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A new conservation law for the Landau-Lifshitz equations

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An explicit expression for a new conservation law, which arises when allowance is made for the magnetic-dipole interactions, is obtained for the case of waves of stationary profile propagating in a uniaxial ferromagnet in a direction perpendicular to the anisotropy axis. It is shown that the new first integral allows the determination of the dependence of the amplitude of stationary-profile waves of all types on the velocity and the characteristic parameter of the magnetic medium. The dependence thus found is in complete agreement with previous numerical calculations.

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The analysis of the self-localized solutions of the Landau–Lifshitz equations is of indubitable interest not only in connection with the development of the theory of moving domain boundaries in magnetically ordered media, but also in connection with the search for magnetic solitons. Our previous[1] qualitative and numerical analysis of the self-localized solutions of the Landau–Lifshitz equations allowed us to give an essentially complete classification of waves of stationary profile. It was shown, in particular, that there exist in uniaxial ferromagnets, besides the well-known solutions, self-localized solutions corresponding to both "slow" and "fast" waves of stationary profile. The slow stationary-profile wave is a solitary wave with respect to one of the angular variables and a wave of the type of a moving domain boundary with respect to the other angular variable. The fast stationary-profile waves are characterized by the fact that the magnetic-moment vector executes precession about the anisotropy axis and the fact that, as the velocity of the stationary-profile wave approaches the lower cutoff velocity for spin waves, the nutational motion of the magnetic-moment vector gets excited.

Further investigations have shown that the Landau–Lifshitz equations for stationary-profile waves propagating in a direction perpendicular to the anisotropy axis admit in the case when the local magnetic fields (the magnetic-dipole interactions) are taken into account of the existence not only of the well-known conservation law connected with the invariance of the Lagrangian under spatial translation, but also of another distinctive law of conservation (of the first integral).

V. I. Arnold proposed the use of the Hénon–Heiles method[2] to demonstrate numerically the existence of the new first integral and the analogy with the dynamics of the material particle to seek the explicit form of the conservation law. The numerical computations confirmed the existence of the new first integral, and simple transformations of the Landau–Lifshitz equations allowed in the case of the simplest form of uniaxial-anisotropy energy the derivation of an explicit expression for it. Knowing the new first integral, we can obtain for the dependence of the characteristic amplitude of the slow or fast magnetic soliton on the velocity and parameter of the uniaxial magnetic medium explicit expressions that are in complete agreement with the results of the numerical analysis.[1]

2. The Landau–Lifshitz equations for stationary-profile waves propagating in a uniaxial ferromagnet in a direction perpendicular to the anisotropy axis can be written in terms of the dimensionless variables obtained by choosing the characteristic dimension of the Bloch–Landau domain boundary as the unit of length and the reciprocal of the precession frequency in the anisotropy field as the unit of time in the form

$$\mu_x'+m_zm_y=-um_x',\quad \mu_y'-(1+\varepsilon)m_zm_x=-um_y',\quad \mu_z'+\varepsilon m_xm_y=-um_z'. \tag{2.1}$$

Here u is the velocity of the stationary-profile wave, $\varepsilon=2\pi M_s^2/K$ is the parameter of the magnetic medium (M_s is the saturation magnetization and K is the uniaxial anisotropy constant), and, finally,

$$\mu=[\mathbf{mm}'] \tag{2.2}$$

is the "angular momentum" of the unit magnetic-moment vector.

The well-known conservation law (first integral) of the Landau–Lifshitz equations (2.1) is the quadratic form

$$(m_x')^2+(m_y')^2+(m_z')^2=(1+\varepsilon)m_x^2+m_y^2+\mathscr{H}, \tag{2.3}$$

where $\mathscr{H}$ is the constant of the first integration. In terms of angular variables with the polar axis directed along the anisotropy axis, the first integral (2.3) assumes the form[1]

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$$(\theta')^2+\sin^2\theta(\varphi')^2=(1+\varepsilon\cos^2\varphi)\sin^2\theta+\mathcal{H}. \qquad (2.4)$$

The scheme of the numerical analysis based on the Hénon–Heiles method[2] is as follows. Let

$$Q(\theta',\varphi';\theta,\varphi)=\text{const} \qquad (2.5)$$

be the new first integral of the Landau–Lifshitz equations. Using the known first integral (2.4), we arrive at the relation

$$Q(\theta',\theta;\varphi)=\text{const}. \qquad (2.6)$$

In the numerical integration of the Landau–Lifshitz equations, we determine the points of intersection of an arbitrary integral curve with the planes

$$\varphi=k\pi,\quad k=0,\pm1,\pm2,\ldots \qquad (2.7)$$

Then if the new first integral (2.5) exists, the above-indicated points in the (θ',θ) plane will, after a sufficiently long calculation, be arranged on some curve, namely, the level curve of the new first integral (2.6). When the initial conditions are changed, the points in the (θ',θ) plane will be arranged on a different curve, corresponding to the new value of the level constant of the first integral (2.6). Figure 1 shows the characteristic level curves of the new first integral that were obtained as a result of the numerical calculations.

