2}\sin^{-3} 2t, \quad 0 \leq t \leq \pi/2. \quad (2.26)$$

In the last two solutions a_0 is an arbitrary constant. For simplicity the analogous constant in Eq. (2.24) has been given a fixed value.

To obtain the soliton solutions with the models (2.24)–(2.26) as backgrounds, it is necessary, in accordance with the procedure described in Ref. 1, that we now determine the wave matrix $\psi(\lambda, t, z)$ corresponding to these metrics; after this, the construction of the solutions reduces to mere algebraic operations. It turns out that for all three models the LA equations can be integrated rather simply and the matrix ψ can be expressed in terms of elementary functions. The details are given in the Appendix, and in what follows we give only the final forms of the resulting expressions, so that if the reader is not interested in the way they are found there is no need to refer to the Appendix or to the previous paper.[1]

In concluding this section we recall that in accordance with the discussion in the Introduction we are considering only the solutions that are associated with a perturbation of the gravitational field. The matter potential φ remains unperturbed in our models, although there is no difficulty in obtaining, by applying precisely the same technique to Eqs. (2.14)–(2.15), exact solutions containing along with the gravitational fields also the soliton fields φ.

§3. SOLITON SOLUTIONS ON BACKGROUND OF FRIEDMANN FIELDS

The one-soliton solution on the background of the flat model (2.24) is

$$\begin{aligned}-ds^2 = {} & l^2s^{-2}t[s^2t^2+(t^2+\mu)^2][l^2t^2+(t^2+\mu)^2]^{-1}(-dt^2+dz^2)\\ & +t[s^2t^2+(t^2+\mu)^2]^{-1}\{[s^2t^2z^2+z^2(t^2+\mu)^2+qz^2(t^2+\mu)-q^2\mu]dx^2\\ & +[s^2t^2+(t^2+\mu)^2-q(t^2+\mu)]dy^2+2qs\mu dxdy\}.\end{aligned} \tag{3.1}$$

Here the quantities s, l, and q are arbitrary constants, related to each other by the equation

$$q=s^2-l^2. \tag{3.2}$$

The quantity μ is a function of the coordinates and is given by the expression

$$\mu=-{}^1/_2(l^2+t^2+z^2)+{}^1/_2[(l^2+t^2+z^2)^2-4t^2z^2]^{1/2}. \tag{3.3}$$

Here the second term contains the arithmetic value of the root.[2)]

We note that the determinant of the matrix g found from Eq. (3.1) is of the same form as in the background solution (2.24): $\det g=\alpha^2$, where $\alpha=tz$. The fluid potential for this solution also retains the unperturbed form

$$\varphi=(3/2)^{1/2}\ln t, \tag{3.4}$$

so that the matter is stationary ($u_3=0$). The energy density can be found easily from Eq. (2.5):

$$\varepsilon={}^3/_4s^2l^{-2}t^{-2}[l^2t^2+(t^2+\mu)^2][s^2t^2+(t^2+\mu)^2]^{-1}. \tag{3.5}$$

The deviation of this value from the background value is due only to the perturbation of the metric (the metric coefficient f) and not to a perturbation of the matter field as such.

From these formulas we see that if we let the parameter q go to zero ($s^2\to l^2$) the solution goes over into the background, Eq. (2.24). We now determine the field of the soliton as the precise deviation of the metric from its background value. This field can be described with a symmetrical perturbation matrix H, which is constructed according to exactly the same rule as in the infinitesimal case:

$$H_{11}=(g_{11}-g_{11}^{(0)})(g_{11}^{(0)})^{-1},\qquad H_{22}=(g_{22}-g_{22}^{(0)})(g_{22}^{(0)})^{-1}, \tag{3.6}$$

$$H_{12}=H_{21}=g_{12}(g_{11}^{(0)}g_{22}^{(0)})^{-1/2}, \tag{3.7}$$

where the quantities with superscript zero relate to the background solution (2.24).

Besides the matrix H, the soliton is also characterized by the perturbation of the metric coefficient f. It is more convenient, however, to consider instead of this the perturbation of an equivalent quantity, the energy density ε, for which we write

$$E=(\varepsilon-\varepsilon_0)\varepsilon_0^{-1}. \tag{3.8}$$

From Eqs. (3.1), (3.5), and (2.24) we get:

$$H=q[s^2t^2+(t^2+\mu)^2]^{-1}\begin{pmatrix} t^2+\mu-q\mu z^{-2} & s\mu z^{-1}\\ s\mu z^{-1} & -t^2-\mu\end{pmatrix}, \tag{3.9}$$

$$E=ql^{-2}(t^2+\mu)^2[s^2t^2+(t^2+\mu)^2]^{-1}. \tag{3.10}$$

Let us examine the behavior of these quantities near the moment $t=0$ of the initial cosmological singularity. It is easy to show that the first nonvanishing terms of the matrix H for $t\to 0$ (and arbitrary values of z) are given by the expression

