can lead to a change in the magnitude and establishment time of an equilibrium charge in an oriented crystal target. The kinetic equations of the equilibrium-charge theory should take correct account of the elementary processes of ionization and recombination of ions in the channeling regime.

1) A generalization of the results to the case of several nearest chains will be presented in Sec. 4.

2) We neglect small shifts of the levels due to their interaction with the states of the continuous spectrum.

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The Kramers-Wannier transformation for spin systems

A. B. Zamolodchikov and M. I. Monastyrskiĭ

Institute of Theoretical and Experimental Physics

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The Kramers-Wannier transformation is constructed for spin systems on a plane lattice. Systems with discrete nonabelian groups are considered, including generalized Potts models. The existence of three different phases in these models is predicted.

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1. INTRODUCTION

In 1941 Kramers and Wannier[1] discovered a special symmetry which relates low-temperature and high-temperature expansions in the plane Ising model. The corresponding transformation, the Kramers-Wannier (KW) transformation, is a definite nonlocal substitution on the variables in the sum over states (partition function). After this substitution the statistical sum involves not the original "spin" variables $\sigma=\pm 1$, defined on the nodes of the lattice, but new "spin" variables $\mu=\pm 1$ defined on the faces of the lattice, or, equivalently, on the nodes of the "dual" lattice. Furthermore the new Hamiltonian, expressed in terms of μ, differ from the original one by the replacement $\sigma\to\mu$ and also a transformation of the temperature parameter:

$$\beta\to\beta^*=\operatorname{arth} e^{-2\beta}. \tag{1.1}$$

The transformation (1.1) establishes a connection between values of the statistical sum in high-temperature and low-temperature phases and, in particular, enables one to find the exact value of the critical temperature β_c^{-1}. In the phase in which the fluctuation of the order parameter σ are large, those of the variables μ are restricted, and conversely. For the "dual" variables μ we therefore use the name "disorder parameter."[2]

The existence of this sort of transformations is evidently a very general property of lattice statistical systems that possess a symmetry group. KW transformations have been constructed explicitly for a number of systems on a plane lattice. These include the N-position models of Potts (see Ref. 3) and generalized Ising models, systems with spins taking values in groups Z_N (Z_N Ising models),[4,5] and in the group $U(1)$ (XY model[6]). For Z_N systems the disorder parameter also takes a value in the group Z_N. Therefore the KW transformation reduces to a transformation of numerical parameters of the Hamiltonian, just as in the ordinary Ising model. In the case of the XY model the disorder parameter is an element of the group Z_N and the KW transformation relates to each other statistical sums of different spin systems.

KW transformations can also be carried out for some systems on many-dimensional lattices. Here new possibilities arise. For example, a KW transformation connects the three-dimensional Ising model with the gauge Ising model, and the four-dimensional gauge

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Ising model is self-dual.[7,8] There is an analogous situation for many-dimensional systems with other commutative symmetry groups (cf., e.g., Refs. 9–11).

The example of the plane Ising model[12] shows that the KW transformation can serve as a powerful instrument for exact study of statistical systems.[2] The KW transformation is of particular interest for four-dimensional lattice gauge theories.[13] There are arguments favoring the idea that the introduction of a disorder parameter would make possible a natural way to describe the phase of nonemerging quarks.[14-17]

Whereas the KW transformation for systems with commutative symmetry groups can be carried out by general methods, the corresponding problem for non-Abelian systems remains an open one.

In the present paper we consider some special examples of plane lattice systems with nonabelian symmetries. These include generalized Potts models, associated with homogeneous spaces of symmetric groups, a spin system running through the group S_3 (the group of transformations of a triangle), and two systems associated with homogeneous spaces of discrete subgroups of the group $O(3)$ (regular polyhedra). In all of these cases the KW transformation can be carried out owing to a special reduction to the commutative case.

We hope that these examples will help to elucidate the general situation with noncommutative groups.

