The energy flux carried away by the electrons from the pinch can be estimated from the equation $Q = j_{ib} S T_e$, where S is the surface area of the pinch. We note that these losses are equal to the energy lost by the hot electrons to heating the cold electrons produced in the pinch as a result of ionization. We obtain $Q \approx 3\cdot 10^{-13} SR \times (T_e/M)^{1/2}$ kW. At $R \approx 1$ atm, $S \approx 1$ cm^2, and $T_e \approx 3 \times 10^5$ K we have $Q \approx 10^2/\sqrt{\mu}$ kW, where μ is the molecular weight of the particle.

In conclusion, we note the following. It was observed in Kapitza's experiments[1] that the light hydrogen and deuterium impurities contribute to the appearance of a hot plasma pinch, and spectroscopic measurements have shown that there are no multiply charged ions in the pinch. The cause of these phenomena can be understood with the aid of the proposed theory. The point is that the only particles that can penetrate into the interior of the hot region of the plasma pinch are those of light impurities, which have a high thermal velocity and a sufficiently high ionization potential. This follows, for example, from Eq. (6) [or (9)]: the larger the parameter $M\beta^2$, the more difficult it is for the particle to land in the interior of the pinch.

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Singularities of the transition to a turbulent motion

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An elementary model of the onset of turbulence, corresponding to Landau's idea of the collapse of the stable limit cycle, is considered. We give qualitative estimates of the conditions for the appearance of the turbulent (stochastic) motion which has the structure of a strange attractor. We show that a strange attractor occurs when the effective coefficient for the stretching of the trajectories in phase space in the dissipationless case becomes larger than the effective dissipation coefficient. We performed a detailed numerical experiment on the model to elucidate the structure of the strange attractor, the transition regime, the local instability, the correlation function, and the stationary distribution function in phase space.

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1. INTRODUCTION

The analysis for the conditions for the transition of a dynamical system from the regime of a regular, conditionally peridoic motion to a regime of an irregular, stochastic motion has become more and more often the topic of study in the physical and mathematical literature. The generation of turbulence from a laminar motion is perhaps the most characteristic field of physics whose content is directly connected with the determination of a criterion for the appearance of stochasticity.

In the case of weak turbulence the dissipative processes in the system are weak and one can describe the dynamics of the system by Hamiltonian equations. An analysis of the conditions for the appearance of turbulence in this kind of system was given by Sagdeev and one of us[1] (see also Ref. 2). It is shown there that the basis for the mechanism for the generation of a stochastic (turbulent) component of the motion is the well known n-wave ($n \geqslant 3$) cluster wave interaction. The wave concept itself is quite well defined, since the dissipation is weak. The situation, however, changes abruptly under conditions of strong dissipation, when the system is no longer Hamiltonian. The analysis of the conditions for the appearance of stochasticity in this kind of system started with Lorenz's well known paper,[3] in which a highly simplified model of thermal convection was studied. A large number of papers, stimulated by Lorenz's paper (see, e.g., Refs. 4 to 9), led to the appearance in the physical literature of a new term—"the strange attractor"—introduced by Ruelle and Takens[8] to denote a particular form of stochasticity occurring in strongly dissipative systems (for details see the reviews by Monin[10] and Rabinovich[11]).

Great hopes were pinned on the study of strange attractors in attempts to construct a theory of the onset of turbulence. One should note that a strange attractor also occurs in problems in other fields of physics: laser systems, plasmas, and so on (see Ref. 11). The study of the stochasticity effect in strongly dissipative systems entails at present considerable difficulties. Real physical systems are as a rule studied numerically. It is also unclear how to apply to these systems the existing rigorous mathematical methods.