$$H=qs^{-2}(l^2+z^2)^{-1}\begin{pmatrix} s^2 & -sz\\ -sz & -l^2\end{pmatrix} \tag{3.11}$$

and for the quantity E the first nonvanishing term is

$$E=ql^2s^{-2}t^2(l^2+z^2)^{-2}. \tag{3.12}$$

It can be seen from Eq. (3.11) that the field of the perturbation H is concentrated, during the first few moments of the evolution, near the axis $z=0$ of the axial symmetry, in a cylindrical volume with the characteristic radius $z\sim l$. The components H_{11} and H_{22} have extrema with respect to the variable z right on the axis $z=0$, and H_{12} has extrema at distance $z=l$ from the axis. The perturbation of the energy density (i.e., the metric coefficient f) is proportional to t^2 in the first nonvanishing approximation, and is already of the next order of smallness as compared with the main terms of the expansion of the matrix H. Nevertheless, as can be seen from Eq. (3.12), the distribution of the quantity E with respect to z also localized on the axis $z=0$ with the characteristic width $z\sim l$.

For simplicity we will suppose that the constants s and l are of the same order of magnitude. Then there is a single characteristic length l in the solution, and the asymptotic expressions (3.11) and (3.12) are the first terms of the expansion of the solution in powers of t/l in the region where $t\ll l$. As in the time t increases we come to the region $t\gg l$, in which all the components of the matrix H go to zero for $t\to\infty$. However, the laws of this dying away are different for points located near the light cone $z=t$ and far from it. If $t\gg l$ and $z<t$, then we get from Eqs. (3.3) and (3.9) the following asymptotic expression for H:

$$H=q(t^2-z^2)^{-1}\begin{pmatrix} 1 & -sz(t^2-z^2)^{-1}\\ -sz(t^2-z^2)^{-1} & -1\end{pmatrix}, \tag{3.13}$$

from which it can be seen that both near $z=0$ and also at any other fixed point in space the perturbation field falls off for t according to the law $H_{11}\sim H_{22}\sim l^2t^{-2}$, $H_{12}\sim l^4t^4$. On the light cone the expressions (3.13) diverge, but this is due only to the fact that they cease to be applicable when we get into the strip $t-z \lesssim l$ adjacent to the light line $z=t$. The behavior of the matrix H inside this strip can be estimated by determining its asymptotic behavior on the cone $z=t$ itself for $t \gg l$. The main term for this is easily found from Eqs. (3.3) and (3.9):

$$H=q(s^2+l^2)^{-1}t^{-1}\begin{pmatrix} l & -s \\ -s & -l \end{pmatrix}. \tag{3.14}$$

Thus we see that for any given time $t \gg l$ the amplitude of the perturbation H at points of the light cone is of the order of lt^{-1} and is very large in comparison with its values at other points of space (where the components of H are of orders l^2t^{-2} and l^4t^{-4}). This means that the initial perturbation H, which for $t \to 0$ was concentrated near the axis $z-0$ with characteristic dimension $z \sim l$, while decreasing with time produces in the later stages a gravitational wave moving out from the axis with the speed of light. The amplitude of this wave also decreases with time, and the field distribution in it, concentrated on the light cone, has the same characteristic width $\delta z \sim l$ as the initial cosmological perturbation of the metric. It must be remembered, however, that these assertions, as always, have only an approximate meaning. Actually the quantities H and E contain, besides the wave part, perturbations relating to the background geometry, and it is hard to give an exact meaning to each of these effects by itself. This fact is well illustrated in an analysis of the relative perturbation E. At the moment when the evolution begins this quantity is vanishingly small and is given by the expression (3.12). For $t \gg l$, in the region $z>t$ the approximation for E is

$$E=ql^2s^{-2}t^2(t^2-z^2)^{-2}. \tag{3.15}$$

At points of the light cone $z=t$ we have for $t \gg l$

$$E=q(s^2+l^2)^{-1}+O(lt^{-1}) \tag{3.16}$$

and finally, in the region $t>z$ with $t \gg l$ we get

$$E=ql^{-2}-ql^{-2}s^2t^2(t^2-z^2)^{-2}. \tag{3.17}$$

It can be seen from these expressions that the fractional perturbation of the metric coefficients f, and along with it the energy density ε, remain small only at the initial moment of the evolution and at the points of space where $z \gg t$, i. e., in regions not yet reached by the gravitational wave. In regions $t>z$, through which the wave has already passed, there remains a final decreasing perturbation $E=ql^{-2} \sim 1$, which reduces to a change of the constant parameters of the background Friedmann solution.