2. SPIN SYSTEMS ON A PLANE LATTICE

Let us consider a plane square lattice with unit edge. Let $x=\{x_\mu\}=\{x_1, x_2\}$ (where x_1 and x_2 are integers) be the coordinates of the nodes, and $e^\alpha_\mu=\{e^1_\mu, e^2_\mu\}=\delta^\alpha_\mu$ be the basis vectors of the lattice. We will often use the notation $x+\hat{\alpha}\equiv\{x_\mu+e^\alpha_\mu\}$. A double index x, α is convenient for denoting the edge in the lattice which connects the nodes x and $x+\hat{\alpha}$. In what follows we shall also need the dual lattice, whose nodes are at the centers of the faces of the original lattice (see Fig. 1). We denote the coordinates of a node of the dual lattice by $\tilde{x}$:

$$\tilde{x}=\{x_\mu+{}^1/_2 e_\mu{}^1+{}^1/_2 e_\mu{}^2\}.$$

We define at the nodes of the original lattice also the spin variables s_x; these take values in some manifold M, which we will call the spin space. We mainly confine ourselves to the case of a finite set M.

The simplest Hamiltonian of such a spin system involves only interactions of nearest neighbors; that is, it is of the form

$$\mathscr{H}=\sum_{x,\alpha} H(s_x, s_{x+\hat{\alpha}}), \tag{2.1}$$

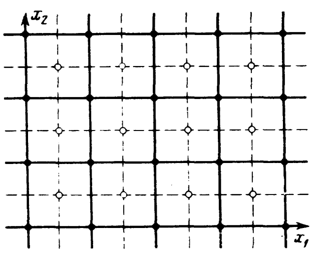

FIG. 1.

where the binary Hamiltonian $H(s, s')$ is a real function of a pair of points from M, with the properties

$$H(s, s')=H(s', s), \tag{2.2a}$$

$$H(s, s')\geqslant 0 \text{ for arbitrary } s, s'\in M, \quad H(s, s)=0. \tag{2.2b}$$

Accordingly, the binary Hamiltonian prescribes on M a structure similar to a metric structure (which in the general case is not a metric, since we nowhere require that the triangle inequality hold), which we shall call the H structure.

Of particular interest are examples in which the manifold M is a homogeneous space, i.e., there exists a group G of transformations of M which preserves the H structure: $H(Gs, Gs')=H(s,s')$ for arbitrary $s, s'\in M$. In this case the spin system has global symmetry with the group G.

Important special cases are systems on groups. For these the spin manifold coincides with a group G: $s_i=g_i\in G$, and the binary Hamiltonian is invariant under left and right translations:

$$H(hg, hg')=H(gh, g'h)=H(g, g') \quad \text{for arbitrary } h\in G. \tag{2.3}$$

The general H function of the system on the group can therefore be put in the form

$$H(g_1, g_2)=H(g_1g_2^{-1})=\sum_p h(p)\chi_p(g_1g_2^{-1}), \tag{2.4}$$

where $\chi_p(g)$ are the characters of the irreducible representations of the group G, and the constants $h(p)$ are chosen so that H has the properties (2.2) and are otherwise arbitrary.

The statistical sum of the general spin system with the Hamiltonian (2.1) is

$$Z=\sum_{s_x\in M}\prod_{x,\alpha} W(s_x, s_{x+\hat{\alpha}}), \tag{2.5}$$

where[1)]

$$W(s, s')=\exp\{-H(s, s')\}. \tag{2.6}$$

According to Eq. (2.2) the function W has the properties

$$W(s, s')=W(s', s), \quad 0\leqslant W(s, s')\leqslant 1, \quad W(s, s)=1. \tag{2.7}$$

For the system on a group we have also

$$W(g_1, g_2)=W(g_1g_2^{-1}), \quad W(g^{-1})=W(g). \tag{2.8}$$

3. SPIN SYSTEMS ON COMMUTATIVE GROUPS

For a spin system on a group G the sum over states (2.5) can be put in the following equivalent form:

$$Z=\sum_{g_{x,\alpha}\in G}\prod_{x,\alpha} W(g_{x,\alpha})\prod_{\tilde{x}}\delta(Q_{\tilde{x}}, I), \tag{3.1}$$

where the summation variables $g_{x,\alpha}$ are defined on the edges of the lattice[2)]

$$Q_{\tilde{x}}=g_{x,1}g_{x+\hat{1},2}g^{-1}_{x+\hat{2},1}g^{-1}_{x,2}, \tag{3.2}$$

and the δ-function is defined by the formula

$$\delta(g, I)=\begin{cases}1, & \text{if } g=I,\\ 0 & \text{otherwise.}\end{cases}$$