The aim of the present paper consists in a detailed numerical study of a greatly simplified model for the

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onset of turbulence, which lends itself also to an analytic treatment. The basic considerations which led to our model are connected with well known arguments by Landau.[12] We shall imagine that there is a single mode which has a stable limit cycle and which is perturbed by all other modes. In the vicinity of the limit cycle one can write down the equations of motion with the action (I) and angle (ϑ) as variables:

$$\dot{I}=-\gamma(I-I_0)+\varepsilon q(I_0,\vartheta)f(t),\qquad \dot{\vartheta}=\omega(I), \tag{1.1}$$

where γ is the dissipation coefficient, I_0 is the action corresponding to the limit cycle, ε is the dimensionless parameter of the interaction with the other modes (we assume that $\varepsilon \ll 1$), q is a function in which I is replaced by I_0 (because $\varepsilon \ll 1$), $\omega(I)$ is the nonlinear mode frequency, while $f(t)$ takes into account the effect of all other modes. It is shown in Ref. 1 that if the number of modes is sufficiently large and that under rather general conditions the function $f(t)$ can be written in the simplified form

$$f(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT), \tag{1.2}$$

where the quantity $2\pi/T$ is of the order of the characteristic distance between the frequencies of neighboring modes. One can reduce Eqs. (1.1) to finite difference transformation equations of the following form:

$$I_{n+1}=I_{n+1}(I_n,\vartheta_n),\quad \vartheta_{n+1}=\vartheta_{n+1}(I_n,\vartheta_n), \tag{1.3}$$

where (I_n,ϑ_n) are the values of (I,ϑ) after the n-th δ-function jolt. It is possible to make a qualitative analytical study of the set (1.3) and to determine explicitly the criterion for the occurrence of a strange attractor. Although the model (1.1) is a very great simplification of those problems which arise in the turbulence studies, its analysis gives rather extensive information, as will be seen in what follows. The main reason is that for the model (1.1) one can obtain for the conditions for the appearance of the strange attractor, for its structure, and for other properties analytical estimates which are confirmed by a numerical analysis. These results are partially given in Ref. 13. Moreover, when $\gamma=0$ the set (1.1) degenerates into the so-called main model of stochasticity of Hamiltonian systems,[2,14,15] which is typical also of the case of a large number of degrees of freedom.[1] One sees easily that the set (1.1) is a natural generalization of the Hamiltonian case when a stable limit cycle is present. Finally, the formal construction of a system with a strange attractor, suggested recently by Vul and Sinai,[16] corresponds apparently to the set (1.1), and this may lead to the possibility of using more rigorous methods.

The main results of the present paper consist of a detailed numerical analysis and of qualitative estimates of the conditions for the appearance of a strange attractor in the system (1.1), and of the establishing of a connection between the model (1.1) and already known models. One of the principal results is the construction of a stationary distribution function on the strange attractor, which opens up a possibility of using thermodynamic methods in systems with strong dissipation.

2. PHYSICAL PREMISES OF THE MODEL

We noted already that the set (1.1) can arise as the result of a strong idealization of the problem of the occurrence of turbulence. The qualitative considerations leading to it consist of the following. Let $a(x,t)$ be a state vector satisfying the hydrodynamic equations of motion and boundary conditions. We expand $a(x,t)$ in a Fourier series:

$$a(x,t)=\sum_k a_k(t)e^{ikx}$$

and write the equations of motion in the form

$$\dot{a}_k=\gamma_k a_k+i\omega_k a_k+\sum_{k_1}V_{kk_1}a_{k_1}a_{k-k_1}, \tag{2.1}$$

where the dissipation coefficients γ_k, the frequencies ω_k, and the interaction matrix elements V_{kk_1} are determined directly from the initial equations of motion.

We assume now that one unstable mode with complex amplitude A is excited. For small values of the time we can write

$$\dot{A}=\bar{\gamma}A+i\omega A, \tag{2.2}$$

where $\bar{\gamma}>0$ and where we have dropped the wave-vector index for simplicity. Landau[12] described qualitatively the transition from the nonperiodic motion (2.2) to a periodic one as follows. With increasing time the amplitude $|A|$ grows and we must take into account in Eq. (2.2) the next terms of the expansion in $|A|$. This leads to the equation

