Theory of nonstationary Josephson effect in short superconducting contacts

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A microscopic theory is used to calculate the connection between the current I with the phase difference φ of the order parameter, as well as the current-voltage characteristic (CVC) of a superconducting bridge of length $2d \ll \xi(T)(1-T/T_c)^{1/4}$. The case of voltages $V \ll \Delta$ at a temperature T close to the critical value T_c, and the case of $V \gg \Delta$ at arbitrary temperature are considered. At low voltages V, the dependence of the Josephson current I_S on φ is determined by the ratio of the characteristic frequency V_c of the variation of φ and the reciprocal time τ_ϵ^{-1} of the energy relaxation. In the case $\tau_\epsilon V_c \gg 1$ the function $I_S(\varphi)$ differs from a sinusoid, and this explains the presence of subharmonic steps on the CVC when monochromatic radiation acts on the bridge. In the case $\tau_\epsilon V_c \ll 1$ the $I_S(\varphi)$ dependence is sinusoidal as in the stationary case. At high voltages and arbitrary temperatures, the $I(V)$ dependence differs from Ohm's law by an amount $[\Delta(\pi^2/4-1)/eR]\tanh(V/2T)$, which determines at $V > T$ the excess current. The response of the bridge to a weak alternating signal in the presence of a direct current smaller than the critical value is also determined.

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INTRODUCTION

The theory of the nonstationary Josephson effect has been developed in sufficient detail only for tunnel junctions. Other types of weakly coupled superconductors have been much less studied. Yet the Josephson effect in them is accompanied by interesting phenomena that do not occur in tunnel junction. The difference is due, in particular, to the fact that flow of a current larger than critical ($I > I_c$) through structures of bridge type with constrictions or through point contacts produces at the center of the junction oscillations of the energy gap and these cause, generally speaking, a strong deviation of the quasiparticle distribution function from equilibrium.

The study of the Josephson effect in bridges with constrictions was initiated by Aslamazov and Larkin[1] on the basis of simplified nonstationary Ginzburg-Landau (GL) equations that are valid only for zero-gap superconductors. They have shown[1] that near T_c the Josephson effect in a short contact [$d \ll \xi(T)$, where $2d$ is the length of the contact and $\xi(T)$ is the correlation length] can be described within the framework of a simple resistive model, in which the current is made up of the quasiparticle conduction current and the Josephson current:

$$I=\frac{1}{2R}\frac{\partial\varphi}{\partial t}+I_c\sin\varphi,\quad \frac{\partial\varphi}{\partial t}=2V(t). \tag{1}$$

Here φ is the difference between the phases of the order parameters and V is the voltage on the contact (the electron charge is set equal to unity).

Likharev and Yakobson[2] analyzed a bridge of variable thickness, likewise on the basis of the GL equations, but with account taken of the time derivative of the order parameter. It was shown that in the case of sufficiently high voltage (and correspondingly currents), such that the characteristic distance $(D\pi/V)^{1/2}$ (D is the diffusion coefficient) over which the electron diffuses over the time π/V becomes shorter than the bridge length $2d$, the current-voltage characteristic (CVC) of the bridge takes the form

$$I=V/R+I_{exc}\,\mathrm{sign}\,V,\quad I_{exc}\sim I_c, \tag{2}$$

where I_{exc} is the so-called excess current, which is independent of the dc voltage V. The voltages V at which (2) holds correspond to currents $I \gtrsim I_c[\xi(T)/d]^2$, i.e., the asymptotic form defined by formula (2) is reached at currents that differ from I_c by the large factor $[\xi(T)/d]^2$. The parameter $d/\xi(T)$ characterizes the weakness of the coupling.

The GL equations used in the theories of Refs. 1 and 2 are valid for zero-gap superconductors. In superconductors with gaps it is necessary to take into account in these equations also the anomalous term that describes the disequilibrium of the quasiparticles. This was done by Aslamazov and Larkin,[3] who considered a bridge with a constriction under rather stringent limitations on the bridge length:

$$\eta=\xi(T)(1-T/T_c)^{1/4}<2d<\xi(T),\quad \tau_\epsilon\Delta^2\gg T$$

(τ_ε is the energy relaxation time). The first of these inequalities means that the retarded (advanced) Green's function satisfies the usual relations

$$g^{R(A)}(\varepsilon)=\varepsilon/\xi^{R(A)},\quad \xi^{R(A)}=\pm((\varepsilon\pm i\delta)^2-\Delta^2)^{1/2}, \tag{b}$$

i.e., at each point inside the bridge the state density takes an equilibrium form with a gap that depends on the coordinates and on the time. They have shown that the oscillations of the gap decrease the number of the quasiparticles at the center of the bridge. This in turn causes and increase in the time-averaged value of the gap and stimulates the superconducting current.

Kulik and Omel'yanchuk[4] have considered the case of a shorter contact of length $2d$, satisfying the inequality

$$2d\ll(D/\Delta)^{1/2}. \tag{3}$$

In this case the expressions given above for $g^{R(A)}$ do not hold. The condition (3) allows us to discard from the equations for the Green's functions all but the gradient terms. Following this idea, Kulik and Omel'yanchuk

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considered the stationary Josephson effect at arbitrary temperatures. What remained unexplained is the more interesting nonstationary effect, wherein deviation from equilibrium takes place. A step in this direction was made by Mitsai.[5] He investigated the linear response of a bridge to a weak alternating current of frequency ω in the presence of a dc superconducting current $I< I_c$ through the bridge.

We present in this paper a theory of the nonstationary Josephson effect in short bridges, when the conditions[1)] (3) and $\Delta\tau_\varepsilon \gg 1$ are satisfied. In the general case of arbitrary frequencies ω, and accordingly voltages V, the problem cannot be solved. We consider therefore the limiting cases $V, \omega \ll \Delta$ and $V, \omega \gg \Delta$, where Δ is the energy gap far from the bridge (in the shores). In the former case we can confine ourselves to a quasiclassical expansion in ω/Δ. We obtain thus an expression for the current I and draw some conclusions concerning the form of the CVC near T_c. The point is that only near T_c is the characteristic voltage on the bridge $V_c \equiv I_c R = \pi\Delta^2/4T$ much smaller than Δ. It will be shown that the expression for I depends strongly on the ratio of V_c and τ_ε^{-1}. In particular, formula (1) is valid only in the limit as $\tau_\varepsilon V_c \to 0$. In the latter case ($V \gg \Delta$) we obtain the form of the CVC $I(V)$ at arbitrary temperatures. It turns out that the $I(V)$ dependence is determined by formula (2), i.e., that the excess current I_{exc} is present. The value $I_{\text{exc}} \sim \Delta/R \sim (T_c/\Delta) I_c$ differs from that obtained in Ref. 2, particularly near T_c we have in accord with experiment[7] $I_{\text{exc}} \gg I_c$. In addition, the asymptotic form of (2) is reached at currents and voltages that are not connected with the coupling-weakness parameter $d/\xi(T)$.

1. BASIC EQUATIONS

We assume the previously analyzed[2,4,5] model of a contact or bridge of variable thickness. We consider a filament with cross section a^2 and length $2d \gg a$, connecting two massive superconductors (shores). All the isotropic functions are assumed on the shore to be in equilibrium and to correspond to a given phase χ and a given potential Φ. The superconductors are assumed dirty ($T\tau \ll 1$). Actually the assumed model describes not only a bridge of variable thickness, but any other short three-dimensional bridge with a constriction. In particular, the problem of a contact in the form of a hyperboloid of revolution reduces to the considered one-dimensional model (see the Appendix), in which all the functions depend on the coordinate x along the filament.