In fact, the general solution of the connection equation $Q_{\bar{x}} = I$ is

$$g_{x,\alpha} = g_x g^{-1}_{x+\hat{\alpha}}$$

and this brings us back to Eq. (2.5).

Systems on commutative groups are a special case, in which the δ-function in Eq. (3.1) can be factorized in the following way:

$$\delta(Q_{\bar{x}}, I) = \sum_p \chi_p(Q_{\bar{x}}) = \sum_p \chi_p(g_{x,1}) \chi_p(g_{x+\hat{1},2}) \chi_p^{-1}(g_{x+\hat{2},1}) \chi_p^{-1}(g_{x,2}). \quad (3.3)$$

This sort of factorization is of decisive importance and enables us to carry through the KW transformation for commutative groups in general form.

We note that for a commutative group G all irreducible representations are one-dimensional and their characters χ_p themselves form a commutative group $\hat{G}$ (the character group) with a group multiplication defined in accordance with the tensor product of representations.[3] By definition

$$\chi_{p_1 p_2}(g) = \chi_{p_1}(g)\chi_{p_2}(g), \quad \chi_{p^{-1}}(g) = \chi_p^{-1}(g),$$

and the unit element of $\hat{G}$ corresponds to the identical representation of G. Accordingly, the summation in Eq. (3.3) can be regarded as a summation over the elements of the dual group $\hat{G}$.

Substituting the expansion (3.3) in Eq. (3.1), we get after an obvious regrouping of factors

$$Z = \sum_{g_{x,\alpha} \in G} \prod_{x,\alpha} W(g_{x,\alpha}) \prod_{\bar{x}} \sum_{p_{\bar{x}}} \chi_{p_{\bar{x}}}(g_{x,1}) \chi_{p_{\bar{x}}}(g_{x+\hat{1},2}) \chi_{p^{-1}_{\bar{x}}}(g_{x+\hat{2},1}) \chi_{p^{-1}_{\bar{x}}}(g_{x,2})$$

$$= \sum_{p_{\bar{x}} \in \hat{G}} \prod_{\bar{x},\alpha} \tilde{W}(p_{\bar{x}} p^{-1}_{\bar{x}+\hat{\alpha}}), \quad (3.4)$$

$$\tilde{W}(p_{\bar{x}} p^{-1}_{\bar{x}+\hat{\alpha}}) = \sum_{g \in G} W(g) \chi_{p_{\bar{x}}}(g) \chi_{p^{-1}_{\bar{x}+\alpha}}(g) = \sum_{g \in G} W(g) \chi_{p_{\bar{x}} p^{-1}_{\bar{x}+\alpha}}(g). \quad (3.5)$$

The expression (3.4) defines a new, dual, spin system on the dual group $\hat{G}$ with a new binary Hamiltonian $\tilde{H}$, which is defined by the formula

$$\exp\{-\tilde{H}(p)\} = \tilde{W}(p). \quad (3.6)$$

The result can be formulated in the following way.

A spin system on a commutative group G with a binary Hamiltonian $H(g)(g \in G)$ is equivalent to a spin system on the character group $\hat{G}$ (and on the dual lattice) with the binary Hamiltonian $\tilde{H}(p)(p \in \hat{G})$ given by the Fourier transformation

$$\exp\{-\tilde{H}(p)\} = \sum_{g \in G} \exp\{-H(g)\} \chi_p(g). \quad (3.7)$$

This is a Kramers-Wannier transformation. In contradistinction to the "order variables" g_x, the name "disorder variables" can be given to the dual spins p_x.[2]

We indicate the character groups for some locally compact commutative groups:

$$G \leftrightarrow \hat{G}$$
$$Z \leftrightarrow U(1)$$
$$Z_N \leftrightarrow Z_N.$$

Here Z is the group of integers under addition, Z_N is the group of integers *modulo* N, and $U(1)$ is the group of rotations of a plane. The character group of a direct product of simple groups is the direct product of the character groups of the simple groups.