$$\dot{A}=(\bar{\gamma}+i\omega)A-\delta|A|^2A \tag{2.3}$$

or

$$\frac{d}{dt}|A|^2=2\bar{\gamma}|A|^2-2\delta|A|^4. \tag{2.4}$$

When $\delta>0$ the stationary state corresponds to the value

$$I_0=|A_0|^2=\bar{\gamma}/\delta.$$

In the vicinity of this value one can rewrite Eq. (2.4) in the form

$$\dot{I}=-2\delta I_0(I-I_0),\quad I=|A|^2, \tag{2.5}$$

where the variable I will be called the action. One can add to Eq. (2.5) an equation for the phase:

$$\dot{\vartheta}=\omega(I), \tag{2.6}$$

where $\omega(I)$ is the frequency of the oscillations of the selected mode and must also take into account nonlinear corrections in the form of an expansion in I. The actual expression (2.6) can be obtained from (2.3) if we write

$$A=|A|e^{i\vartheta}. \tag{2.7}$$

Equation (2.5) describes the motion in the vicinity of a stable limit cycle to which corresponds the value I_0. In the normal situation $\gamma\propto R-R_{\rm cr}$, where $R_{\rm cr}$ is the critical characteristic number of the problem.

Equation (2.4) does not contain oscillating terms and arises as the result of some procedure for smoothing the motion of the selected mode A. We now take into account in some appropriate manner the effect of the other modes a_k on the evolution of A. Using (2.1) and (2.3) we can write:

$$\dot{A}=(\bar{\gamma}+i\omega)A-\delta|A|^2A+\sum_k V_{kk_0}a_k a_{k_0-k}, \tag{2.8}$$

where the wave number k_0 corresponds to the mode A. Multiplication of (2.8) by A^* gives

$$\dot{I}=-2\delta I_0(I-I_0)+2\,\mathrm{Re}\Big(A^*\sum_k V_{kk_0}a_ka_{k_0-k}\Big). \tag{2.9}$$

We assume now that the difference $R-R_{cr}$ is small and that for the modes a_k in the sum in (2.9) the instability either does not develop or develops rather slowly. We can then write approximately:

$$a_k=|a_k|\exp(i\omega_k t),\quad a_{-k}=a_k^*$$

and

$$\sum_k V_{kk_0}a_ka_{k_0-k}\approx\sum_{k>k_0}V_{kk_0}|a_k||a_{k_0-k}|\exp\{i(\omega_k-\omega_{k_0-k})t\}+\sum_{k<k_0}V_{kk_0}|a_k||a_{k_0-k}|\exp\{i(\omega_k+\omega_{k_0-k})t\}. \tag{2.10}$$

We consider the expression

$$\sum_{n=-\infty}^{\infty}\delta(t-nT)=\frac{1}{T}\sum_{n=-\infty}^{\infty}\exp\Big(\frac{2\pi int}{T}\Big). \tag{2.11}$$

We drop from the right-hand side of (2.10) the fast oscillating terms occurring in the sum with $k<k_0$ and we assume that the k-dependences of V_{kk_0}, $|a_k|$, and $|a_{k_0-k}|$ are weak. If we further assume that a rather large number of modes with $k>k_0$ with a weak dispersion $d\omega/dk$ is excited, we get from a comparison of (2.10) and (2.11)

$$\sum_{k>k_0}V_{kk_0}a_ka_{k_0-k}\sim\mathrm{const}\sum_{n=-\infty}^{\infty}\delta(t-nT), \tag{2.12}$$

where the quantity T has the meaning

$$2\pi/T\sim(d\omega/dk)\Delta k$$

(Δk is the characteristic distance between neighboring wave numbers of the excited modes a_k). Substitution of (2.7) and (2.12) into (2.9) leads to the model (1.1). It is useful to emphasize that from a qualitative point of view the representation (2.12) works rather well for broad excitation spectra and weak dispersion not only in the single-mode approximation.[1]

3. MODEL HIERARCHY

We make in (1.1) some slight simplifications without changing the basic properties of this set. We put:

$$q(I_0,\vartheta)=I_0\cos\vartheta,\quad \omega(I)=\omega_0+\alpha\omega_0(I-I_0)/I_0, \tag{3.1}$$

i.e., we retain in $q(I_0,\vartheta)$ only the first harmonic in the phase, the factor I_0 is introduced from dimensional considerations, and α is a dimensionless nonlinearity parameter. We introduce the dimensionless quantities:

$$y=(I-I_0)/I_0,\quad \vartheta=2\pi x,\quad \Omega=\omega_0T,\quad \Gamma=\gamma T. \tag{3.2}$$