To find the current in the bridge it is necessary to solve a system of equations for Green's functions integrated with respect to the variable $\xi_p = (p - p_F)p/m$.[8-10] This system takes the matrix form[10]

$$\frac{\mathbf{p}}{m}\frac{\partial \check{G}}{\partial \mathbf{R}} + \hat{\sigma}_z\frac{\partial \check{G}}{\partial t} + \frac{\partial \check{G}}{\partial t'}\hat{\sigma}_z + \check{H}(t)\check{G} - \check{G}\check{H}(t') + \check{\Sigma}\check{G} - \check{G}\check{\Sigma} = \check{0}. \quad (4)$$

The Green's function $\check{G}$ and the self-energy part $\check{\Sigma}$ are here matrices in the form

$$\check{G} = \begin{pmatrix} \hat{G}^R & \hat{G} \\ \hat{0} & \hat{G}^A \end{pmatrix}, \quad \check{\Sigma} = \begin{pmatrix} \hat{\Sigma}^R & \hat{\Sigma} \\ \hat{0} & \hat{\Sigma}^A \end{pmatrix}. \quad (5)$$

In turn, $\check{G}^{R(A)}$ and $\check{G}$ are (2×2) matrices made up of ordinary Green's functions g and Gor'kov functions f:

$$\hat{G} = \begin{pmatrix} g & f \\ -f^+ & \bar{g} \end{pmatrix}.$$

The notation in (4) is

$$\check{G} = \check{G}_\mathbf{p}(\mathbf{R}; t, t') = \frac{i}{\pi}\int d\xi_p d\mathbf{r}\, \check{G}\left(\mathbf{R}+\frac{\mathbf{r}}{2}, \mathbf{R}-\frac{\mathbf{r}}{2}; t, t'\right) e^{-i\mathbf{pr}},$$

$$\check{H}(t) = -\frac{ie}{m}\mathbf{p}\mathbf{A}(t)\hat{\sigma}_z - i\check{\Delta}(t) + ie\Phi(t).$$

In our case the vector potential can be neglected. The matrix product $\check{G}\check{\Sigma}$ means convolution with respect to the internal time variable. The function G satisfies the additional normalization condition[10]

$$\check{G}^2 \equiv \check{G}\check{G} = \check{1}\delta(t-t'). \quad (6)$$

The current density is expressed in terms of the function $\check{G}$ via the formula

$$J(t) = -\frac{ep}{4\pi}\int \frac{d\Omega}{4\pi}\,\mathrm{Sp}\,\hat{\sigma}_z(\mathbf{p}\hat{G}). \quad (7)$$

In the considered case of a dirty superconductor, the Green's function $\check{G}$ can be represented in the form

$$\check{G} = \check{G}_0 + (\mathbf{p}/p)\check{\mathbf{G}}_1. \quad (8)$$

The condition (3) allows us to neglect in Eq. (4) for $\check{\mathbf{G}}_1 = (\check{G}_x, 0, 0)$ all but the gradient terms

$$\frac{\partial}{\partial x}\check{G}_x = 0. \quad (9)$$

It follows therefore that $\check{G}_x$ does not depend on x. The constant $\check{G}_x$ can be found from the normalization condition (6) by using the boundary conditions

$$\check{G}_0(\pm d) = \begin{pmatrix} \hat{G}_0^R & \hat{G}_0 \\ \hat{0} & \hat{G}_0^A \end{pmatrix}_{x=\pm d}, \quad (10)$$

where

$$\hat{G}_0^{R(A)} = \hat{S}(t)\hat{g}^{R(A)}\hat{S}^+(t')$$

are isotropic retarded and advanced Green's functions in the shores. The matrix $\hat{S}(t)$ takes into account the presence of the phase $\chi(x, t)$:

$$\hat{S} = \begin{pmatrix} e^{i\chi/2} & 0 \\ 0 & e^{-i\chi/2} \end{pmatrix}. \quad (11)$$

The functions $\hat{g}_0^{R(A)}$ are the equilibrium Green's functions at $\chi = 0$. They are obtained from one of the equations in (4), which takes in the homogeneous and stationary case the form

$$[\hat{g}_0^R, \varepsilon\hat{\sigma}_z + i\Delta\hat{\sigma}_y + i\hat{\gamma}^R] = \hat{0}, \quad (12)$$

where the square brackets stand for the commutator, and the matrix $\hat{\gamma}^R$:

$$\hat{\gamma}^R = \alpha_{ph}\int_{-\infty}^{\infty} d\varepsilon'(\varepsilon'-\varepsilon)|\varepsilon'-\varepsilon|\frac{\mathrm{ch}(\varepsilon/2T)\theta(\varepsilon'^2-\Delta^2)(\varepsilon'\hat{\sigma}_z + i\Delta\hat{\sigma}_y)}{\mathrm{sh}((\varepsilon'-\varepsilon)/2T)\mathrm{ch}(\varepsilon'/2T)(\varepsilon'^2-\Delta^2)^{1/2}} \equiv \gamma_1^R\hat{\sigma}_z + \gamma_2^R i\hat{\sigma}_y, \quad (12')$$

determines the damping via the electron-phonon collisions $\alpha_{ph} = (\pi/4)\xi_{ph}/\Theta_D^2$, $\Theta_D = sp$, and s is the speed of sound.

From (12) it follows, when the normalization condition $(\hat{g}_0^R)^2 = \hat{1}$ is taken into account

$$\hat{g}_0^R = \frac{(\varepsilon + i\gamma_1^R)\hat{\sigma}_z + (\Delta + i\gamma_2)i\hat{\sigma}_y}{((\varepsilon+i\gamma_1^R)^2 - (\Delta+i\gamma_2^R)^2)^{1/2}} \equiv g^R(\varepsilon)\hat{\sigma}_z + f^R(\varepsilon)i\hat{\sigma}_y. \quad (13)$$

The function $\hat{g}_0^A$, for which $\hat{\gamma}^A = -\hat{\gamma}^R$, is similar in form. The function $\hat{G}_0^r$ is expressed in terms of $\hat{G}_0^{R(A)}$ (Refs. 7, 8):

$$\hat{G}_0^r = \hat{G}_0^R \hat{n} - \hat{n} \hat{G}_0^A, \quad n = \hat{S} n \hat{S}^+, \quad n(\varepsilon) = \text{th}(\varepsilon/2T). \tag{14}$$

From the normalization condition (6) and from the boundary condition (10) we get an equation for $\hat{G}_x$

$$(\hat{G}_0^R \hat{G}_x + \hat{G}_x \hat{G}_0^A + \hat{G}_x^R \hat{G}_0^r + \hat{G}_0^r \hat{G}_x^A)_{x=\pm d} = \hat{0}. \tag{15}$$

Thus, to determine the sought function $\hat{G}_x$ we must find $\hat{G}_x^{R(A)}$. We use a relation that follows from (t) (Ref. 10):

$$\check{G}_x = -l \check{G}_0 \frac{\partial}{\partial x} \check{G}_0, \tag{16}$$

where $l = (p/m)\tau$ is the mean free path. A solution of (16) is the function

$$\check{G}_0(x) = \check{G}_0(0) \exp(-\check{G}_x x/l).$$

We add and subtract the values of the functions $\check{G}_0(x)$ at $x = \pm d$. Then, introducing the notation

$$\check{G}_\pm = {}^1/_2 (\check{G}_0(d) \pm \check{G}_0(-d)), \tag{17}$$

we get

$$\check{G}_+ = \check{G}_0(0) \,\text{ch}(\check{G}_x d/l), \quad \check{G}_- = -\check{G}_0(0) \,\text{sh}(\check{G}_x d/l). \tag{18}$$

As follows from (6), the matrices $\check{G}_0(0)$ and $\check{G}_x$ anticommute with each other, and $[\check{G}_0(0)]^2 = \check{1}$. Multiplying $\check{G}_+$ and $\check{G}_-$ we get from (18)

$$\check{G}_x = -\frac{l}{2d} \text{Arsh}(2\check{G}_+ \check{G}_-) = -\frac{l}{2d} \sum_{k=0}^{\infty} \alpha_{2k+1} (\check{G}_+ \check{G}_-)^{2k+1}. \tag{19}$$