4. THE GENERALIZED POTTS MODEL

Another spin system for which the KW transformation is known is the Potts model. The spin space of the N-position Potts model, which we call P_N, contains N points $s_1, s_2, \ldots, s_N$ and has an H structure of maximal symmetry:

$$H(s_i, s_j) = \begin{cases} 0 & i=j \\ \beta & i \neq j \end{cases}. \quad (4.1)$$

In this case the spin space is isomorphic to the factor space S_N/S_{N-1}, where S_N is the group of permutations of N objects.

The simplest way to carry out the KW transformation of a statistical sum is as follows. We consider an arbitrary commutative group of order N, for example Z_N. The Potts model is obtained if we choose the function W of the spin model on G in the form

$$W(g) = (1-\gamma)\delta(g, I) + \gamma, \quad \gamma = e^{-\beta}. \quad (4.2)$$

Carrying out the transformation of the statistical sum by the regular procedure, explained in Sec. 2, we get a spin system on the dual group $\hat{G}$ with the function

$$\tilde{W}(p) = N^{-1} \sum_{g \in G} W(g) \chi_p(g) = 1 - \gamma + N\gamma\delta(p, I) \propto (1-\tilde{\gamma})\delta(p, I) + \tilde{\gamma}, \quad (4.3)$$

where $p \in \hat{G}$ and

$$\tilde{\gamma} = e^{-\tilde{\beta}} = \frac{1-\gamma}{1+(N-1)\gamma}. \quad (4.4)$$

Accordingly, the dual system is also a model of P_N, but characterized by a new parameter $\tilde{\beta}$ from Eq. (4.4); that is, the model P_N is self-dual[4]:

$$P_N \overset{\text{KW}}{\leftrightarrow} P_N.$$

This transformation of the model P_N is related to the KW transformation of a system on a commutative group. For a commutative group G of order N it is possible to make a special choice of W functions so that the symmetry of the H structure is increased to S_N and we get the model P_N. Such a situation is rather typical for spin systems with nonabelian symmetry groups. In this and the following sections we shall present several more examples of this type.

The simplest generalization of the model is a spin system on the factor space $S_{N_1} \times S_{N_2}/S_{N_1-1} \times S_{N_2-1}$, which we call the generalized Potts model $P_{N_1 N_2}$. The spin space $P_{N_1 N_2}$ contains $N_1 N_2$ points $S_i^a, i = 1, 2, \ldots, N_2$; $a = 1, 2, \ldots, N_1$, and the H function is

$$H(s_i^a, s_j^b) = \begin{cases} 0, & \text{if} \quad i=j \text{ and } a=b, \\ \beta_1, & \text{if} \quad i=j, \text{ but } a \neq b, \\ \beta_2, & \text{if} \quad i \neq j. \end{cases} \quad (4.5)$$

It is easy to understand that $P_{N_1 N_2}$ can be regarded as a special case of a model on a group $G_1 \otimes G_2$, where the groups G_1 and G_2 are commutative and of orders N_1 and N_2, respectively. The following form of the W function is chosen ($g \in G_1 \otimes G_2, g = g_1 g_2, g_1 \in G_1, g_2 \in G_2$):

$$W(g) = (1-\gamma_1)\delta(g, I) + (\gamma_1 - \gamma_2)\delta(g_2, I) + \gamma_2, \quad (4.6)$$

where $p_1 = e^{-\beta_1}, p_2 = e^{-\beta_2}$. The Fourier transformation on the group $G_1 \otimes G_2$ gives

$$\tilde W(p_1\times p_2)=\sum_{g\in G} W(g_1\times g_2)\chi^{(1)}_{p_1}(g_1)\chi^{(2)}_{p_2}(g_2)\propto$$
$$\propto(1-\tilde\gamma_1)\delta(p_1\times p_2, I)+(\tilde\gamma_1-\tilde\gamma_2)\delta(p_1,I)+\tilde\gamma_2, \tag{4.7}$$

where $p_1\in\hat G_1, p_2\in\hat G_2$, and

$$\tilde\gamma_1=\frac{1+(N_1-1)\gamma_1-N_1\gamma_2}{1+(N_1-1)\gamma_1+N_1(N_2-1)\gamma_2},\ \tilde\gamma_2=\frac{1-\gamma_1}{1+(N_1-1)\gamma_1+N_1(N_2-1)\gamma_2}. \tag{4.8}$$