Using (1.2), (3.1), and (3.2) we can by a simple integration from (1.1) obtain the set (1.3) in explicit form:

$$y_{n+1}=e^{-\Gamma}(y_n+\varepsilon\cos 2\pi x_n),\quad x_{n+1}=\Big\{x_n+\frac{1}{2\pi}\Omega(1+\mu y_n)+\frac{1}{2\pi}K\mu\cos 2\pi x_n\Big\}, \tag{3.3}$$

where the brackets $\{\ldots\}$ denote the fractional part of the argument,

$$K=\varepsilon\alpha\Omega,\quad \mu=(1-e^{-\Gamma})/\Gamma. \tag{3.4}$$

Absence of dissipation corresponds to $\Gamma=0(\mu=1)$. In that case the set (3.3) takes the form

$$y_{n+1}=y_n+\varepsilon\cos 2\pi x_n,\quad x_{n+1}=\Big\{x_n+\frac{1}{2\pi}\Omega(1+y_n)+\frac{1}{2\pi}K\cos 2\pi x_n\Big\}, \tag{3.5}$$

which is the basic model for stochasticity of Hamiltonian systems.[2] In what follows we assume everywhere

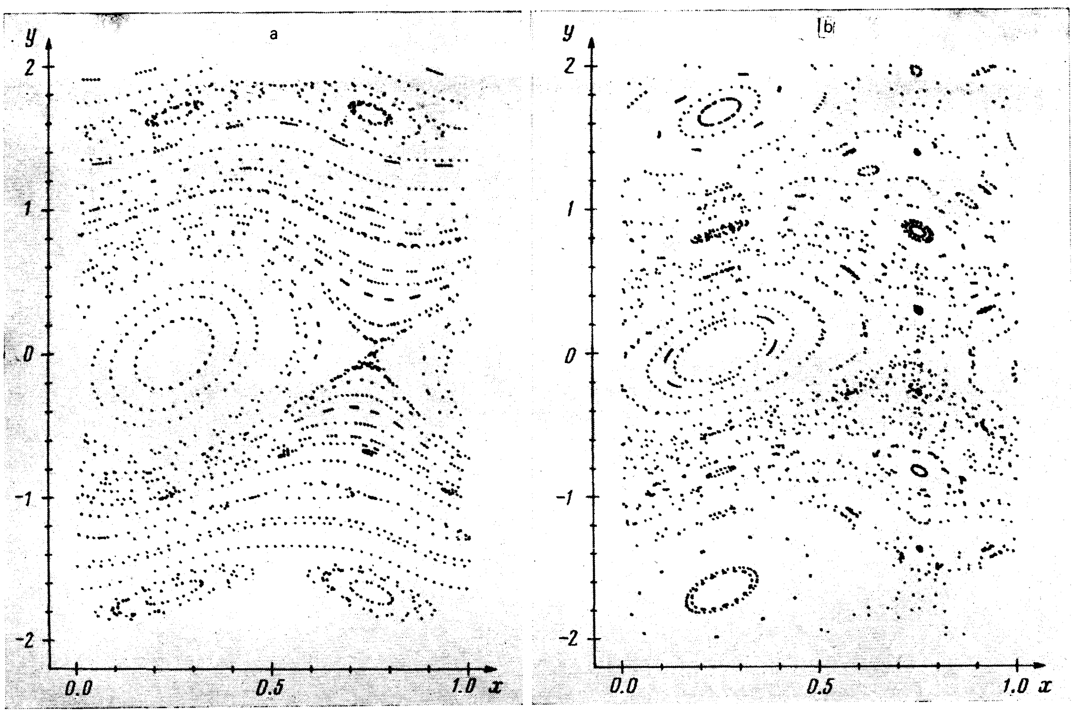

FIG. 1. Trajectories in phase space when there is no dissipation ($\Gamma=0$) for $\varepsilon=0.3$, $\alpha=0.3$ and different values of K: a) $K=0.09$; b) $K=0.18$.