We separate the matrix components (1, 1) and (2, 2). Then, as can be easily verified, we obtain

$$G_x^{R(A)} = -\frac{l}{2d} \text{Arsh}(2G_+^{R(A)} G_-^{R(A)}). \tag{20}$$

We write down also an expression for $\hat{G}_x$ in series form. We shall find it useful in the determination of the CVC in the case of high voltages ($V \gg \Delta$). To this end we separate the component $(\check{G}_x)_{12}$ in (19). After simple transformations and taking (20) into account we obtain

$$\hat{G}_x = -\frac{l}{2d} \sum_{k=0}^{\infty} \alpha_{2k+1} \sum_{m=0}^{2k} (\hat{G}_+^R \hat{G}_-^R)^{2k-m} \hat{A} (\hat{G}_+^A \hat{G}_-^A)^m + \hat{G}_x^R \hat{n}_+ - \hat{n}_+ \hat{G}_x^A,$$
$$\hat{A} = (\hat{G}_+^R)^2 \hat{n}_- - \hat{G}_+^R \hat{n}_- \hat{G}_+^A + \hat{G}_-^R \hat{n}_- \hat{G}_-^A - \hat{n}_- (\hat{G}_-^A)^2. \tag{21}$$

The functions $\hat{n}_\pm$ and $\hat{G}_\pm^{R(A)}$ are defined in terms of $\hat{n}$ and $\hat{G}_0^{R(A)}$ in analogy with (17).

Thus, the function $\hat{G}_x$ has singularities in both the upper and lower ε half-planes. We shall obtain also an expression for $\hat{G}_x^R$ in the case when the phase difference $\varphi = \chi(d) - \chi(-d)$ is either constant in time or varies slowly in comparison with Δ^{-1}. We express $\hat{G}_0^R$ in terms of $\hat{g}_0^R$ (13) with the aid of the matrix $\hat{S}$ and recognize that $\chi(d) = -\chi(-d) = \varphi/2$. Then

$$\hat{G}_x^R = -\frac{l}{2d} i \sum_{k=0}^{\infty} \alpha_{2k+1} (\hat{G}_+^R \hat{\sigma}_z) \frac{(2f^R((g^R)^2 - (f^R)^2 \cos^2(\varphi/2))^{1/2})^{2k+1}}{((g^R)^2 - (f^R)^2 \cos^2(\varphi/2))^{1/2}}$$
$$= -\frac{l}{2d} b^R (\Delta \hat{\sigma}_z \cos(\varphi/2) + i\hat{\sigma}_y \varepsilon); \tag{22}$$
$$b^R = \frac{i}{\tilde{\xi}^R} \text{Arsh}\left(\frac{\Delta \sin(\varphi/2)}{\xi^R}\right), \quad \tilde{\xi}^R = \left((\varepsilon + i\delta)^2 - \Delta^2 \cos^2 \frac{\varphi}{2}\right)^{1/2},$$
$$\xi^R = ((\varepsilon + i\delta)^2 - \Delta^2)^{1/2}$$

Formula (22) was obtained earlier[4,5] by another method.

The function $\hat{G}_x$ at low voltages (the quasiclassical case) is easier to obtain by starting not from formula (21) but from Eq. (15). We seek the solution of this equation in the form $\hat{G}_x = \hat{G}_x^R \hat{F} - \hat{F} \hat{G}_x^A$. We then obtain from (15)

$$(\hat{G}_0^R \hat{F} - \hat{F} \hat{G}_0^A)_{x=\pm d} = (\hat{G}_0^R \hat{n} - \hat{n} \hat{G}_0^A)_{x=\pm d}. \tag{23}$$

Adding and subtracting the equations (23) at $x = \pm d$ we arrive at the system

$$\hat{G}_\pm^R \hat{F}^a - \hat{F}^a \hat{G}_\pm^A = \hat{G}_\pm^R \hat{n}_- - \hat{n}_- \hat{G}_\pm^A, \tag{24}$$

where $\hat{F}^a = \hat{F} - \hat{n}_+$. We represent the current density J_x in the form

$$J_x = J_x^r + J_x^a = (\pi\sigma/2l) \,\text{Sp}\, \hat{\sigma}_z (\hat{G}_x^R \hat{n}_+ - \hat{n}_+ \hat{G}_x^A + \hat{G}_x^a), \tag{25}$$

where the current J_x^r is determined by the first two terms in the parentheses, and the anomalous current J_x^a is determined by the function $\hat{G}_x^a = \hat{G}_x^R \hat{F}^a - \hat{F}^a \hat{G}_x^A$, which has complicated analytic properties and describes the deviation from equilibrium.

2. LINEAR RESPONSE OF A BRIDGE AT A CURRENT SMALLER THAN CRITICAL

Assume that a direct current $I_0 < I_c$ and a weak alternating current $I_1 \ll I_c$ flow through the bridge. The phase difference is then $\varphi = \varphi_0 + \varphi_1$, where $\varphi_1 \ll 1$. In the calculation of $\hat{F}^a$ we take into account the fact that

$$\hat{n}_- \sim \hat{\sigma}_z \sin \frac{\varphi_1(t) - \varphi_1(t')}{4} \approx \hat{\sigma}_z \frac{\varphi_1(t) - \varphi_1(t')}{4},$$

i.e., $F^a \sim \varphi_1$. Linearizing (24), we get

$$\hat{F}^a(\varepsilon_+, \varepsilon_-) = \frac{\varphi_1(\omega)}{4} \frac{(g^R(\varepsilon_+) - g^A(\varepsilon_-))\hat{\sigma}_z - (f^R(\varepsilon_+) + f^A(\varepsilon_-)) \cos(\varphi_0/2) \hat{1}}{(f^R(\varepsilon_+) - f^A(\varepsilon_-)) \sin(\varphi/2)}$$
$$\times (\text{th}(\varepsilon_+/2T) - \text{th}(\varepsilon_-/2T)), \tag{26}$$

where $\varepsilon_\pm = \varepsilon \pm \omega/2$. With the aid of (25), (26), and (22) we determine the current

$$I^a(\omega) = \frac{\varphi_1(\omega)}{8R} \int d\varepsilon \Big\{ b^R(\varepsilon_+) \Big[\varepsilon_+ (g^R(\varepsilon_+) - g^A(\varepsilon_-))$$
$$-\Delta (f^R(\varepsilon_+) + f^A(\varepsilon_-)) \cos^2 \frac{\varphi_0}{2} \Big] + b^A(\varepsilon_-) [\varepsilon_- (g^R(\varepsilon_+) - g^A(\varepsilon_-))$$
$$+\Delta (f^R(\varepsilon_+) + f^A(\varepsilon_-)) \cos^2(\varphi_0/2)] \Big\} [\text{th}(\varepsilon_+/2T) - \text{th}(\varepsilon_-/2T)]$$
$$\times [\sin(\varphi_0/2) (f^R(\varepsilon_+) - f^A(\varepsilon_-))]^{-1}. \tag{27}$$

We obtain the function $\hat{G}_{x,1}^{R(A)} = \hat{G}_x^{R(A)} - \hat{G}_{x,0}^{R(A)}$ by linearizing the equation [the function $G_x^{R(A)}(\varphi_0)$ is given by (22)]

$$\hat{G}_-^R \hat{G}_x^R + \hat{G}_x^R \hat{G}_-^R = \hat{0}. \tag{28}$$

Multiplying (28) by $i\hat{\sigma}_y$ and calculating the trace, we get

$$\text{Sp}(\hat{\sigma}_z \hat{G}_{x,1}^R(\varepsilon_+, \varepsilon_-)) = \frac{i\varphi_1(\omega) l}{4d \sin(\varphi_0/2)} \{b^R(\varepsilon_+) [(g^R(\varepsilon_+) - g^R(\varepsilon_-))\varepsilon_+$$
$$+ (f^R(\varepsilon_+) + f^R(\varepsilon_-)) \Delta \cos^2(\varphi_0/2)] + b^R(\varepsilon_-) [(g^R(\varepsilon_+) - g^R(\varepsilon_-))\varepsilon_-$$
$$- (f^R(\varepsilon_+) + f^R(\varepsilon_-)) \Delta \cos^2(\varphi_0/2)]\} [f^R(\varepsilon_+) - f^R(\varepsilon_-)]^{-1}. \tag{29}$$