Accordingly, in the final stages of the expansion for $t \to \infty$ we have instead of Eq. (2.24) the following asymptotic behavior:

$$-ds^2=l^2s^{-2}t(-dt^2+dz^2)+t(z^2dx^2+dy^2), \tag{3.18}$$

$$\varepsilon={}^3/_4\, s^2l^{-2}t^{-3}. \tag{3.19}$$

This phenomenon illustrates the interaction between the wave and background parts of the solution. It may be possible to speak here of an exchange of energy between the gravitational wave and the background, but this would require that one give some satisfactory definition of these concepts.

Finally, we must discuss the physical meaning of the arbitrary constants contained in the solutions (3.1)–(3.5). The foregoing analysis has shown that the constant l is the characteristic width of the initial distribution of the soliton field. After this we can associate the constant s with the amplitudes of this distribution. A different, and not less clear, physical meaning of the constant s can be obtained if we examine in more detail the development in time of the profile of the component H_{22} of the perturbation. For $t \to 0$ the shape of this profile follows from Eq. (3.11).

Let us now determine the extrema of H_{22} with respect to the variable z at an arbitrary time t, by considering the equation $\partial H_{22}/\partial z=0$. It is easy to show that this equation has two solutions: one of them is $z=0$, independently of the time t (which corresponds to the smooth behavior of H_{22} on the axis of symmetry), and the second solution gives the following world line: $z^2 = s^{-1}(st+l^2)(t-s)$ (we assume $s>0$; otherwise the formula must be written with s replaced with $|s|$). It follows that there is a second extremum on the profile of H_{22}, but it appears only after a finite time interval $t=s$ after the beginning of the evolution. Up to the time $t=s$ the distribution of H_{22} with respect to z has a smooth nature as in Eq. (3.11). After the time $t=s$ the world line of the second extremum[3)] moves out toward increasing values of the coordinate z, and for $t \to \infty$ it asymptotically approaches the light line $z=t-q/2s$. This means that the time $t=s$ marks the beginning of the wave stage of the evolution of the soliton, i. e., the generation of the gravitational wave. Thus the variable s has the meaning of the delay time, or the time of embryonic development of the wave. The pattern of the behavior of the component H_{22} is shown in Fig. 1.

Let us now pass on to the one-soliton solutions with the open and closed Friedmann models as backgrounds. These metrics can be written in the following unified form:

$$\begin{aligned} -ds^2=&a_0^2rs^{-2}k^{-1}\sin 2ktQL^{-1}(-dt^2+dz^2)\\ &+(2k)^{-1}\sin 2ktQ^{-1}[(2k^{-2}a_0^2L\sin^2kz+\sigma\mu\cos^2\gamma+\sigma s^{-2}\mu R\sin^2\gamma)dx^2\\ &+s^2r^{-1}(2a_0^2L\cos^2kz-\sigma k^2\mu\sin^2\gamma-\sigma k^2s^{-2}\mu R\cos^2\gamma)dy^2\\ &+\sigma k^2s^{-1}\mu(R-s^2)\cos 2\gamma\, dxdy], \end{aligned} \tag{3.20}$$

$$Q=s^2\sin^2 kt+R\cos^2 kt, \quad L=r\sin^2 kt+R\cos^2 kt. \tag{3.21}$$

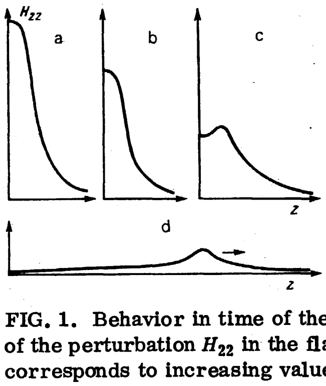

FIG. 1. Behavior in time of the profile of the absolute value of the perturbation H_{22} in the flat model. The sequence of plots corresponds to increasing values of the time t: a) distribution of the perturbation at the time the evolution starts, $t \to 0$; b) the initial perturbation begins to die away; c) profile near the critical time $t-s$, beginning of production of the wave; d) the wave recedes to infinity with the speed of light, its amplitude decreasing as $|q/2st|$.

The fluid potential is of the form

$$\varphi=(^3/_2)^{1/2}\ln(k^{-1}\,\mathrm{tg}\,kt) \tag{3.22}$$

and from Eq. (2.5) we get the energy density

$$\varepsilon=3a_0^{-2}s^2r^{-1}k^3LQ^{-1}\sin^{-3}2kt. \tag{3.23}$$

In these formulas a_0, s, k, σ, r, and γ are arbitrary constants, connected by two relations:

$$r=-k^{-2}\,\mathrm{tg}^2\,2\gamma,\qquad \sigma=s^2-r. \tag{3.24}$$

The quantities μ and R are functions of the coordinates given by

$$\mu=a_0^2(2k^2)^{-1}\{\cos 2\gamma+\cos 2kt\cos 2kz-[(\cos 2\gamma+\cos 2kt\cos 2kz)^2-\sin^2 2kt\sin^2 2kz]^{1/2}\} \tag{3.25}$$

$$R=k^6a_0^{-4}\cos^{-2}2\gamma[a_0^2k^{-4}\,\mathrm{tg}^2\,kt(\cos 2\gamma-\cos 2kz)-\mu k^{-2}\cos^{-2}kt]^2, \tag{3.26}$$

where the square bracket taken to the power in $\frac{1}{2}$ in Eq. (3.25) means the arithmetic root.