that $\varepsilon \ll 1$. In this case the parameter K plays a fundamental part. The limit for the occurrence of stochasticity is the value $K \sim 1$. When $K \lesssim 1$ the motion is stable while for $K \gtrsim 1$ the motion has the property of local instability which can be expressed as follows. We denote by

$$D(t)=[(x(t)-\bar{x}(t))^2+(y(t)-\bar{y}(t))^2]^{1/2}$$

the distance between two trajectories in phase space. If $D(0)$ is very small ($D \ll 1$), the local instability manifests itself in the relation

$$D(t) \sim D(0)e^{ht} \tag{3.6}$$

until $D \sim 1$ is reached. The instability growth rate h is proportional to the Kolmogorov entropy and for the set (3.5) is of the order of[2] $h \sim T^{-1} \ln K$. Local instability (3.6) leads to an exponentially fast mixing of the trajectories in phase space and, hence, to turbulent motion.

The qualitative aspect of the appearance of stochasticity consists in the following.[2] For small ε one can to a first approximation neglect the change in the variable y and when $K \gg 1$ there occurs a "stretching" of trajectories with respect to phase such that

$$|\delta x_{n+1}/\delta x_n| \sim K \gg 1. \tag{3.7}$$

Exceptions are the stability islands in the vicinity of the stable singular point $x_n \approx 0$, where the derivative $\partial x_{n+1}/\partial x_n$ vanishes. What we have just said is illustrated in Fig. 1 by the data from a numerical analysis. We give in that figure the points (x_n, y_n) of a trajectory for the set (3.5). At small values of K the trajectories are periodic (Fig. 1a). Increasing K leads to their partial destruction in the neighborhood of the hyperbolic point (Fig. 1b). Large stability islands are still preserved. Finally, at sufficiently large values of K practically the whole of phase space is a region of stochastic motion.

The considerations presented here show that the basic mechanism for the appearance of stochasticity is the mixing of the phases of the system. This means also that the phase correlation function

$$R(m)=\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N}\cos 2\pi x_{n+m}\cos 2\pi x_n \tag{3.8}$$

must vanish as $m \to \infty$ if $K \gg 1$. More precisely[15]:

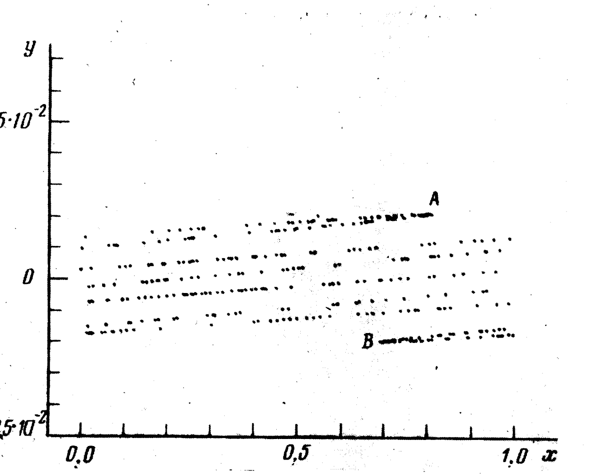

FIG. 2. Strange attractor for $\varepsilon=0.3$, $\alpha=0.3$, $K=9.03$, $\Gamma=5$.