The current I^r is equal to

$$I^r(\omega) = \frac{d}{2R} \int d\varepsilon \,\text{Sp}\, \hat{\sigma}_z \Big\{ [\hat{G}_x^R(\varphi_0) - \hat{G}_x^A(\varphi_0)] \text{th} \frac{\varepsilon}{2T} \cdot 2\pi\delta(\omega)$$
$$+ \hat{G}_{x,1}^R(\varepsilon_+, \varepsilon_-) \text{th} \frac{\varepsilon_-}{2T} - \hat{G}_{x,1}^A(\varepsilon_+, \varepsilon_-) \text{th} \frac{\varepsilon_+}{2T} \Big\}. \tag{30}$$

The first term under the integral sign determines the static part of the superconducting current calculated in Ref. 4. From (27), (29), and (30) we obtain in the case $\omega \ll \Delta$ and $\Delta \ll T$, in the principal approximation in Δ/T,

$$I(\omega)=I_c\Big\{\sin\varphi_0 2\pi\delta(\omega)+\varphi_1(\omega)\cos\varphi_0 + \frac{1}{2}\varphi_1(\omega)\frac{\omega}{\omega+i\nu_\varepsilon}\left(\varphi_0-\sin\varphi_0\right)\operatorname{ctg}\frac{\varphi_0}{2}\Big\}+\frac{i\omega\varphi_1(\omega)}{2R}, \tag{31}$$

where account is taken of the fact that near T_c

$$2\gamma_1=\nu_\varepsilon\equiv\tau_\varepsilon^{-1}=\alpha_{ph}\int_0^\infty d\varepsilon'\frac{\varepsilon'^3}{\operatorname{sh}(\varepsilon'/2T)}=7\zeta(3)T^3\alpha_{ph},\qquad \gamma_2\sim\frac{\Delta}{T}\gamma_1. \tag{31'}$$

At $\omega \gg \nu_\varepsilon$ formula (31) goes over into the formula obtained by Mitsaĭ[5] by another method.[2)]

3. DISCUSSION OF RESULTS AND CHEMO-EXCITATION MODEL

We now obtain the solution of (24). It is convenient to introduce a new matrix $\hat{X}$ connected with $\hat{F}^a$ by the relation

$$\hat{F}^a=\hat{G}_-^R\hat{X}+\hat{X}\hat{G}_-^A. \tag{32}$$

Substituting (32) in (24) and using the fact that the matrices $\hat{G}_+^R$ and $\hat{G}_-^R$ anticommute [this follows from (6)], we obtain for $\hat{X}$ the equation

$$\hat{G}_+^R\hat{X}+\hat{X}\hat{G}_+^A=-\hat{n}_-=-n(t-t')i\sin\left(\frac{\varphi(t)-\varphi(t')}{4}\right)\hat{\sigma}_z. \tag{33}$$

We take into account the fact that in our case of low frequencies ($V \ll \Delta$) the dependence of all the matrices on the time difference is faster than the dependence on the time sum $(t+t')/2$. Using this circumstance, we carry out a Fourier transformation with respect to the difference variable $(t-t')$. Then, accurate to terms $\sim V/\Delta$, Eq. (33) reduces to a partial differential equation

$$\begin{aligned}&\left(g^R\hat\sigma_z+f^R\cos\frac{\varphi}{2}i\hat\sigma_y\right)\hat{X}+\hat{X}\left(g^A\hat\sigma_z+f^A\cos\frac{\varphi}{2}i\hat\sigma_y\right)\\&+\frac{i}{2}\Big[\frac{\partial}{\partial\varepsilon}\left(g^R\hat\sigma_z+f^R\cos\frac{\varphi}{2}i\hat\sigma_y\right)\frac{\partial}{\partial t}\hat{X}-\frac{\partial}{\partial t}\hat{X}\frac{\partial}{\partial\varepsilon}\Big(g^A\hat\sigma_z\\&+f^A\cos\frac{\varphi}{2}i\hat\sigma_y\Big)-\left(\frac{\partial}{\partial t}\cos\frac{\varphi}{2}\right)\left(f^R i\hat\sigma_y\frac{\partial\hat{X}}{\partial\varepsilon}-\frac{\partial\hat{X}}{\partial\varepsilon}f^A i\hat\sigma_y\right)\Big]\\&=-\frac{1}{4}\frac{\partial\varphi}{\partial t}\frac{\partial}{\partial\varepsilon}\operatorname{th}\frac{\varepsilon}{2T}\hat\sigma_z,\end{aligned} \tag{34}$$

where φ and $\hat{X}$ are functions of the time t. It follows from (34) that $\hat{X}=X_0\hat{1}+X_1\hat{\sigma}_x$.

The character of the solution of (34) varies with $|\varepsilon|$. At $|\varepsilon|<\Delta$ we have

$$g^R-g^A=\nu_\varepsilon\frac{\partial g}{\partial\varepsilon},\qquad g=\frac{\varepsilon}{(\Delta^2-\varepsilon^2)^{1/2}},$$
$$f^R-f^A=\nu_\varepsilon\frac{\partial f}{\partial\varepsilon},\qquad f=\frac{\Delta}{(\Delta^2-\varepsilon^2)^{1/2}},$$

where ν_ε is the frequency of the energy relaxation [see (31′)]. Multiplying (34) by $\hat\sigma_z$ (or by $\hat\sigma_y$) and calculating the trace, we obtain two equations

$$2gX_0+i\cos\frac{\varphi}{2}\frac{\partial f}{\partial\varepsilon}\left(\frac{\partial X_1}{\partial t}+\nu_\varepsilon X_1\right)-i\left(\frac{\partial}{\partial t}\cos\frac{\varphi}{2}\right)f\frac{\partial X_1}{\partial\varepsilon}=-\frac{1}{8T}\frac{\partial\varphi}{\partial t},$$
$$2f\cos\frac{\varphi}{2}X_0+i\frac{\partial g}{\partial\varepsilon}\left(\frac{\partial X_1}{\partial t}+\nu_\varepsilon X_1\right)=0. \tag{35}$$

Recognizing that $g\partial g/\partial\varepsilon=f\partial f/\partial\varepsilon$, we obtain from (35) for the function

$$\tilde{X}_1=X_1\exp(-\nu_\varepsilon t)$$

the equation

$$\sin\frac{\varphi}{2}\frac{\partial f}{\partial\varepsilon}\frac{\partial\tilde{X}_1}{\partial t}-\frac{\partial\sin(\varphi/2)}{\partial t}f\frac{\partial\tilde{X}_1}{\partial\varepsilon}=-\frac{i}{8T}\frac{\partial\varphi}{\partial t}\exp(\nu_\varepsilon t)\operatorname{ctg}\frac{\varphi}{2}. \tag{36}$$

The first integral of (36) is readily obtained

$$\varkappa(\varepsilon,t)=if\sin\frac{\varphi}{2}=\frac{\Delta\sin(\varphi/2)}{(\Delta^2-\varepsilon^2)^{1/2}}. \tag{37}$$

The solution of (36) is

$$\tilde{X}_1=-\frac{\Delta}{8T\varkappa^2(\varepsilon,t)}\int_{-\infty}^{t}dt_1\frac{\partial\varphi(t_1)}{\partial t_1}\frac{\sin\varphi(t_1)\exp(\nu_\varepsilon t_1)}{2(\varkappa^2(\varepsilon,t)-\sin^2(\varphi(t_1)/2))^{1/2}}. \tag{38}$$