The solution depends on two essentially new constant parameters, γ and s. The constant a_0 is of the same nature as in the background models (2.25) and (2.26), and the constant k (if $k\neq 0$) can be eliminated by a transformation of the constants and a scale transformation $(kt, kz)\to(t, z)$. The choice of the constant k determines the type of the model. For real values of k (in this case we can take $k=1$) the solution describes the evolution of a soliton on the background of the closed Friedmann model. For imaginary k (here we can set $k=i$) we get the analogous solution on the background of the open model, and the case $k=0$ reduces to the soliton perturbation of the flat model, which we have already discussed. It is not hard to carry out the passage to the limit $k\to 0$, by setting $a_0^2=\frac{1}{2}$ and renaming the constants in the following way:

$$s=2s',\ \sigma=4q,\ \cos 2\gamma=-1-2k^2l^2,$$

where the constants s', l, and q are to be regarded as independent of k. In this case we get from Eqs. (3.24)–(3.26) in the limit $k\to 0$:

$$r=4l^2,\ q=s'^2-l^2,\ R=4k^2(t^2+\mu)^2,$$

and for the function μ we get the formula (3.3), if we consider $k^2<0$ (for $k^2>0$ we get a result analogous to Eq. (3.3), but with the minus sign for the square root). It can now be verified that the limit of the metric (3.20) for $k\to 0$ exists and can be reduced by a simple transformation to the form (3.1), in which s' will appear instead of s.

It is easy to see that the solution (3.20)–(3.26) goes over into the background solutions (2.24)–(2.26) through taking the limit with respect to the parameter $\sigma(\sigma\to 0)$. As has already been pointed out, we shall consider here only those regions of variation of the arbitrary constants in which our solutions have no additional singularities beyond the initial cosmological singularities that are already present in the background models. The solution (3.20)–(3.26) in fact has this property if the constant r is positive:

$$r=-k^{-2}\,\mathrm{tg}^2\,2\gamma>0. \tag{3.27}$$

This condition means that for the closed model (real k) we must choose a purely imaginary γ, and for the open model vice versa: imaginary k, but real γ. The constant s is always real, and consequently, with the condition $r>0$ the parameter $\sigma=s^2-r$ can in fact go to zero. It is easy to see that for $\sigma=0$ the metric gives

$$-ds^2=a_0^2k^{-1}\sin 2kt(-dt^2+dz^2+k^{-2}\sin^2 kz\,dx^2+\cos^2 kz\,dy^2), \tag{3.28}$$

and it follows from Eq. (3.23) that

$$\varepsilon=3a_0^{-2}k^3\sin^{-3}2kt. \tag{3.29}$$

For $k=0$ ($a_0^2=\frac{1}{2}$), $k=i$, and $k=1$ the form (3.28) becomes identical with the respective metrics (2.24), (2.25), and (2.26). There is similar agreement for the potential φ and the energy density ε.

We note also that the determinant of the matrix g, i. e., of the two-row block g_{ab}, has the following simple form:

$$\det g=\alpha^2,\ \alpha=a_0^2(2k^2)^{-1}\sin 2kt\sin 2kz, \tag{3.30}$$

and, as can be seen from Eq. (3.28), remains the same as in the unperturbed background metrics.

For the case of the open model the solution (3.20)–(3.26) describes approximately the same pattern of evolution of the soliton as is found in the flat model. With the closed model, on the other hand, there are naturally qualitative differences because there are no infinite values for either the time or the space coordinates. For this reason we confine ourselves here to closed model only. In all further formulas the parameter k is regarded as real, and the parameter γ, as imaginary. In the closed space the evolution of the model occupies a finite time interval from the moment $kt=0$ (the big bang) to the time of collapse of the Universe, $kt=\pi/2$. It is not hard to show that near the initial instant $kt\to 0$ the asymptotic form of the solution (3.20)–(3.26) is

$$\begin{gathered}
g_{11}=a_0^2k^{-3}\sin 2kt\sin^2 kz[1+\sigma s^{-2}\sin^2\gamma\sin^2 kz(\cos^2 kz-\sin^2\gamma)^{-1}],\\
g_{22}=a_0^2k^{-1}\sin 2kt\cos^2 kz[1-\sigma r^{-1}\sin^2\gamma\sin^2 kz(\cos^2 kz-\sin^2\gamma)^{-1}],\\
g_{12}=-a_0^2\sigma(4ks)^{-1}\cos 2\gamma\sin 2kt\sin^2 2kz(\cos 2\gamma+\cos 2kz)^{-1},\\
f=a_0^2k^{-1}\sin 2kt,\quad \varepsilon=3k^3a_0^{-2}\sin^{-3}2kt,
\end{gathered} \tag{3.31}$$