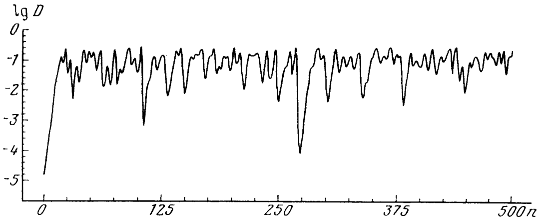

FIG. 3. Distance between two initially close trajectories ($D(0)=10^{-5}$) as a function of the dimensionless time for motion on the strange attractor.

$$R(m) \propto \exp(-{}^1/_2 m \ln K), \tag{3.9}$$

and the phase correlation is uncoupled after a time

$$\tau=2T/\ln K \sim 1/h. \tag{3.10}$$

Hence it follows that the mechanism for the occurrence of stochasticity can be traced using the even simpler cosine (sine) model:[1)]

$$x_{n+1}=\{K\varkappa^{-1}\cos \varkappa x_n\}, \tag{3.11}$$

where $\varkappa$ is a certain constant. A study[17] has shown that when $K \gtrsim 1$ the mapping (3.11) has the following properties: 1) the motion is stochastic and after finite times the local instability (3.6) appears; 2) a stationary distribution function $\rho(x)$ sets in after long times and is close to a constant (~1) everywhere except for a small region in the vicinity of the point x_0 of the first resonance:

$$x_0=\{K\varkappa^{-1}\cos \varkappa x_0\}.$$

Similar properties are possessed also by the set (3.5) in the Hamiltonian case which was studied in detail by Chirikov.[14]

4. STUDY OF THE DISSIPATIVE CASE

Considerations similar to the ones given earlier can be advanced also for the dissipative system (3.3) of interest to us. Using (3.7) we find from (3.3) and (3.4) the condition for the appearance of stochasticity in the form

$$K_\Gamma = K\mu \gg 1. \tag{4.1}$$

When $\Gamma \gg 1$ the condition (4.1) becomes

$$K_\Gamma = K/\Gamma \gg 1. \tag{4.2}$$

However, the dissipative nature of the system (3.3) causes the invariant set on which the stochastic motion is realized to have a structure that differs from that in the Hamiltonian case.

The numerical analysis confirms Eq. (4.1). We give in Fig. 2 a typical phase-plane pattern which corresponds to the appearance of a strange attractor. We recall that a strange attractor is an invariant region in

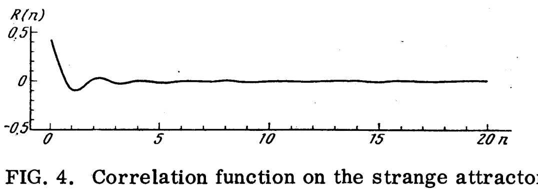

FIG. 4. Correlation function on the strange attractor for the same data as in Fig. 2.

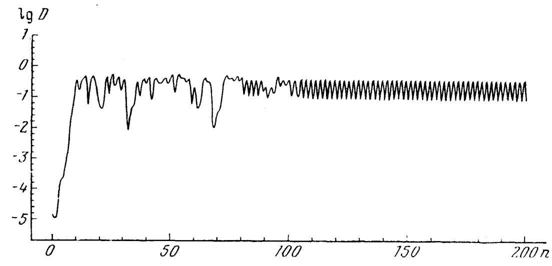

FIG. 5. Local properties of the motion in the case of degeneracy of the strange attractor to the usual limit cycle.

phase space, and has one less dimension than the whole phase space and a Cantor type structure. The latter property manifests itself in that when the scale increases each of the lines in Fig. 2 subdivides into a structure similar to that in Fig. 2. Moreover, the region depicted in Fig. 2 is attractive, i.e., for any initial condition the point of a trajectory lands after some time on the structure depicted in Fig. 2.[2)] The motion of the representative point on the strange attractor is stochastic and corresponds to a transition from laminar to turbulent motion. The ensuing property of local instability is illustrated in Fig. 3. The development of local instability corresponds for short times to Eq. (3.6) and for long times to random fluctuations in the vicinity of $D \sim 1$. The correlation function evaluated from Eq. (3.8) is shown in Fig. 4 and its relaxation time corresponds to Eq. (3.10).

In those cases where $K \gg 1$, $\Gamma \gg 1$, but condition (4.2) is not satisfied, i.e., $K/\Gamma < 1$, the motion of the system in the initial stage is close to stochastic (since $K \gg 1$). Subsequently, however, the trajectory is attracted to some periodic cycle. This statement is illustrated by numerical data in Fig. 5.