With the aid of (38), (32), and (25) we obtain the contribution made to the current by energies $|\varepsilon|<\Delta$:

$$I_1^a=\frac{d}{2Rl}\int_0^\Delta d\varepsilon\operatorname{Sp}(\hat\sigma_z\hat{G}_x^a)=\frac{d\Delta i}{4TRl}\int_0^\Delta d\varepsilon f\sin\left(\frac{\varphi}{2}\right)X_1\operatorname{Sp}\hat\sigma_z(\hat{G}_x^R-\hat{G}_x^A)=I_cP\{\varphi\}, \tag{39}$$

$$P\{\varphi\}=\sin\varphi\int_0^1 dk\frac{k^3}{4(1-k^2)^{1/2}(1-k^2\sin^2(\varphi/2))^{1/2}}\times\int_{-\infty}^{t}dt_1\frac{\partial\varphi(t_1)\sin\varphi(t_1)\exp[-\nu_\varepsilon(t-t_1)]}{\partial t_1\,(1-k^2\sin^2(\varphi(t_1)/2))^{1/2}}. \tag{39'}$$

We now calculate $\hat{G}_x^a$ at $|\varepsilon|>\Delta$. From (24) we determine

$$\hat{F}^a=\frac{\varepsilon}{4\Delta\sin(\varphi/2)}\frac{\partial}{\partial\varepsilon}\operatorname{th}\frac{\varepsilon}{2T}\hat\sigma_x.$$

Therefore

$$\begin{aligned}I_2^a&=\frac{d}{2Rl}\int_\Delta^\infty d\varepsilon\operatorname{Sp}(\hat\sigma_z\hat{G}_x^a)\\&=\frac{1}{2R}\frac{\partial\varphi}{\partial t}\int_\Delta^\infty d\varepsilon\frac{\varepsilon^2\operatorname{Arsh}[\Delta\sin(\varphi/2)/(\varepsilon^2-\Delta^2)^{1/2}]}{\Delta(\varepsilon^2-\Delta^2\cos^2(\varphi/2))^{1/2}\sin(\varphi/2)}\frac{\partial}{\partial\varepsilon}\operatorname{th}\frac{\varepsilon}{2T},\\ I_2^a&=I^r-I_1^a.\end{aligned} \tag{40}$$

The regular part I^r yields the usual superconducting current[4]

$$I_s=I_c\sin\varphi,\quad I_c=\pi^2\Delta^2/4TR. \tag{41}$$

Gathering together (39), (40), and (41), we obtain in the principal approximation in Δ/T

$$I=\frac{1}{2R}\frac{\partial\varphi}{\partial t}+I_c\sin\varphi+I_cP\{\varphi\}\equiv\frac{1}{2R}\frac{\partial\varphi}{\partial t}+I_c\tilde{P}\{\varphi\}. \tag{42}$$

The last term in (42), which contains the function $P\{\varphi\}$, is due to the deviation from the simple resistive model described by formula (1). From expression (39) for $P\{\varphi\}$ it follows that this deviation does not depend on the length $2d$ of the bridge (of course, if the condition (3) is satisfied), and is determined by the relation between ν_ε and the characteristic frequency of the change of the phases $\varphi(t)$.

We consider now the limiting cases.

(a) $\lambda=V_c\tau_\varepsilon\ll1$, i.e., $\pi\Delta^2\tau_\varepsilon/4T\ll1$. At low voltages $V\ll\nu_\varepsilon=V_c/\lambda$ the functions that depend on $\varphi(t_1)$ in (39′) can be taken outside the integral with respect to the time t_1 [making the substitution $\varphi(t_1)\to\varphi(t)$], and integration with respect to the variable k yields

$$P\{\varphi\}=\frac{1}{2}\frac{\partial\varphi}{\partial t}\tau_\varepsilon \operatorname{ctg}\frac{\varphi}{2}([\varphi]-\sin\varphi),\qquad (43)$$

where $[\varphi]$ is a function with period 2π, equal to φ at $|\varphi|<\pi$. If $\varphi=\varphi_0+\varphi_1$, expression (43) coincides with the next to the last term in (31) in the limit when $\omega\tau_\varepsilon \ll 1$. The function $P\{\varphi\}$ can be represented in the form

$$P\{\varphi\}=\frac{1}{2}\frac{\partial\varphi}{\partial t}\tau_\varepsilon\sum_{k=0}^{\infty} b_k\cos k\varphi,\qquad (44)$$

where

$$b_0=2\ln 2-1=0.39,\quad b_1=4\ln 2-3=-0.23;$$

$$b_k=(-1)^{k+1}k^{-2}\int_0^\infty dx\,\frac{e^{-x}}{\operatorname{ch}^2(x/2k)};\quad k\geqslant 2.$$

The first term of the series (44) determines the change of the contact resistance, and the remaining terms yield the contribution to the interference current. We note that the sign of b_1 is opposite the sign of the coefficient of the interference term $(\partial\varphi/\partial t)\cos\varphi$ in a tunnel junction.

We obtain the form of the CVC at voltages $V\ll\nu_\varepsilon$, using (42), (43), and the formula

$$V=\pi\Big/\int_{-\pi}^{\pi}d\varphi\Big/\frac{\partial\varphi}{\partial t},\qquad (45)$$

$$\frac{\partial\varphi}{\partial t}=2R(I-I_c\sin\varphi)\left(1+\lambda\operatorname{ctg}\frac{\varphi}{2}(\varphi-\sin\varphi)\right)^{-1}.$$

The calculation of expression (45) yields a sum of integrals in the form

$$\int_{-\pi}^{\pi}d\varphi\frac{\varphi\operatorname{ctg}(\varphi/2)}{a-\sin\varphi}+\int_{-\pi}^{\pi}d\varphi\frac{2\cos^2(\varphi/2)}{\sin\varphi-a}\equiv u(a),\quad a=\frac{I}{I_c},$$

the first of which is calculated by integration in the complex plane along the contour shown in Fig. 1. We get

$$V=R(I^2-I_c^2)^{1/2}/(1+\lambda u(I/I_c)),\qquad (46)$$

where

$$u(a)=\frac{(a^2-1)^{1/2}}{a}\ln\frac{2(a+(a^2-1)^{1/2})}{a}+\frac{1}{a}\operatorname{arctg}\frac{1}{(a^2-1)^{1/2}}-1.$$

At currents $I\gg I_c$, the CVC takes the asymptotic form $I=V/\tilde{R}$, where

$$\tilde{R}=R/(1+\lambda(2\ln 2-1))<R.\qquad (46')$$

Thus, the bridge conductivity is increased by the deviation from equilibrium. We note that a similar expression follows from the series (44). The conductivity increase obtained by Golub[6] differs from the one given here. In Golub's paper the second term in the denominator of formula (46) contains, besides a weaker (logarithmic) dependence on τ_ε, also at the small parameter $[d/\xi(T)]^2\ll 1$.

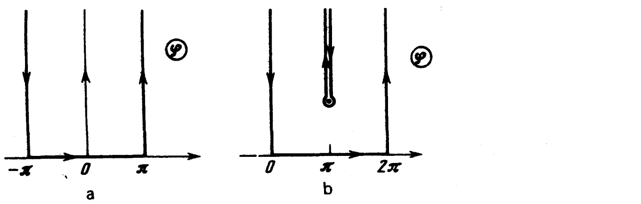

FIG. 1. Contours in complex φ plane, which are used to calculate the form of the CVC (a) to calculate the Fourier-expansion coefficients of the function $\tilde{P}\{\varphi\}$ (b).