3. Let us proceed to the derivation of the explicit expression for the new first integral of the Landau–Lifshitz equations. Let us first consider the case of zero stationary-profile-wave velocity. The first pair of the Landau–Lifshitz equations (2.1) can, after simple transformations and allowance for the known first integral (2.3), be written in the form

$$\begin{aligned} &m_x''+[\mathcal{H}-1-\varepsilon+2(1+\varepsilon)m_x^2+2m_y^2]m_x=0,\\ &m_y''+[\mathcal{H}-1+2(1+\varepsilon)m_x^2+2m_y^2]m_y=0. \end{aligned} \qquad (3.1)$$

The transformation

$$m_x\to\tilde{m}_x=(1+\varepsilon)^{1/2}m_x,\quad m_y\to\tilde{m}_y=m_y \qquad (3.2)$$

reduces the system (3.1) to the form

$$\begin{aligned} &\tilde{m}_x''+[\mathcal{H}-1-\varepsilon+2(\tilde{m}_x^2+\tilde{m}_y^2)]\tilde{m}_x=0,\\ &\tilde{m}_y''+[\mathcal{H}-1+2(\tilde{m}_x^2+\tilde{m}_y^2)]\tilde{m}_y=0, \end{aligned} \qquad (3.3)$$

which clearly indicates the existence of the first integral

$$(\tilde{m}_x')^2+(\tilde{m}_y')^2+(\mathcal{H}-1)(\tilde{m}_x^2+\tilde{m}_y^2)-\varepsilon\tilde{m}_x^2+(\tilde{m}_x^2+\tilde{m}_y^2)^2=Q\equiv\text{const} \qquad (3.4)$$

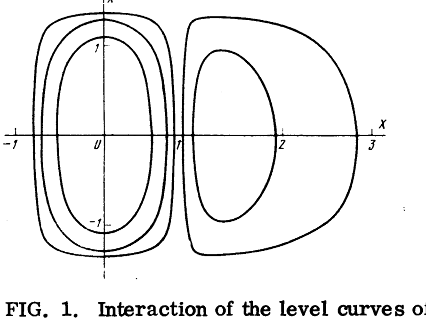

FIG. 1. Interaction of the level curves of the new first integral $Q(\vartheta',\varphi',\vartheta,\varphi)$ with the (X',X) plane for the case $\varepsilon=1$, $\mathcal{H}=0$; $X=\ln\operatorname{tg}(\vartheta/2)$.

or (in terms of the original variables)

$$\begin{aligned} &(1+\varepsilon)(m_x')^2+(m_y')^2+(\mathcal{H}-1)[(1+\varepsilon)m_x^2+m_y^2]\\ &\quad-\varepsilon(1+\varepsilon)m_x^2+[(1+\varepsilon)m_x^2+m_y^2]^2=Q. \end{aligned} \qquad (3.5)$$

Thus, in the language of mechanics, to the new first integral (3.5) corresponds the motion in a potential field of a material particle with an anisotropic mass. The anisotropic mass is connected with the allowance for the local magnetic fields. The above-presented result can easily be generalized to the case of nonzero stationary-profile-wave velocities.

However, we can propose a simpler derivation of the new first integral. To wit, multiplying the first and last equations of the system (2.1) respectively by the quantities $\varepsilon(\mu_x+um_x)$ and (μ_z+um_z) and adding the results, we arrive at a different expression for the new first integral in the form of a quadratic form in the variables (μ, m):

$$(\mu_z+um_z)^2-\varepsilon(\mu_x+um_x)^2+\varepsilon m_y^2=Q. \qquad (3.6)$$

The two first integrals (2.3) and (3.6) allow us to determine the connection between the maximum amplitude of the magnetic solitons (the maximum value, θ_m, of the polar angle) and the corresponding magnitude, φ_m, of the azimuthal angle:

$$\cos\theta_m=\frac{u(1+\varepsilon\cos^2\varphi_m)^{1/2}-(1+\varepsilon)^{1/2}}{1+\varepsilon\cos^2\varphi_m}. \qquad (3.7)$$

In particular, for the two types of symmetric solitons we find that

$$\begin{aligned} &\cos\theta_m=(u-1)/(1+\varepsilon)^{1/2},\quad \varphi_m=0,\\ &\cos\theta_m=u-(1+\varepsilon)^{1/2},\quad \varphi_m=\pi/2. \end{aligned} \qquad (3.8)$$