Equation (4.7) means that

$$P_{N_1N_2}\overset{\text{KW}}{\leftrightarrow}P_{N_2N_1} \tag{4.9}$$

with the transformation of parameters of Eq. (4.8). In particular, P_{NN} models are self-dual.

In the model $P_{N_1N_2}$, depending on the values of the parameters γ_1 and γ_2, we can expect that there will be three phases, characterized by different behaviors of the distribution function of an individual spin s_x:

$$\rho(s_x)=Z^{-1}\sum_{\substack{s_y\in M,\\ y\neq x}}\prod_{y,\alpha}W(s_y, s_{y+\hat\alpha}). \tag{4.10}$$

I. A phase with fully broken $S_{N_1}\times S_{N_2}$ symmetry:

$$\rho_I(s_x=s_i^a)=\rho_1+\rho_2\delta_{ii_0}+\rho_3\delta_{ii_0}\delta_{aa_0},\quad N_1N_2\rho_1+N_1\rho_2+\rho_3=1,$$

where $s_{i_0}^{a_0}$ is some chosen position of the spin.

II. A phase with broken S_{N_2} symmetry:

$$\rho_{II}(s_x=s_i^a)=\rho_1'+\rho_2'\delta_{ii_0},\qquad N_1N_2\rho_1'+N_1\rho_2'=1.$$

III. A "high-temperature" completely symmetrical phase.

$$\rho_{III}(s_x=s_i^a)=(N_1N_2)^{-1}.$$

Let us consider the self-dual models P_{NN} in more detail. Strictly speaking, the presence of two connecting parameters, γ_1 and γ_2, in this model does not allow us to find exact critical values of γ_1 and γ_2 [which would have to be distributed along certain critical lines in the (γ_1,γ_2) plane], since the KW symmetry transformation can connect different critical points. We note, however, that: a) in the case $N_1=N_2=N$ the transformation (4.8) of the parameters has a line of fixed points determined by the equation

$$\gamma_1+N\gamma_2=1, \tag{4.11}$$

b) for $\gamma_1=\gamma_2=\gamma$ the model P_{NN} coincides with P_{N^2} and has one phase-transition point $\gamma_c=(N+1)^{-1}$; c) for $\gamma_2=0$ we are dealing with N noninteracting models P_N, and consequently $\gamma_{1,c}=(N^{1/2}+1)^{-1}$; d) for $\gamma_1=1$ the model P_{NN} reduces to models P_N with $\gamma=\gamma_2$, and therefore $\gamma_{2,c}=(N^{1/2}+1)^{-1}$.

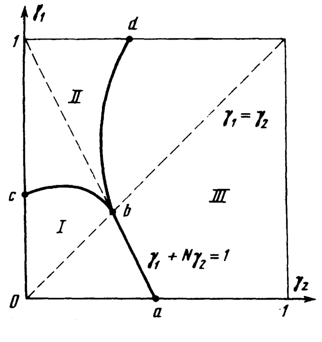

FIG. 2.

On the basis of these remarks we suppose that the phase diagram of the model P_{NN} in the (γ_1,γ_2) plane is of the form shown in Fig. 2; the segments bc and bd of the phase-separation line are only qualitatively correct [they transform into each other under the formulas (4.8)], and ab is the self-duality line (4.11). The coordinates of the points a, b, c, d marked on Fig. 2 are

$$a:\ \gamma_1=0,\ \gamma_2=N^{-1};\quad b:\ \gamma_1=\gamma_2=(N+1)^{-1};$$
$$c:\ \gamma_1=(\sqrt{N}+1)^{-1},\ \gamma_2=0;\qquad d:\ \gamma_1=1,\ \gamma_2=(\sqrt{N}+1)^{-1}.$$

In the general case $N_1\neq N_2$ the KW translation in itself gives no nontrivial quantitative information about the phase picture for the model.