At voltages $V\gg\nu_\varepsilon$ we can obtain from (39)

$$P\{\varphi\}=\sin\varphi\left\{\int_0^1 dk\frac{k}{(1-k^2)^{1/2}}\frac{\langle(1-k^2\sin^2(\varphi/2))^{1/2}\rangle}{(1-k^2\sin^2(\varphi/2))^{1/2}}-1\right\}.\qquad (47)$$

The angle brackets denote here averaging over the period $T=\pi/V$, which is effected with the aid of the expression for $\partial\varphi/\partial t$ (42):

$$\langle(1-k^2\sin^2(\varphi/2))^{1/2}\rangle=\frac{2V}{\pi}\int_0^{\pi/2}d\varphi\frac{(1-k^2\sin^2(\varphi/2))^{1/2}}{R(I-I_c\tilde{P}\{\varphi\})}.\qquad (48)$$

In the considered limiting case $\lambda\ll 1$ the inequality $V\gg\nu_\varepsilon$ means that $I\gg I_c$. We therefore obtain from (48) in the principal approximation

$$\langle(1-k^2\sin^2(\varphi/2))^{1/2}\rangle=(2/\pi)\mathrm{E}(k),$$

where $E(k)$ is a complete elliptic integral. For the total current we get from (42) and (47)

$$I=\frac{1}{2R}\frac{\partial\varphi}{\partial t}+I_c\frac{2}{\pi}\sin\varphi\int_0^1 dk\frac{k\mathrm{E}(k)}{(1-k^2)^{1/2}(1-k^2\sin^2(\varphi/2))^{1/2}}.\qquad (49)$$

As seen from (49), the total current contains besides the ohmic part also a component $I_c\tilde{P}\{\varphi\}$, which depends on the difference of the phases φ in a nonsinusoidal manner. The function $\tilde{P}\{\varphi\}$ obtained by numerically calculating the integral in (49) is shown in Fig. 2.

We expand $\tilde{P}\{\varphi\}$ in a series

$$\tilde{P}\{\varphi\}=\sum_{n=1}^{\infty}a_n\sin n\varphi,$$

$$a_n=\frac{2}{\pi^2}\int_0^1 dk\frac{k\mathrm{E}(k)}{(1-k^2)^{1/2}}\int_{-\pi}^{\pi}d\varphi\times\frac{\sin n\varphi\sin\varphi}{(1-k^2\sin^2(\varphi/2))^{1/2}}.\qquad (50)$$

Using for the integration of the integral with respect to φ the contour shown in Fig. 1b, we can show that at large n the quantity a_n decreases like $(-1)^n n^{-2}$. We note that since $\tilde{P}\{\varphi\}$ contains the harmonics $\sin n\varphi$, subharmonic steps at voltages $2V=\omega/n$ appear on the CVC when external radiation of frequency ω is applied to the junction.

An analytic expression for the CVC can be obtained in the considered case $V\tau_\varepsilon\ll 1$ in the entire range of variation of V. At $V_c\ll\nu_\varepsilon$, the function $I(V)$ is determined by formula (46), and at $V\gg V_c$ this relation can be obtained directly from (39′), noting that in the principal approximation $\varphi=2Vt$. To this end we must substitute the expression for $\varphi(t_1)$ in (39′), expand the latter in a Fourier series in t and t', and average over t. As a result we obtain an expression for the CVC in series

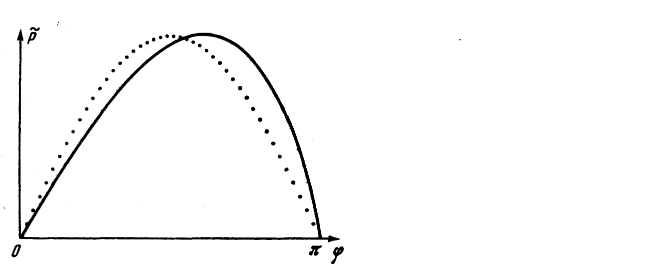

FIG. 2. Plot of $\tilde{P}\{\varphi\}$, obtained by numerical calculation of the integral in Eq. (49). The points show the plot of $\sin\varphi$.

form. In the limiting case $V \ll \nu$ the CVC coincides with (46), and at $V \gg \nu_\varepsilon$ the deviation of the CVC from Ohm's law decreases in proportion to ν_ε/V.

(b) $\lambda \gg 1$. In this case at practically all voltages $V > \nu_\varepsilon$ the function $P\{\varphi\}$ is given by (47). Substituting (48) in (47) we obtain an integral equation that determines the form of the function

$$P\{\varphi\}=\int_{-\pi}^{\pi} d\varphi_1 \frac{Q(\varphi,\varphi_1)}{I-I_c P\{\varphi_1\}},$$

$$Q(\varphi,\varphi_1)=\frac{V}{2\pi R}\sin\varphi \int_0^1 dk \frac{k}{(1-k^2)^{1/2}}\left(\frac{1-k^2\sin^2(\varphi_1/2)}{1-k^2\sin^2(\varphi/2)}\right)^{1/2}, \tag{51}$$

$$V=2\pi \Big/ \int_{-\pi}^{\pi}\frac{d\varphi}{R(I-I_c P\{\varphi\})}.$$

Thus, at $\lambda \gg 1$ even the form of the function $\tilde{P}\{\varphi\}$ in (42) depends on the current I that flows through the bridge.

If $I \sim I_c$, the function $\tilde{P}\{\varphi\}$ can apparently be determined only numerically. However, definite conclusions can be drawn concerning the form of $I(V)$. In particular, $\tilde{P}\{\varphi\}$ can again be represented in the form of the series (50), whose coefficients at $n \gg 1$ also decrease like $(-1)^n/n^2$. In the case $I \gg I_c$ the expression (49) for the current is again valid, and the expansion of $I(V)$ in powers of V_c/V takes the form

$$I=\frac{V}{R}+\frac{V_c}{2\pi V}\int_{-\pi}^{\pi} d\varphi P^2\{\varphi\}, \tag{52}$$

where the function $\tilde{P}\{\varphi\}$ is determined by the second term in (49), and its form is shown in Fig. 2. Thus, the value of the integral in (52) is of the order of unity and the CVC approaches Ohm's law, just as in the case of the simple resistive model (1), like V_c/V.

At voltages lower than V_c but higher than ν_ε, the form of the CVC can be obtained analytically from (51). If $\nu_\varepsilon \ll V \ll V_c$ the main contribution to the integral is made by the region near the maximum of the function $\tilde{P}\{\varphi\}$, which can be represented in this vicinity in the form

$$P\{\varphi\}=P\{\varphi_0\}-{}^1/_2|P_{\varphi\varphi}|(\varphi-\varphi_0)^2.$$

We then get from (51)

$$P\{\varphi\}=\sin\varphi_0\int_0^1 \frac{k\,dk}{(1-k^2)^{1/2}}\left(\frac{1-k^2\sin^2(\varphi_0/2)}{1-k^2\sin^2(\varphi/2)}\right)^{1/2}, \quad P\{\varphi_0\}=\sin\varphi_0.$$

The angle φ_0 is determined from the condition that the derivative $\partial\tilde{P}/\partial\varphi$ vanish at the point φ_0, and turns out to equal 1.92. For the CVC we get from (51)

$$V=R[2|P_{\varphi\varphi}''|(I-0.94I_c)]^{1/2}.$$

Thus, at finite voltages the current in the junction can be less than the critical value, i.e., hysteresis takes place.

To conclude this section, we calculate the change of the critical current induced by a weak alternating voltage. We consider the most interesting case of sufficiently high frequencies $\omega \gg \nu_\varepsilon$. The phase φ consists of a constant part φ_0 and a small alternating one $\tilde{\varphi}_1$:

$$\varphi=\varphi_0+\tilde{\varphi}_1, \quad \tilde{\varphi}_1=\varphi_1\sin\omega t.$$

The current is obtained from (42) and (47)

$$I=\frac{1}{2R}\frac{\partial\tilde{\varphi}_1}{\partial t}+I_c\sin\varphi\int_0^1 dy\frac{\langle q(\varphi,y)\rangle-q(\varphi,y)}{q(\varphi,y)}, \tag{53}$$

$$q(\varphi,y)=[(1+y^2)+(1-y^2)\cos\varphi]^{1/2},$$

and the averaging is over the period $2\pi/\omega$.