and near the finite cosmological singularity $kt\to\pi/2$ we get for these same quantities:

$$\begin{gathered}
g_{11}=a_0^2k^{-3}\sin 2kt\sin^2 kz(1+\sigma s^{-2}\sin^2\gamma\cos^2 kz(\sin^2 kz-\sin^2\gamma)^{-1}],\\
g_{22}=a_0^2k^{-1}\sin 2kt\cos^2 kz[1-\sigma r^{-1}\sin^2\gamma\cos^2 kz(\sin^2 kz-\sin^2\gamma)^{-1}],\\
g_{12}=a_0^2\sigma(4ks)^{-1}\cos 2\gamma\sin 2kt\sin^2 2kz(\cos 2\gamma-\cos 2kz)^{-1},\\
f=rs^{-2}a_0^2k^{-1}\sin 2kt,\quad \varepsilon=3s^2r^{-1}k^3a_0^{-2}\sin^{-3}2kt.
\end{gathered} \tag{3.32}$$

In Eq. (3.31) one must take $\sin 2kt\approx 2kt$, and analogously in (3.32) $\sin 2kt\approx\pi-2kt$.

The field of the soliton for the solution (3.20)–(3.26) is determined as before by the perturbation matrix H and the fractional change E of the energy density. These components are given by the same formulas (3.6)–(3.8), in which the quantities g_{ab} and ε must be taken to mean the expressions shown in Eq. (3.20) and (3.23), and the corresponding quantities with the index zero refer to the background solution (3.28), (3.29). Setting

$$k=1,\quad \sin\gamma=i\Delta,\quad s=2p\Delta(1+\Delta^2)^{1/2}(1+2\Delta^2)^{-1}, \tag{3.33}$$

where Δ and p are new arbitrary constants (and Δ is already real), we get from Eqs. (3.24) and (3.31) the asymptotic values of the perturbation fields H and E for $t \to 0$:

$$H_{11}=(1-p^2)p^{-2}\Delta^2\sin^2 z(\cos^2 z+\Delta^2)^{-1},\quad H_{22}=(p^2-1)\Delta^2\sin^2 z(\cos^2 z+\Delta^2)^{-1},$$
$$H_{12}=(1-y^2)p^{-1}\Delta(1+\Delta^2)^{1/2}\sin z\cos z(\cos^2 z+\Delta^2)^{-1},\quad E=0, \tag{3.34}$$

and from Eqs. (3.24) and (3.32) we get their asymptotic forms for $t \to \pi/2$:

$$H_{11}=(1-p^2)p^{-2}\Delta^2\cos^2 z(\sin^2 z+\Delta^2)^{-1},\quad H_{22}=(p^2-1)\Delta^2\cos^2 z(\sin^2 z+\Delta^2)^{-1},$$
$$H_{12}=(p^2-1)p^{-1}\Delta(1+\Delta^2)^{1/2}\sin z\cos z(\sin^2 z+\Delta^2)^{-1},\quad E=p^2-1. \tag{3.35}$$

These formulas show clearly the distribution of the perturbation H in the initial and final moments of the evolution. Near the initial instant the absolute values of the components H_{11} and H_{22} are largest at $z=\pi/2$ and equal to zero for $z=0$. The absolute value of H_{12} has its maximum in the region $\pi/4<z<\pi/2$. With the passage of time the maxima in the distributions of the quantities H_{11} and H_{22} are shifted in space, and at the finite time $t=\pi/2$ they are at $z=0$, while H_{11} and H_{22} go to zero at the former position of the maxima, $z=\pi/2$. The extremum of the component H_{12} also shifts during the cycle of evolution, through a finite distance in the direction of smaller values of z, and for $t\to\pi/2$ it is in the range $0<z<\pi/4$. Figures 2 and 3 show the initial and final profiles of the perturbations H_{22} and H_{12} as functions of the value of the parameter Δ, which determines the widths of the corresponding distributions (for definiteness we consider $p>1$, $\Delta>0$ and use a fixed value of the parameter p).