The obtained relations (3.8) are in complete agreement with the results of our earlier numerical computations.[1]

It should be noted that, on going over to those degenerate solutions of the Landau–Lifshitz equations which correspond to a fixed—in space—orientation of the plane of rotation of the magnetic moment ($\varphi\equiv\text{const}$), there arises a dependence of the first integrals (2.3) and (3.6) that reflects the transition to a system with one angular degree of freedom. As a consequence, there arises the well-known relation between the velocity and the orientation of the plane of rotation of a magnetic moment. When the magnetic-dipole interactions are neglected (i.e., for $\varepsilon\to0$), the new first integral leads to the conservation of the component of the angular momentum $\boldsymbol{\mu}$ along the anisotropy axis.

4. In conclusion, let us note that the numerical analysis of the Landau–Lifshitz equations for the case of a uniaxial anisotropy of a more general form, e.g.,

$$K(1+\beta\sin^2\theta)\sin^2\theta,\quad \beta\geqslant0 \qquad (4.1)$$

or for the case of a constant external field directed along the anisotropy axis, also indicates the existence of a new first integral. However, the attempts to obtain an explicit expression for the conservation law in these cases have not met with success.

Let us note that, with the aid of the substitution

$$\xi=\frac{m_x'}{m_x},\quad \eta=\frac{m_y'}{m_y},\quad \zeta=\frac{m_z'}{m_z}, \tag{4.2}$$

which defines as new variables the logarithmic derivatives of the components of the unit magnetic-moment vector, we can write the Landau–Lifshitz equations in the above-indicated cases in the form

$$\begin{aligned}\zeta'-\eta'&=-1-\zeta^2+\eta^2+R_1(m_x,\ m_y,\ m_z),\\ \xi'-\zeta'&=1+\varepsilon-\xi^2+\zeta^2+R_2(m_x,\ m_z,\ m_z),\\ \eta'-\xi'&=-\varepsilon-\eta^2+\xi^2+R_3(m_x,\ m_y,\ m_z).\end{aligned} \tag{4.3}$$

The explicit form of the functions $R_i(\mathbf{m})$ is determined by the choice of the specific expression for the anisotropy energy or the external fields. What is important is the fact that the functions $R_i(\mathbf{m})$ satisfy the condition

$$R_1(\mathbf{m})+R_2(\mathbf{m})+R_3(\mathbf{m})=0. \tag{4.4}$$

Consequently, the system of three first-order equations (4.3) and its solutions in the three dimensional (ξ,η,ζ) space are determined up to some two-dimensional surface

$$\Gamma(\xi,\ \eta,\ \zeta)=0, \tag{4.5}$$

The obvious relation

$$m_xm_x'+m_ym_y'+m_zm_z'=0 \tag{4.6}$$

and the known first integral

$$\mathcal{H}(m_x',\ m_y',\ m_z',\ m_x,\ m_y,\ m_z)=\text{const} \tag{4.7}$$

assume, when the substitution (4.2) is taken into account, the form of two mixed forms:

$$\begin{gathered}m_x^2\xi+m_y^2\eta+m_z^2\zeta=0,\\ \mathcal{H}(\xi,\ \eta,\ \zeta;\ m_x,\ m_y,\ m_z)=\text{const.}\end{gathered} \tag{4.8}$$

The relations (4.8) can be solved for the unknown pair of magnetic-moment-vector components (e.g., m_x and m_y), and, consequently, allow us to express the functions R_i in terms of the new variables (ξ,η,ζ). However, the three new variables (ξ,η,ζ) should satisfy the equation (4.5) of some surface.

Thus, we arrive at the conclusion that the existence of a third mixed form, similar to the forms (4.8), and, consequently, of a new first integral, is possible. Such a situation arises in the above-investigated case of stationary-profile waves for ferromagnets with the simplest type of uniaxial anisotropy. To wit, the new first integral (3.6) determines the third missing mixed form

$$Q(\xi,\ \eta,\ \zeta;\ m_x,\ m_y,\ m_z)=\text{const} \tag{4.9}$$

and, consequently, the surface (4.5) in the (ξ,η,ζ) phase space.

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Note added in proof (20 April 1979). As a result of a more accurate numerical analysis for an anisotropy energy of the form (4.1), we have discovered the decay of the level surfaces of the new first integral, an effect which corresponds to the earlier studied[1] lifting of the accidental degeneracy of the separatrix solutions. This indicates that the Landau–Lifshitz equations are not a completely integrable system in the case of uniaxial anisotropy of the general type.

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