Effective permittivity of a highly inhomogeneous plasma

B. S. Abramovich

Gorky Radiophysical Scientific Research Institute
(Submitted 13 October 1978; resubmitted 19 February 1979)
Zh. Eksp. Teor. Fiz. 77, 87–91 (July 1979)

The value of the effective permittivity of a nonregular plasma at the resonance frequency and with inhomogeneities distributed according to the Cauchy law is found on the basis of an analysis of the entire iteration series for the mass operator of the Dyson equation for the mean field.

PACS numbers: 52.25.Mq

An important role is played in the electrodynamics of randomly inhomogeneous media by the effective-permittivity tensor $\hat{\varepsilon}^{eff}$, which relates the mean (over the inhomogeneity ensemble) induction and electric field in the nonregular medium:

$$\langle \mathbf{D}\rangle=\hat{\varepsilon}^{eff}\langle \mathbf{E}\rangle. \tag{1}$$

In a statistically homogeneous medium the operator $\hat{\varepsilon}^{eff}$ is an integral operator with a difference kernel uniquely determining the mean Green function of the system. The knowledge of such characteristics as $\hat{\varepsilon}^{eff}$ allows us to reduce the problem of finding the mean field to the corresponding problem of finding the field in an "effective" homogeneous dissipative medium, to compute the losses due to radiation emission by the prescribed currents (e.g., the emission of a charged particle moving in the inhomogeneous medium, the intensity of the thermal fluctuation field, etc.).

There have in recent years been published a considerable number of papers on the theory of multiple wave scattering in which equations of the Dyson and Bethe-Salpeter types are used to solve general and specific problems.[1] In these papers, besides the investigation of the main problem involving the determination of the asymptotic form of the kernels of the equations, the Dyson equation for the mean field in a statistically homogeneous and isotropic medium is solved. In this case $\hat{\varepsilon}^{eff}$ essentially coincides with the mass operator of the Dyson equation. However, in concrete computations use is made of one or another approximation for the mass operator (the Foldy approximation,[2] the Bourret approximation,[3] the Finkel'berg approximation[4]), since it is virtually impossible even in the case of homogeneous fields to sum the topologically complex perturbation theory series and obtain the exact value of the mass operator (and, with it, $\hat{\varepsilon}^{eff}$).

A similar problem arises in the theory of phase transitions near a critical point,[5] in the theory of percolation near the flow threshold,[6] in the description of strong turbulence, etc., where the summation problem has been tackled with the aid of the renormalization-group[7] and ε-expansion[8] methods.

Besides this, use is often made in the static problem of the "physical" solution obtained by the self-consistent field method, in which the effective permittivity (or the effective conductivity) of an isotropic medium is determined by the solution to the Maxwell-Odelevskiĭ equation[6,9,10]:

$$\left\langle \frac{\varepsilon-\varepsilon^{eff}}{\varepsilon+(n-1)\varepsilon^{eff}}\right\rangle=0. \tag{2}$$

Here n is the dimensionality of the problem ($n=1,2,3$) and the averaging is performed over the realizations of the random field $\varepsilon(\mathbf{r},\omega)$, the local permittivity of the medium. The results obtained by this method are graphic and, in a number of cases (especially for two-dimensional systems), are excellently confirmed by experiment.[6,11] But the well-known derivations of Eq. (2) in percolation theory (see, for example, Ref. 6 for a review) and in multiple-scattering theory, where the solution to the Maxwell-Odelevskiĭ equation is the principal approximation for $\hat{\varepsilon}^{eff}$ of a medium with small-scale inhomogeneities,[12] are to a large extent intuitive and not well grounded. Estimates obtained for the corrections to the formula (2) for a medium with three-dimensional inhomogeneities ($n=3$) in the random-phase approximation indicate the marked limitedness of this result.[10]

A classical example of a highly inhomogeneous medium is a turbulent plasma placed in an electric field $\mathbf{E}_0=\mathbf{E}_{00}e^{-i\omega t}$ of frequency close to the mean plasma frequency: $\omega\sim\langle\omega_p(\mathbf{r})\rangle$. In fact, in this case the mean permittivity $\langle\varepsilon\rangle\sim 0$, and even relatively weak fluctuations in the particle concentration of the plasma lead to huge relative permittivity fluctuations $\sigma_\varepsilon/\langle\varepsilon\rangle\gg 1$ (σ_ε is the variance of the permittivity fluctuations).