The qualitative criterion for the appearance of a strange attractor can thus be formulated as follows. Let the initial system be reduced to some "diagonal" form, i.e., to variables such that the fast and slow motions are effectively separated. In that case the ratio of the stretching parameter K of the fast variable (phase) when there is no dissipation to the characteristic dissipation coefficient Γ of the slow variable (action) must be larger than unity.

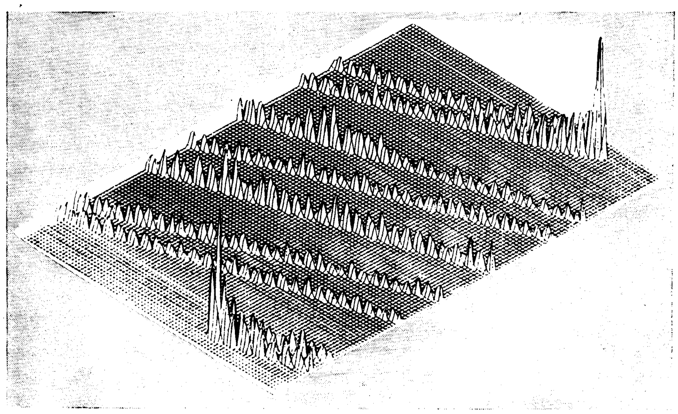

FIG. 7. Distribution function $\rho(x,y)$ on the strange attractor depicted in Fig. 2.

We now dwell on the particular features of Eqs. (3.3) when the stochasticity parameter K_Γ changes in the transition region $K_\Gamma \gtrsim 1$ where the random (turbulent) motion originates. When $\Gamma < 1$ and $K \gg 1$ the picture in the phase plane becomes complicated (Fig. 6) and is, in some sense, intermediate between the distribution of phase points when $\Gamma = 0$ and the distribution when $\Gamma > 1$ (Fig. 2). The appearance of a weak structure in the form of almost horizontal straight lines is seen in Fig. 6.

Increasing Γ in the region $\Gamma > 1$ leads to the appearance of a number of bifurcations in the regimes of motion of the system. For some values of K_Γ the strange attractor vanishes and becomes a limit cycle. This effect is analogous to the appearance of a stability region in the variable K in the Hamiltonian case. A numerical analysis shows that for the values $\varepsilon = 0.3$, $\alpha = 0.3$, and $K = 9.03$ (i.e., for the same values as for the case in Fig. 2) stability islands appear in the range $5.9 \lesssim \Gamma \lesssim 6$. When $\Gamma > 6$ the strange attractor appears again and vanishes when $\Gamma > 14$.

In concluding this section we consider the distribution

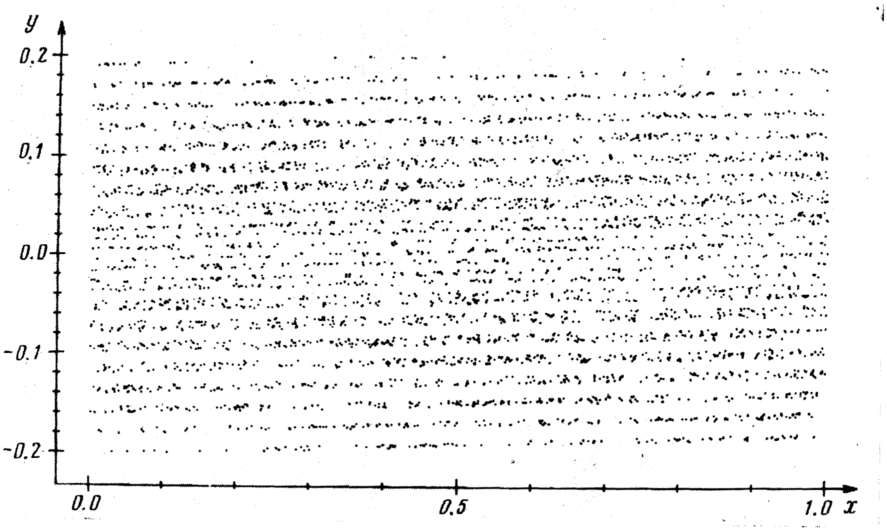

FIG. 6. Strange attractor for weak dissipation ($\varepsilon = 0.3$, $\alpha = 0.3$, $K = 9.03$, $\Gamma = 0.85$).