In a closed space ($0\leqslant z\leqslant\pi/2$) we can speak of localization of the perturbations only for a sufficiently small value of Δ. If $\Delta\ll 1$, then we can see from Eqs. (3.34) and (3.35) and the figures that the field of the soliton at the beginning and at the end of the evolution is concentrated near $z=\pi/2$ and $z=0$, respectively, in narrow ranges of width $\delta z\sim\Delta$, which are much smaller than the number $\pi/2$, i.e., than the linear extent of the Universe in the coordinate z. With this condition the picture of the evolution of the soliton in the stage of expansion of space is partially similar to what happens in the flat model; during a short time interval after the beginning ($t\lesssim\Delta$) the pertrubation H in the region around $z=\pi/2$

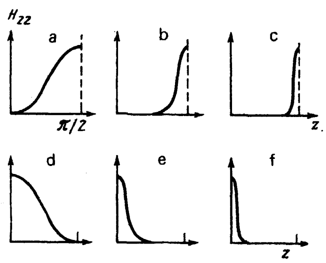

FIG. 2. Profiles of the initial and final distributions of the perturbation component H_{22} in closed models. Curves a, b, and c correspond to the beginning of the evolution, $t\to 0$, and d, e, and f, to the final time $t\to\pi/2$; a and d correspond to very large values of the parameter Δ, for which the width of the soliton is comparable with the size of the Universe; b,d and c,e show the change of shape of the initial and final distributions as Δ is made smaller. The value of H_{22} at the maximum is p^2-1 throughout.

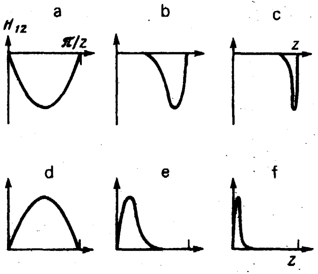

FIG. 3. Profiles of the initial and final distributions for the component H_{12}. The upper row shows profiles near the time $t=0$, and the lower row shows them near the final moment $t=\pi/2$. Curves a and d are for very large values of the parameter Δ, and b,e and c,f show the change of shape of the distributions as Δ is made smaller. The respective extreme values of H_{12} on the upper and lower diagrams are $(1-p^2)/2p$ and $(p^2-1)/2p$, and the coordinate values at which they occur are given by the equations $\cos 2z_0=-(1+2\Delta^2)^{-1}$ and $\cos 2z_0=(1+2\Delta^2)^{-1}$, respectively.

will die away, without changing the general shape and width of its profiles. Near the points with $z=0$, on the other hand, the perturbation H begins to grow. After a critical time $t\sim\Delta$ this process will continue, but along with it a gravitational wave appears from the region near $z=\pi/2$ and is propagated toward $z=0$ with the speed of light. At the time of maximum expansion, $t=\pi/4$, it has passed through a "quarter of the Universe" and reaches the region with $z=\pi/4$.

With further increase of the time from $t=\pi/4$ to $t=\pi/2$ the perturbation H becomes concentrated in the region at $z=0$, and after a time $t\sim\pi/2-\Delta$ it absorbs the wave which has arrived there. The final distribution of the field of the soliton is given by Eqs. (3.35) and again has a small width $\delta z\sim\Delta$. It can be shown that the distribution of the field in the gravitational wave itself is similarly small in width during the entire time of its propagation from the region $z=\pi/2$ to the region at $z=0$. The process is shown schematically in Fig. 4.

For large values of the parameter Δ both the initial and the final distributions of the field of perturbations has a width of the order of the size of the whole Universe (corresponding to the profiles shown in Fig. 2, a and d). For any observer to study the profile of the soliton will require a time of the order of the entire cycle of evolution of the Universe, and the usual interpretation of a soliton as a single localized disturbance can be applied in this case only in a conventional sense.

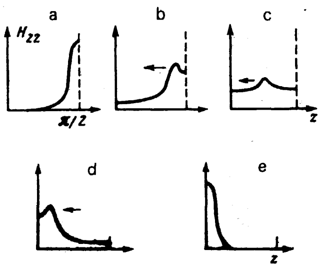

FIG. 4. Schematic representation of the evolution of a soliton in the closed model. The sequence a–e corresponds to variation of the time from $t=0$ to $t=\pi/2$. The picture corresponds to rather small values of the parameter Δ.

As for the perturbation E of the energy density, in the approximation considered here it is zero at the beginning of the evolution, and Eg. (3.35) shows that at the concluding stage of collapse it becomes constant in space, producing a change of the parameters of the background Friedmann model. Here we again encounter the same phenomenon as was described in the analysis of the perturbations on the background of the flat model.

In conclusion we note that the geometrical loci of the points $z=\pi/2$ and $z=0$ are circles in the closed three-dimensional space of the Universe, and are great circles on this hypersphere. It can be seen from Eqs. (2.18) and (2.19) that in standard four-dimensional spherical coordinates the equation of the circle $z=\pi/2$ is $\chi=\pi/2$, $\theta=\pi/2$ and it may be arbitrarily called the equator. The equation of the circle $z=0$ is $\theta=0$ and $\theta=\pi$, and it can be called the polar axis. These circles have no points in common. The equator and polar axis so defined are completely equivalent and can be interchanged by a suitable transformation of the four-dimensional coordinates. These closed curves are the equivalent of what qas an infinite axis of cylindrical symmetry of the soliton in the open models.