Let us consider the behavior of an electric field in an isotropic plasma-like medium, at each point of which a nonregular permittivity, $\varepsilon(\omega,\mathbf{r})$, relating the local induction $\mathbf{D}(\omega,\mathbf{r})$ and electric field $\mathbf{E}(\omega,\mathbf{r})$ is defined:

$$\mathbf{D}(\omega,\mathbf{r})=\varepsilon(\omega,\mathbf{r})\mathbf{E}(\omega,\mathbf{r}).$$

The last relation is valid if the characteristic spatial scales connected with the thermal motion of the carriers in the medium are small in comparison with the inhomogeneity dimension. As the basic equations, let us, for simplicity, take the equations of quasi-electrostatics:

$$\operatorname{div}\mathbf{D}=4\pi\rho_0,\qquad \operatorname{rot}\mathbf{E}=0, \tag{3}$$

where ρ_0 is the charge density of the external field sources.

Assuming that the permittivity fluctuations are statistically homogeneous and isotropic, we write the first of the Eqs. (3) in the form

$$\varepsilon_0(\omega)\frac{\partial E_i}{\partial x_i}=\frac{\partial}{\partial x_i}[\varepsilon_0(\omega)-\varepsilon(\omega,\mathbf{r})]E_i+4\pi\rho_0,\quad i=1,2,\ldots n. \tag{4}$$

 0038-5646/79/070044-03$02.40

The quantity $\varepsilon_0(\omega)$ is the mean permittivity's renormalized value, which determines the renormalization of the unperturbed Green function of the system in question.[1)] To the Eq. (4) corresponds the integral stochastic equation:

$$E_i(\omega,\mathbf{r})=\int d\mathbf{r}'[\varepsilon(\omega,\mathbf{r}')-\varepsilon_0(\omega)]\frac{\partial^2 G(\omega,\mathbf{r}-\mathbf{r}')}{\partial x_i\partial x_j}E_j(\omega,\mathbf{r}')+F_i^{(0)}(\omega,\mathbf{r}). \quad (5)$$

Here $G(\omega,\mathbf{r})$ is the Green function for the n-dimensional Poisson equation in a medium with permittivity $\varepsilon=\varepsilon_0$ and

$$F_i^{(0)}(\omega,\mathbf{r})=-4\pi\int d\mathbf{r}'\frac{\partial G(\omega,\mathbf{r}-\mathbf{r}')}{\partial x_i}\rho_0(\mathbf{r}').$$

The differentiation in (5) should be understood in the sense of the differentiation of generalized functions; therefore, it is expedient to represent the singular function $\partial^2 G/\partial x_i\partial x_j$ in the regularized form:

$$\varepsilon_0\frac{\partial^2 G}{\partial x_i\partial x_j}=g_{ij}(\mathbf{r})-\frac{\delta_{ij}}{n}\delta(\mathbf{r}), \quad (6)$$

where $g_{ij}(\mathbf{r})$ is a tensor having in the k representation the following form:

$$g_{ij}(\mathbf{k})=\frac{\delta_{ij}}{n}-\frac{k_ik_j}{k^2}.$$

Using (6), we can reduce Eq. (5) to the form

$$F_i(\omega,\mathbf{r})=\int d\mathbf{r}' g_{ij}(\mathbf{r}-\mathbf{r}')\xi(\omega,\mathbf{r}')F_j(\omega,\mathbf{r}')+F_i^{(0)}(\omega,\mathbf{r}), \quad (7)$$

where we have introduced the new variables

$$\mathbf{F}=\frac{\varepsilon+(n-1)\varepsilon_0}{n\varepsilon_0}\mathbf{E},\quad \xi=n\frac{\varepsilon-\varepsilon_0}{\varepsilon+(n-1)\varepsilon_0}. \quad (8)$$