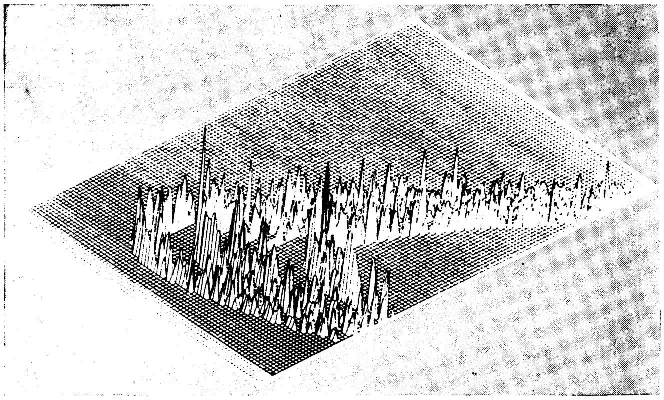

FIG. 8. Subdivision of the distribution function $\rho(x,y)$ in the vicinity of the point B on the strange attractor depicted in Fig. 2.

function $\rho(x, y, t)$ of the states of the system. A numerical analysis shows that there exists a stationary function $\rho(x, y)$ on the strange attractor. We give in Fig. 7 an example of such a function (the numerical data are the same as in Fig. 2). It has two sharp maxima in the neighborhood of the points A and B (Fig. 2) and in the remaining region fluctuates about a constant value. With good accuracy we may assume that the function smoothed over a small region is

$$\bar{\rho}(x, y) \approx \text{const}.$$

This conclusion is completely analogous to the properties of the distribution $\rho(x)$ in the cosine (sine) model (3.11). If, however, we increase the scale of the mapping of $\rho(x, y)$, there occurs a subdivision of the lines of the attractor and correspondingly a subdivision of the "crests" of the relief in Fig. 7. An example of such a subdivision for the region near the point B is given in Fig. 8. The sharp maximum occurring in Fig. 7 is strongly diminished and the amplitude of the distribution becomes more uniform.

The conclusion about the existence of a stationary distribution function on the strange attractor and about its properties has, in our opinion, the following fundamental significance. Notwithstanding the presence of strong dissipation in the system, one can describe the turbulent motion by means of an appropriate invariant distribution similar to what is done for Hamiltonian systems. In other words, for the range of parameters for which a strange attractor appears one might construct an "equilibrium thermodynamics" and study the relaxation to equilibrium by means of a suitable kinetic equation.

5. CONCLUSION

Using a very simplified model we have shown how turbulent motion can be generated under the influence of a perturbation in the vicinity of an unperturbed stable limit cycle. The simplicity of the model enables us to give an intuitive, qualitative analysis which was confirmed by the data from a numerical experiment. One may expect that the basic features of the appearance of turbulence, discovered above, are possessed also by more complex systems. We note two aspects whose study, in our opinion, would enable us to reinforce the "position" of the strange attractor in turbulence theory. The first is connected with a study of a large number of interacting dissipative modes and with making more precise the turbulence evolution picture suggested by Landau. The second is connected with an analysis of the picture we have obtained of the appearance of a strange attractor and with a comparison of it with existing mathematical models[8,16] and others. An answer to this question would enable us to elucidate the formal side of the problem. In connection with this last remark we note that, for instance, the formal model of turbulence suggested by Ruelle and Takens[18] bears, apparently, no relation to the real situation.

From the results of the present paper it follows also that the main stochastic characteristics of a dissipative system are similar to those of Hamiltonian systems. These include the property of local instability, the exponential damping of the correlation function, and the existence of a stationary distribution function.

[1)] Replacing the cosine by a sine does not play any role.

[2)] An analysis of the structure of the strange attractor which appears (number of lines, size, position of the points A and B, and so on) is given in Ref. 13.

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