§4. VACUUM SOLUTIONS

It was shown in Sec. 2 that for any solution of the form (1.1) in a space with matter described by a potential φ there is a corresponding solution of the gravitational equations in vacuum, of the form (2.10). By using Eq. (2.9) this solution can be written in the form

$$-ds^2=fF^{-1}(-dt^2+dz^2)+g_{ab}dx^a dx^b, \tag{4.1}$$

where the functions f and g_{ab} are precisely the same as in the solution with matter, and the coefficient F is determined by the equations (2.15). It is easy to find this coefficient for the solutions (3.20)–(3.26) by substituting in Eq. (2.15) the expressions for the potential φ [Eq. (3.22)] and the function α [Eq. (3.30)]. A simple integration gives

$$F=F_0(\sin 2kt)^{3/2}(\sin 4k\zeta)^{-3/4}(\sin 4k\eta)^{-3/4}, \tag{4.2}$$

where ζ and η are the light variables (2.8) and F_0 is an arbitrary constant. Substituting this result along with the metric coefficients f and g_{ab} of Eq. (3.20) in the expression (4.1), we get the desired vacuum solution.

We note here that an interesting qualitative study of closed vacuum cosmological models with metrics of the type (1.1) has been given by Gowdy.[5]

APPENDIX

We shall here describe briefly the method for deriving the solutions presented in Sec. 3. As shown in the previous paper,[1] the main step in finding them is the determination of the matrix functions $\psi(\lambda,\zeta,\eta)$ corresponding to the background metrics (3.28). Such a function satisfies the equations

$$D_1\psi=(\lambda-\alpha)^{-1}A\psi, \quad D_2\psi=(\lambda+\alpha)^{-1}B\psi, \tag{A.1}$$

where the operators D_1 and D_2 are given by

$$D_1=\partial_\zeta-2(\lambda-\alpha)^{-1}\alpha_{,\zeta}\lambda\partial_\lambda, \quad D_2=\partial_\eta+2(\lambda+\alpha)^{-1}\alpha_{,\eta}\lambda\partial_\lambda. \tag{A.2}$$

Here λ is a complex spectral parameter, ζ and η are the variables (2.8), α^2 is the determinant of the matrix g of the background solution, which is of the form (3.30), and the matrices A and B are defined from Eq. (2.7) with the same matrix g, which, as shown in Eq. (3.28), is

$$g=\mathrm{diag}\,(a_0^2k^{-3}\sin 2kt\sin^2 kz,\ a_0^2k^{-1}\sin 2kt\cos^2 kz) \tag{A.3}$$

(the commas in Eq. (A.2) and the letter ∂ denote ordinary differentiation).

Integration of Eqs. (A.1) and (A.2) with $k\neq 0$ leads to the following diagonal matrix ψ:

$$\psi_{11}=[a_0^4k^{-6}\sin^2 2kt\sin^2 kz-a_0^2k^{-4}\lambda\cos 2kt-k^{-2}(\alpha^2+2\beta\lambda+\lambda^2)]^{1/2},$$
$$\psi_{22}=(\alpha^2+2\beta\lambda+\lambda^2)\psi_{11}^{-1}, \quad \psi_{12}=0, \tag{A.4}$$

where

$$\alpha=a_0^2(2k^2)^{-1}\sin 2kt\sin 2kz, \quad \beta=-a_0^2(2k^2)^{-1}\cos 2kt\cos 2kz. \tag{A.5}$$

The limit $k\to 0$ cannot be taken directly in Eqs. (A.4) and (A.5), as in the case of the flat model, but it is easy to find a solution for which it is possible. The point is that the matrix ψ and the function β (the second solution of the wave equation satisfied by α) are not uniquely determined. The function β is determined up to an arbitrary additive constant, and the matrix ψ, up to multiplication from the right by an arbitrary matrix of the argument $w=\frac{1}{2}(\alpha^2\lambda^{-1}+2\beta+\lambda)$. Using this freedom, we can reconstruct the solution (A.4), (A.5) so that it has a limit for $k=0$. We have not done this, however, and in constructing the solutions (3.20)–(3.26) for $k\neq 0$ we have used just the formulas (A.4), (A.5) (the indicated transformation for $k=0$ would not change anything in the solution except to redefine the constants). The matrix ψ for the flat model can be found either by the method indicated or by direct integration of Eqs. (A.1) and (A.2). The result, which we have used in constructing the solution (3.1), can be written in the form

$$\psi_{11}=[(z^2+\lambda)(\alpha^2+2\beta\lambda+\lambda^2)]^{1/2}, \quad \psi_{22}=(\alpha^2+2\beta\lambda+\lambda^2)\psi_{11}^{-1},$$
$$\psi_{12}=0, \quad \alpha=tz, \quad \beta=1/2(t^2+z^2). \tag{A.6}$$