Solving Eq. (7) by the method of successive iterations, and averaging the resulting series under the assumption that the $\xi(\omega,\mathbf{r})$ field is statistically homogeneous and isotropic, we can easily derive with the aid of the standard closure procedure an expression for the mass operator, $\hat{\Sigma}$, of the Dyson equation for the mean field $\langle\mathbf{E}\rangle$. In this case the kernel, $\Sigma_{ij}(\omega,\mathbf{k})$, of the mass operator is represented by the following functional series in terms of the moments of the random field ξ:

$$\Sigma_{ij}^{-1}(\omega,\mathbf{k})=\sigma_{ij}^{-1}(\omega,\mathbf{k})+g_{ij}(\mathbf{k}),$$

$$\sigma_{ij}(\omega,\mathbf{k})=\sum_{\alpha=0}^{\infty}\frac{1}{(n\pi)^{2\alpha}}\int d\varkappa_1\ldots d\varkappa_\alpha\, g_{il}(\varkappa_1)\ldots g_{pj}(\varkappa_\alpha)\Phi_\xi(|\varkappa_1-\varkappa_2|\ldots|\varkappa_\alpha-\mathbf{k}|),$$

where $\Phi_\xi(\mathscr{H}_\beta)$ is the spectrum of the multivariate moment of the $\xi(\omega,\mathbf{r})$ field.

It follows from (8) that $\hat{\varepsilon}^{eff}$ can be expressed in terms of $\hat{\Sigma}$ as follows:

$$\varepsilon_{ij}^{eff}(\omega,\mathbf{k})=\varepsilon_0(\omega)[\delta_{ip}-\Sigma_{ip}(\omega,\mathbf{k})]^{-1}[\delta_{pj}+(n-1)\Sigma_{pj}(\omega,\mathbf{k})].$$

The exact determination of $\hat{\varepsilon}^{eff}$ is possible if we are able to compute the mass operator $\hat{\Sigma}$. The simplest way of accomplishing the latter is to find the conditions under which $\hat{\Sigma}=0$. Then

$$\varepsilon_{ij}^{eff}=\varepsilon^{eff}\delta_{ij},\quad \varepsilon^{eff}=\varepsilon_0 \quad (9)$$

and the problem of finding $\hat{\varepsilon}^{eff}$ reduces to the problem of the self-consistent determination of the quantity ε_0.

The possibility of the mass operator's vanishing is connected with the analytic properties of the transformation, (8), of $\xi=\xi(\varepsilon/\varepsilon_0)$. Indeed, let us assume that $\varepsilon_0=i|\varepsilon_0|$ is an imaginary quantity. Then it follows from (8) that the domain of variation of the quantity $\xi(\omega,\mathbf{r})$ in the complex ξ plane is the circumference of the circle

$$\xi=\frac{n}{2(n-1)}(n-2+ne^{i\varphi}), \quad (10)$$

where the random "phase" $\varphi(\omega,\mathbf{r})$ is distributed over the range $|\varphi|\leq\pi$. Let us assume that the random phase is so "randomized" that its values at noncoincident points are not correlated. Then all the moments of the random $\xi(\omega,\mathbf{r})$ field are equal to zero, and the vanishing of all the moments of the quantity $\xi(\omega,\mathbf{r})$ is a sufficient condition for the vanishing of the mass operator. It follows from the formula (10) that $\langle\xi^N\rangle=0$ if there exists a probability distribution, $W(\varphi)$, for the phase, such that the relation

$$\langle e^{iN\varphi}\rangle=\left(\frac{2-n}{n}\right)^N,\quad N=1,2,\ldots. \quad (11)$$

is fulfilled for any real N. It is easy to see that the right-hand side of the expression (11) determines the coefficients of the expansion of the distribution function $W(\varphi)$ into a Fourier series, on summing which, we obtain

$$W(\varphi)=\frac{(2\pi)^{-1}}{(n-1)\cos^2(\varphi/2)+(n-1)^{-1}\sin^2(\varphi/2)},\quad |\varphi|\leq\pi.$$