We expand (53) in powers of $\tilde{\varphi}_1$ accurate to second order inclusive:

$$I=\frac{1}{2R}\frac{\partial\tilde{\varphi}_1}{\partial t}+I_c\sin\varphi_0+I_c\tilde{\varphi}_1\left(\cos\varphi_0-\sin\varphi_0\int_0^1 dy\frac{\partial}{\partial\varphi_0}\ln q(\varphi_0,y)\right)$$
$$+\frac{1}{2}I_c\Big[\tilde{\varphi}_1^2\sin\varphi_0+\sin\varphi_0\int_0^1 dy\frac{\partial^2 q(\varphi_0,y)}{\partial\varphi_0^2}(\langle\tilde{\varphi}_1^2\rangle-\tilde{\varphi}_1^2)$$
$$+2\tilde{\varphi}_1^2\int_0^1 dy\frac{\partial}{\partial\varphi_0}\ln q(\varphi_0,y)\left(\cos\varphi_0-\sin\varphi_0\frac{\partial}{\partial\varphi_0}\ln q(\varphi_0,y)\right)\Big]. \tag{54}$$

We note that the terms linear in φ_1 in (54) coincide with (31) in the limit $\omega\tau_\varepsilon \gg 1$. Averaging over the time, we obtain from (54) the dc current as a function of φ_0. To determine the maximum current we must substitute $\varphi_0=\pi/2$. As a result we get

$$I_{max}=I_c(1-{}^1/_{16}\pi\varphi_1^2).$$

Thus, in the presence of a weak alternating signal the critical current decreases.

4. REGIONS OF HIGH VOLTAGES $V \gg \Delta$

The condition $V \gg \Delta$ means that the voltage V on the bridge exceeds the characteristic frequency $V_c = I_c R$ of the Josephson oscillations. Therefore the phase shift φ is given in the principal approximation by $\varphi = 2Vt$. We calculate the current in this case on the basis of Eqs. (7), (20), and (21). Using the form $\hat{g}^R$ and formulas (11) and (17), we determine $\hat{G}_\pm^{R(A)}$:

$$\hat{G}_+^{R(A)}={}^1/_2[g(\varepsilon+v)+g(\varepsilon-v)]\delta(\varepsilon-\varepsilon')\hat{\sigma}_z$$
$$+{}^1/_2[f(\varepsilon+v)\delta(\varepsilon-\varepsilon'+V)+f(\varepsilon-v)\delta(\varepsilon-\varepsilon'-V)]i\hat{\sigma}_y,$$
$$\hat{G}_-^{R(A)}={}^1/_2[g(\varepsilon+v)-g(\varepsilon-v)]\delta(\varepsilon-\varepsilon')\hat{1}$$
$$+{}^1/_2[f(\varepsilon+v)\delta(\varepsilon-\varepsilon'+V)-f(\varepsilon-v)\delta(\varepsilon-\varepsilon'-V)]\hat{\sigma}_x.$$

For brevity we have left out here the indices $R(A)$, and $v = V/2$.

On the basis of these expressions we have

$$\hat{\rho}=\hat{G}_+^{R(A)}\hat{G}_-^{R(A)}=\rho_2 i\hat{\sigma}_y+\rho_3\hat{\sigma}_z, \tag{55}$$

where

$$2\rho_2=[g(\varepsilon-v)+g(\varepsilon+3v)]f(\varepsilon+v)\delta(\varepsilon-\varepsilon'+V)-f(\varepsilon-v)$$
$$\times[g(\varepsilon+v)+g(\varepsilon-3v)]\delta(\varepsilon-\varepsilon'-V),$$
$$2\rho_3=f(\varepsilon+v)f(\varepsilon+3v)\delta(\varepsilon-\varepsilon'+2V)-f(\varepsilon-v)f(\varepsilon-3v)\delta(\varepsilon-\varepsilon'-2V).$$

In the calculation of the sums of the series (20) and (21) we separate the sets of terms in which the singularities accumulate near energies ε that satisfy the relation $|\varepsilon \pm v| = \Delta$. We calculate first the direct current. To this end we determine that part of the sum of the series (20) and (21) which is proportional to $\delta(\varepsilon-\varepsilon')$. Separating the principal part, we get

$$\hat{\rho}^{2k}=[f^{2k}(\varepsilon+v)+f^{2k}(\varepsilon-v)]\delta(\varepsilon-\varepsilon')\hat{1},$$
$$\hat{A}=\hat{\sigma}_z A(\varepsilon)\delta(\varepsilon-\varepsilon'), \quad A(\varepsilon)=2\left[\operatorname{th}\frac{\varepsilon+v}{2T}-\operatorname{th}\frac{\varepsilon-v}{2T}\right]N_s(\varepsilon+v)N_s(\varepsilon-v), \tag{56}$$

where $N_s = (1/2[g^R(\varepsilon) - g^A(\varepsilon)]$ is the reduced state density of the homogeneous superconductor.

We represent each term of the sum in (21) in the form

$$\hat{A}_{2k+1}=\sum_{m=0}^{2k}(\hat{\rho}^R)^{2k-m}\hat{A}(\hat{\rho}^A)^m=\sum_{n=0}^{k}(\hat{\rho}^R)^{2(k-n)}\hat{A}(\tilde{\rho}^A)^{2n}$$
$$+\sum_{n=0}^{k-1}(\hat{\rho}^R)^{2(k-1-n)}(\rho^R\hat{A}\hat{\rho}^A)(\hat{\rho}^A)^{2n}, \qquad (57)$$

where

$$\hat{\rho}^R\hat{A}\hat{\rho}^A=[f^R(\varepsilon+v)f^A(\varepsilon+v)A(\varepsilon+v)+f^R(\varepsilon-v)f^A(\varepsilon-v)A(\varepsilon-v)]\delta(\varepsilon-\varepsilon').$$

Taking (56) into account, we can reduce the sums in (57) to a sum of the terms of a geometric progression, which adds up to

$$\hat{A}_{2k+1}=[D_{2k+1}(\varepsilon-v)A(\varepsilon)+\tilde{D}_{2k+1}(\varepsilon-v)A(\varepsilon-v)+D_{2k+1}(\varepsilon+v)A(\varepsilon)$$
$$+\tilde{D}_{2k+1}(\varepsilon+v)A(\varepsilon+v)]\hat{\sigma}_z\delta(\varepsilon-\varepsilon');$$
$$D_{2k+1}(\varepsilon)=\frac{(f^R)^{2(k+1)}-(f^A)^{2(k+1)}}{(f^R)^2-(f^A)^2},$$
$$\tilde{D}_{2k+1}(\varepsilon)=\frac{f^A(f^R)^{2k+1}-f^R(f^A)^{2k+1}}{(f^R)^2-(f^A)^2}.$$

Substituting the expression obtained for $\hat{A}_{2k+1}$ in (21) we obtain for the dc part of the current

$$I_{dc}=\frac{1}{2R}\int d\varepsilon\, N_s(\varepsilon+v)N_s(\varepsilon-v)[D(\varepsilon+v)+D(\varepsilon-v)-1]$$
$$\times\left[\operatorname{th}\frac{\varepsilon+v}{2T}-\operatorname{th}\frac{\varepsilon-v}{2T}\right], \qquad (58)$$
$$D(\varepsilon)=\frac{\operatorname{Arsh} f^R(\varepsilon)-\operatorname{Arsh} f^A(\varepsilon)}{f^R(\varepsilon)-f^A(\varepsilon)}.$$