5. SPIN SYSTEM ON THE GROUP S_3

Let us now consider the KW transformation for a spin system on the noncommutative group S_3, the permutation group of three objects. It is isomorphic with the symmetry group of an equilateral triangle. It has six elements; an arbitrary element can be represented in the form

$$g_{n,m}=r^n\sigma^m,\quad n=0,1,2;\ m=0,1, \tag{5.1}$$

where r is a rotation of the triangle through the angle $2\pi/3$, $r^3=1$, and σ is a reflection, $\sigma^2=1$.

There are three irreducible unitary representations of the group S_3; two of them are one-dimensional,

$$R_1(g_{n,m})=1,\ R_2(g_{n,m})=(-1)^m, \tag{5.2}$$

and one is two-dimensional:

$$R_3(g_{n,m})=\Omega^n\Sigma^m; \tag{5.3}$$

Here Ω and Σ are matrices:

$$\Omega=\begin{pmatrix} e^{2\pi i/3} & 0\\ 0 & e^{-2\pi i/3}\end{pmatrix},\quad \Sigma=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{5.4}$$

It is easy to show, using Eqs. (4.6), (5.3), and (5.4), that the general form of the W function for a spin system on the group S_3 is

$$W(g_{n,m})=\begin{cases}1, & n=m=0\\ \gamma_1, & m=0,\ \ n=1,2,\\ \gamma_2, & m=1,\ \ n=0,1,2\end{cases} \tag{5.5}$$

where γ_1 and γ_2 are positive constants not larger than one. The spin manifold of this system is represented schematically in Fig. 3.

It is easy to note that owing to the structure (5.5) of the W function the model on S_3 is identical with the generalized Potts model $P_{3,2}$ (see Sec. 4). This observation spares us the need to carry out separately the KW transformation on the group S_3. According to Eq. (4.9), the result of the process will be the model $P_{2,3}$, for which the spin manifold can be parametrized

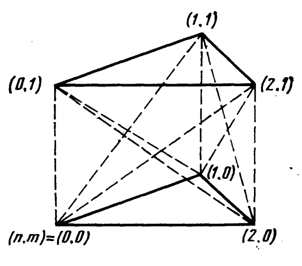

FIG. 3.

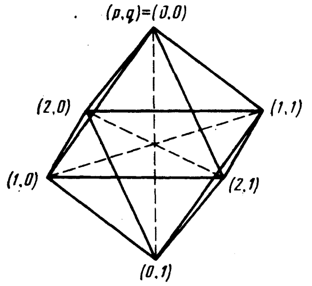

FIG. 4.

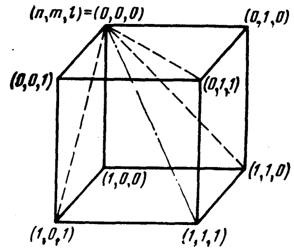

FIG. 5.

with two integers: $p=0,1$ and $q=0,1,2$. Then the $\tilde{W}$ function is

$$\tilde{W}(p,q)=\begin{cases}1, & p=0, \quad q=0\\ \tilde{\gamma}_1, & p=0, \quad q=1,\\ \tilde{\gamma}_2, & p=1,2, \quad q=0,1\end{cases} \tag{5.6}$$

where the dual parameters $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are connected with the original parameters as follows [cf. Eq. (4.9)]:

$$\tilde{\gamma}_1=\frac{1+2\gamma_1-3\gamma_2}{1+2\gamma_1+3\gamma_2}, \quad \tilde{\gamma}_2=\frac{1-\gamma_2}{1+2\gamma_1+3\gamma_2}. \tag{5.7}$$

It is also interesting to note that the dual model $P_{2,3}$ can be regarded as a spin system in which the individual spins take values at the vertices of a regular octahedron. In fact, the elements of the spin manifold can be identified with the vertices of the octahedron in such a way that elements with the same value of the index p correspond to opposite vertices of the octahedron, as shown in Fig. 4.