Using (8) and (10), we obtain $\varepsilon=-(n-1)|\varepsilon_0|\cot(\varphi/2)$, from which it is not difficult to determine the probability distribution, $W(\varepsilon)$, of the permittivity to which the distribution (12) corresponds:

$$W(\varepsilon)=\frac{|\varepsilon_0|}{\pi(|\varepsilon_0|^2+\varepsilon^2)}.$$

This distribution is none other than the Cauchy distribution with variance $\sigma=|\varepsilon_0|$ and mean value $\langle\varepsilon\rangle=0$.

Thus, the effective permittivity of a chaotically inhomogeneous plasma at the resonance frequency ($\langle\varepsilon\rangle=0$), with the inhomogeneities distributed according to the Cauchy law, can be determined exactly, and is equal, in accordance with (9), to

$$\varepsilon^{eff}(\omega=\langle\omega_p\rangle)=i\sigma_\varepsilon$$

regardless of the dimensionality of the inhomogeneities. The imaginariness of the quantity $\varepsilon^{eff}(\langle\omega_p\rangle)$ is due to the resonance damping resulting from the conversion of the mean-field energy into fluctuation-plasma-oscillation energy.[12]

In conclusion, let us note that, according to the optical theorem, the mean Green function is uniquely connected with the kernels of the mass operator and the intensity of the Bethe-Salpeter equation for the coherence function of the field. Therefore, the possibility should exist in the present model, which has enabled us to determine the mass operator, of finding the intensity operator also. This, however, falls outside the limits of the present investigation.

The author expresses his gratitude to V.I. Tatarskiĭ for his attention to the work and for his valuable comments and to S.N. Gurbatov for a useful discussion.

[1)]In the case of anisotropic fluctuations $\varepsilon_0(\omega)$ is a tensor.

[1]Yu. N. Barabanenkov, Yu. A. Kravtsov, S. M. Rytov, and V. I. Tatarskiĭ, Usp. Fiz. Nauk 102, 3 (1970) [Sov. Phys. Usp. 13,

551 (1971)].
[2]L. L. Foldy, Phys. Rev. **67**, 107 (1945).
[3]R. C. Bourret, Can. J. Phys. **40**, 782 (1962).
[4]V. M. Finkel'berg, Zh. Eksp. Teor. Fiz. **46**, 725 (1964); **53**, 401 (1967) [Sov. Phys. JETP **19**, 494 (1964); **26**, 268 (1968)].
[5]L. P. Kadanoff, W. Götze, D. Hamblen, R. Hecht, E. A. S. Lewis, V. V. Palciauskas, M. Payl, and J. Swift, Rev. Mod. Phys. **39**, 395 (1967).
[6]S. Kirpatrick, Rev. Mod. Phys. **45**, 574 (1973). V. K. S. Shante and S. Kirpatrick, Adv. Phys. **20**, 325 (1971).
[7]K. G. Wilson and J. Kognt, Phys. Rep. **10**, 2078 (1974).
[8]A. Z. Patashinskiĭ and V. L. Pokrovskiĭ, Usp. Fiz. Nauk **D21**, 55 (1977) [Sov. Phys. Usp. **20**, 31 (1977)].
[9]H. Stachoweiak, Bull. Acad. Pol. Sci. Ser. Math. **15**, 631 (1967).
[10]V. M. Finkel'berg, Zh. Tekh. Fiz. **34**, 509 (1964) [Sov. Phys. Tekh. Phys. **9**, 396 (1964)].
[11]J. Anderson, Phys. Rev. Lett. **27**, 1719 (1971).
[12]Yu. A. Ryzhov, V. V. Tamoĭkin, and V. I. Tatarskiĭ, Zh. Eksp. Teor. Fiz. **48**, 656 (1965) [Sov. Phys. JETP **21**, 375 (1965)].

Translated by A. K. Agyei