The further operations that lead to the solution are merely algebraic and are explained in Ref. 1. We shall not repeat them here, but we point out the following important features. Starting from the background metric (3.28) and the ψ function (A.4)–(A.6), we arrive at solutions in which the natures of the variables t and z are in a certain sense reversed. Whereas in the background solutions (3.28) the matrix g has an isotropic cosmological singularity with respect to t and fictitious coordinate singularities with respect to z, in the one-soliton solutions an isotropic physical singularity appears with respect to the space variable z, and fictitious ones with respect to t. When we try to take the limit with respect to a parameter to obtain the background metric we get instead of the metric coefficients g_{ab} from Eq. (3.28) the same functions except that t and z are interchanged. Accordingly, to recover the cosmological character of the model, one must interchange the coordinates t and z and at the same time choose the correct sign of the metric coefficient f (so that the variable t will actually be timelike). This must be done first in the vacuum solution, and then one can turn to the solution with matter.

The final sequence of operations is: 1) with the metric (3.28) and the potential (3.22) we determine the vacuum background solution [the coefficient F for changing from f to f_V is given by Eq. (4.2)], 2) we apply the one-soliton perturbation to the vacuum solution, 3) in the result so found we make the interchange $t \to z$, $z \to t$ and choose the correct sign of f_V, 4) we again return to the solution with matter with the same potential (3.22). [The transition coefficient F remains unchanged, since the function α is not changed by the interchange $t \to z$, $z \to t$, and Eq. (2.15) is also unchanged when there is no change of the potential φ. But the solution of these equations, i. e., the coefficient F itself, does not have this symmetry, and $F(t,z) \neq F(z,t)$, which is important in this sequence of transformations.] After these operations we obtain a solution which can be reduced to the form (3.20) by a certain linear transformation (with constant coefficients) of the variables x and y.

Analogous operations in the case of the flat model give the metric (3.1). As was pointed out in Sec. 3, a transition from (3.20) to (3.1) in the limit $k \to 0$ exists, although it does not exist in explicit form between Eqs. (A.4), (A.5), and (A.6). The linear transformation of the coordinates x, y is made from considerations of convenience of the final result; only after this transformation do we get the metric (3.20), in which: a) there is a transition in the limit with respect to a parameter to the form (3.28), b) the coefficient g_{11} goes to zero at $z = 0$, and c) for the closed model the coefficient g_{22} goes to zero for $kz = \pi/2$. For the flat model the analogous transformation serves to satisfy conditions a) and b) and to make the behavior of the metric for $z \to \infty$ the same as in the background solution (2.24).

In the investigation of the properties of the solution (3.20)–(3.26) it is necessary to use certain identities connecting the funtions μ [Eq. (3.25)] and R [Eq. (3.26)]. We give them here. The function μ is a solution of the quadratic equation

$$\mu^2-(a_0^2k^{-2}\cos 2\gamma-2\beta)\mu+\alpha^2=0, \tag{A.7}$$

where α and β are given by Eq. (A.5). Besides this, the following two identities hold:

$$\begin{aligned} &r\alpha^2\mu^{-1}\,\mathrm{tg}^2\,kt+\mu R=a_0^2k^{-2}(\cos 2\gamma+\cos 2kz)L,\\ &r\mu+\alpha^2\mu^{-1}R\,\mathrm{ctg}^2\,kt=a_0^2k^{-2}(\cos 2\gamma-\cos 2kz)L, \end{aligned} \tag{A.8}$$

from which one further relation can easily be derived:

$$\begin{aligned} &r\mu(\cos^2\gamma-\sin^2 kz)+\mu R(\sin^2\gamma-\sin^2 kz)\\ &=a_0^2(2k^2)^{-1}\sin^2 2kz(r\sin^2 kt-R\cos^2 kt). \end{aligned} \tag{A.9}$$

The quantity L in Eq. (A.8) is determined by Eq. (3.21).

[1] A system of units is used in which the speed of light and the gravitational constant are equal to unity. The interval is written in the form $-ds^2=g_{ik}dx^idx^k$, where g_{ik} has the signature $(-, +, +, +)$.

[2] We impose this condition only for definiteness. The opposite sign of the square root in Eq. (3.3) leads to the same physical results. The same is true for the function μ in the expressions (3.25) (see further discussion).

[3] Throughout its entire extent this world line remains spacelike and corresponds to the phase velocity of propagation of the wave. The physical velocity of the wave is equal to the speed of light for large times, and in other regions it is not uniquely definable. The values of the quantity H_{22} at the extremal points $z^2=s^{-1}(st+l^2)(t-s)$ are given by the simple expression $H_{22}=-q/2st$, from which it is apparent that as t increases this extreme value decays in the same way as the field H on the light cone $z=t$, Eq. (3.14).

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Translated by W. H. Furry