Accordingly, we arrive at the following result: The spin system on the group S_3 is connected by a KW transformation with the spin system on an octahedron:

$$S_3 \overset{\text{KW}}{\leftrightarrow} \text{ocatahedron}. \tag{5.8}$$

Furthermore, the formulas (5.7) connect the parameters of the two systems.

We note that the group S_3 is a special case of the dihedral group D_N of the symmetry of a regular polygon of N sides. The KW transformation for the spin system on D_N with arbitrary N can be performed analogously to the case of S_3. Namely, if we consider the system on the direct product $Z_2\otimes Z_N$, it is possible by a special choice of the W function to obtain the H structure of the system on D_N. The structure of the spin spaces of the corresponding dual systems is rather cumbersome, and we do not present it.

6. THE SPIN SYSTEM ON A THREE-DIMENSIONAL CUBE

In the preceding section a regular polyhedron, the octahedron, appeared as the spin manifold of a system dual to that on the group S_3. Other regular polydedra can also be regarded as spin manifolds. The spin systems so obtained and be interesting, for example, as "discrete approximations" to a continuous model (the Heisenberg ferromagnet) whose spin space is the sphere S^2.

The KW transformation for the spin system on the simplest regular polyhedron, the tetrahedron, was considered in Ref. 4. The system on the tetrahedron turns out to be self-dual,[5] which makes it possible to find the exact temperature for the phase transition. The KW transformation of the system on the octahedron does not give this kind of information; as we saw in the preceding section, the transformtion connects it with a different spin system.

The next polyhedron, as to number of vertices, is the cube. We can parametrize the points of the spin space of the system on the cube with integers: $n=0,1$, $m=0,1$, and $b=0,1$ (see Fig. 5). The general form of the W function for this system is:

$$W(n,m,l)=\begin{cases}1, & n=m=l=0\\ \gamma_1, & n+m+l=1\\ \gamma_2, & n+m+l=2'\\ \gamma_3, & n+m+l=3\end{cases} \tag{6.1}$$

where $0\leqslant\gamma_1,\gamma_2,\gamma_3\leqslant 1$.

To carry out the KW transformation of the statistical sum of the system on the cube, it is convenient to regard it as a special case (in the sense of a special choice of the W function) of the spin system on the group $Z_2\otimes Z_2\otimes Z_2$. The irreducible representations of this group can be numbered with three integers: $p=0,1$, $q=0,1$, and $r=0,1$, and are of the form

$$\chi_{p,q,r}(n,m,l)=(-1)^{pn+qm+rl}, \tag{6.2}$$

where n, m, l number the elements of the group itself in an obvious way. It follows from the results of Sec. 2 that the system on the group $Z_2\otimes Z_2\otimes Z_3$ is self-dual. Also if the W function of the original system is given by Eq. (6.1), then the normalized $\tilde{W}$ function of the dual system is

$$\tilde{W}(p,q,r)=\Big\{\sum_{l,m,n=0,1}W(n,m,l)\Big\}^{-1}\Big\{\sum_{l,m,n}W(n,m,l)\chi_{p,q,r}(n,m,l)\Big\}=\begin{cases}1, & p+q+r=0\\ \tilde{\gamma}_1, & p+q+r=1\\ \tilde{\gamma}_2, & p+q+r=2\\ \tilde{\gamma}_3, & p+q+r=3\end{cases} \tag{6.3}$$

$$\tilde{\gamma}_1=\frac{1+\gamma_1-\gamma_2-\gamma_3}{1+3\gamma_1+3\gamma_2+\gamma_3}, \quad \tilde{\gamma}_2=\frac{1-\gamma_1-\gamma_2+\gamma_3}{1+3\gamma_1+3\gamma_2+\gamma_3}, \quad \tilde{\gamma}_3=\frac{1-3\gamma_1+3\gamma_2-\gamma_3}{1+3\gamma_1+3\gamma_2+\gamma_3}. \tag{6.4}$$

The structure of the W function, Eq. (6.3) coincides with the original structure, Eq. (6.1). Thus the spin system on the cube is self-dual. This means that the statistical sum of the spin system on the cube is symmetric under the involutive transformation (6.4).

We note that the transformation (6.4) has a line of fixed points given by the equations

$$\gamma_2+\sqrt{2}(\gamma_1+\gamma_2)=1, \quad 3\gamma_1+\sqrt{2}(1-\gamma_3)=1. \tag{6.5}$$

In the general case it is not clear whether the symmetry relative to the transformation (6.4) can serve as a source of exact data on the positions of the singularities of the statistical sum in the space of the parameters γ_1, γ_2, and γ_3. We note only the special case of the cube, in which these parameters are connected by the relations

$$\gamma_2=\gamma_1^2, \quad \gamma_3=\gamma_1^2. \tag{6.6}$$

It corresponds to one of the discrete approximations for the classical Heisenberg model; the sum of states can be written in the form

$$Z(\beta)=\sum_{\mathbf{n}_x}\exp\left\{\beta\sum_{x,\alpha}[\mathbf{n}_x\mathbf{n}_{x+\hat{\alpha}}-1]\right\}, \tag{6.7}$$

where the three-dimensional vectors $\mathbf{n}_x$ run through the vertices of a cube inscribed in the unit sphere. The connection (6.6) between the parameters is preserved after the transformation (6.4), and the transition temperature for the model (6.7) can be calculated exactly. This case is trivial, however, since the statistical sum (6.7) reduces to three noninteracting Ising models.

[1)] We omit the factor $\beta=(kT)^{-1}$ in the exponent in Eq. (2.6), including it in the definition of H.

[2)] If we regard $g_{x,\alpha}$ as connectivities in a G-stratification over the lattice, the constraints $Q_{\tilde{x}}=I$ are the condition for zero curvature.

[3)] It can easily be seen that the relation between the groups G and $\hat{G}$ is of the nature of a duality; i. e., if $\hat{G}$ is the group of characters of G, then $\hat{\hat{G}}=G$ is the group of characters of the group $\hat{G}$.

[4)] In the model P_N the existence of two different phases is possible. The KW symmetry enables us to determine the critical value β_c, which satisfies the equation $\beta_c=\tilde{\beta}(\beta_c)$. This gives $\beta_c=\ln(1+N^{1/2})$. Some rigorous results of the existence of two phase in P_N have been obtained recently.[18]

[5)] It is easily seen that the spin system on the tetrahedron is equivalent to the model P_4 (see Sec. 4).

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Anomaly of the magnetic susceptibility near the ferroelectric phase transition point in narrow-gap A_4B_6 semiconductors

V. I. Litvinov and V. K. Dugaev

Chernovtsy Semiconductor Materials Technology Division of the Institute of Applied Mechanics and Mathematics Problems, Academy of Sciences of the Ukrainian SSR

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The magnetic susceptibility in a system that is unstable against singlet electron-hole pairing or a structural phase transition is computed. The magnetic susceptibility has a fluctuation-induced singularity near the transition temperature: it undergoes a finite jump in the region of applicability of the Landau-Ginzburg theory and obeys a power law with an exponent equal to $d\nu-1$, where ν is the exponent of the correlation length, in the scaling region. The diamagnetism is found in the mean-field approximation to decrease smoothly below the transition temperature. The results qualitatively explain the experiments that have been performed on SnTe samples in the vicinity of the ferroelectric transition point.

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1. INTRODUCTION

The occurrence of a ferroelectric phase transition in A_4B_6 crystals and solid solutions based on them has been well established. In particular, the dependence of the transition temperature, T_c, for SnTe on the carrier concentration is well known.[1,2] A structural-phase-transition-induced anomaly has been observed[3] in the magnetic susceptibility (MS) of SnTe at $T=T_c$.

The existence of such an anomaly is not *a priori* apparent within the framework of the conventional notions about a structural transition, which does not affect the magnetic properties of the system. Since it

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