When integrating in (58) we recognize that

$$\operatorname{Arsh} f^R-\operatorname{Arsh} f^A=2\operatorname{Arth}\frac{\varepsilon}{\Delta}\theta(\Delta-|\varepsilon|)+2\operatorname{Arcth}\frac{\varepsilon}{\Delta}\theta(|\varepsilon|-\Delta),$$
$$\frac{g^R-g^A}{f^R-f^A}=\frac{\varepsilon}{\Delta}\theta(|\varepsilon|-\Delta)+\frac{\Delta}{\varepsilon}\theta(\Delta-|\varepsilon|).$$

Then

$$I_{dc}=\frac{2}{R}\operatorname{th}\frac{V}{2T}\left[\int_0^\Delta d\varepsilon\left(\frac{\Delta}{\varepsilon}\operatorname{Arth}\frac{\varepsilon}{\Delta}-1\right)\right.$$
$$\left.+\int_\Delta^\infty d\varepsilon\left(\frac{\varepsilon}{\Delta}\operatorname{Arcth}\frac{\varepsilon}{\Delta}-1\right)\right]+\frac{1}{R}\int_0^\infty d\varepsilon\,[\operatorname{th}\beta(\varepsilon+v)-\operatorname{th}\beta(\varepsilon-v)]$$
$$=\frac{V}{R}+\frac{\Delta}{R}\left(\frac{\pi^2}{4}-1\right)\operatorname{th}\frac{V}{2T}. \qquad (59)$$

In the expression for I^r, the terms that are constant in time are of the order of Δ/V and can be neglected. The second term of (59) yields at $V\gg 2T$ the excess current I_{exc}.

We note that the solution (19) of Eq. (4) was obtained neglecting terms that contain time derivatives. Expression (59), however, which determines the for- of the CVC at $V\gg\Delta$, remains valid also at $V\gg D/d^2$. This can be verified by substituting in (4) the solution (19) in which the principal terms in Δ/V are retained.

To calculate the oscillating terms of the current we must separate in expressions (20) and (21) the terms with the δ functions $\delta(\varepsilon-\varepsilon'\pm 2V)$. The terms containing $\delta(\varepsilon-\varepsilon'\pm 2nV)$, where $n\geq 2$, are of next order of smallness in the parameter Δ/V. As a result we obtain for the oscillating part of the current

$$I_{osc}=I_1(V)\sin(2Vt)+I_2(V)\cos(2Vt),$$

where in the case $\Delta\sim T$

$$I_1(V)\sim\frac{\Delta}{V},\quad I_2(V)\sim\frac{\Delta}{V}\ln\frac{V}{\Delta},$$

i.e., the amplitude of the "Josephson current" decreases with increasing V, as is the case also in tunnel junctions.

CONCLUSION

The theory developed enables us to describe the behavior of superconducting point contacts or sufficiently short bridges of variable thickness. In the latter case the condition (3) on the length of the bridge is easiest to satisfy by using in the experiment materials having a low energy relaxation frequency ν_ε (e.g., aluminum[11]). At low voltages $V\ll\Delta$ the dependence of the current I on the phase difference φ is essentially determined by the relation between ν_ε and $V_c=I_cR$. In particular, at $\nu_\varepsilon\ll V_c$ the dependence of the current

$$I-\frac{\partial\varphi}{\partial t}\frac{1}{2R}$$

on φ near T_c deviates substantially from sinusoidal. At the same time, the CVC of the bridge differs little from the relation[1] $I=R(V^2+V_c^2)^{1/2}$ (see Figs. 3 and 4). The most distinguishing features of the considered model at low V are a certain increase of the bridge conductivity at $\tau_\varepsilon V_c\ll 1$ [see Eq. (46)] and the presence of hysteresis at $\tau_\varepsilon V_c\gg 1$. The general form of the CVC at a temperature close to T_c is shown in Fig. 3. Figure 4 shows the initial sections of the CVC in enlarged scale at different values of λ. These curves were obtained by numerically integrating Eq. (42), in which the functional $P\{\varphi\}$ was replaced by the model functional

$$P_M\{\varphi\}=\sin\varphi\int_{-\infty}^{t}dt_1\sin\varphi(t_1)\frac{\partial\varphi}{\partial t_1}\exp[\nu_\varepsilon(t_1-t)],$$

which reflects the main properties of $P\{\varphi\}$. It is seen that at low V the CVC differs little from the relation $I=R(V^2+V_c^2)^{1/2}$. At $V>\Delta$ the CVC is determined by formula (59) (Fig. 3) and the $I(V)$ curves becomes asymptotically a straight line shifted relative to the ohmic straight line by an amount equal to the excess current $I_{exc}=(\pi^2/4-1)\Delta/eR$. At low temperatures $I_{exc}\sim I_c\sim\Delta/eR$, while near T_c we have $I_{exc}\sim\Delta/eR\gg I_c\sim(\Delta/eR)(\Delta/T)$. The quantity I_{exc}, its temperature dependence, and the law that governs the change of the current to the asymptotic form (59) agrees with the experimental data obtained for point contacts.[7]

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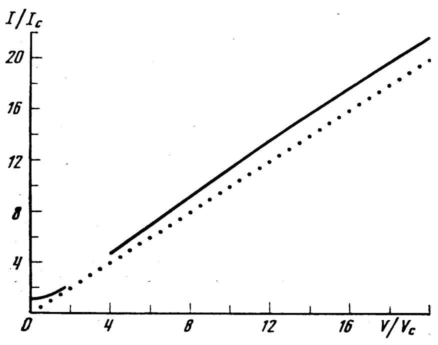

FIG. 3. Form oc the CVC at temperatures close to critical ($\Delta/T=1/2$). The points show the plot of $I=V/R$.

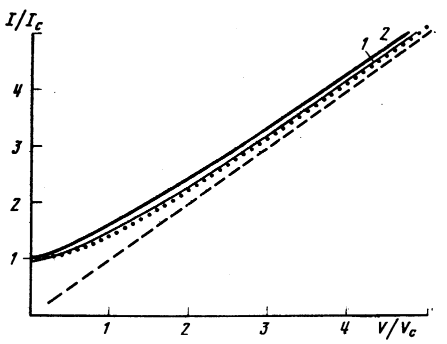

FIG. 4. Form of the CVC at $V \ll \Delta$ and different values of $\lambda = V_c\tau_\varepsilon$: 1) $\lambda = 20$; 2) $\lambda = 1$. The point show the plot of $I = (V^2 + V_c^2)^{1/2}/R$, and the dashed line the plot of $I = V/R$.

APPENDIX

The derived expressions for the current in the bridge contains no characteristics whatever connected with the bridge geometry, other than its resistance R. This suggests that the results do not depend on the shape of the bridge (or of the point contact). We examine below a contact in the shape of a single-cavity hyperboloid of revolution, and show that the total current in it is described by the same expressions as in the case of a filament that joins bassive superconducting shores.

The general expression from which the total current in the contact is determined is obtained by multiplying (19) by the cross section area of the filament:

$$\check{G}_1 = S\check{G}_z = -\frac{l\rho}{R}\,\mathrm{Arsh}(2\check{G}_+\check{G}_-). \qquad (A.1)$$

We obtain the equivalent expression for a contact in the form of a hyperboloid. To do so we change to the coordinates σ, τ, φ of an oblate ellipsoid of revolution

$$x^2 = a^2(1+\sigma^2)(1-\tau^2)\cos^2\varphi, \quad y^2 = a^2(1+\sigma^2)(1-\tau^2)\sin^2\varphi,$$
$$z = a\sigma\tau, \quad -\infty < \sigma < \infty, \quad 0 < \tau < 1, \quad 0 < \varphi < 2\pi.$$

Let the surface of the contact be determined by the apex angle θ of the cones and by the smallest radius r_0 of the neck. Then $r_0 = a\sin(\theta/2)$, and the contact surface is given by the condition $\tau = \cos(\theta/2)$. Under condition (3) (which now takes the form $r_0/\sin\theta \ll \eta$) the Green's functions that determine the current are obtained from Eq. (9) and take the form

$$\check{G} = (\check{G}_\sigma[(\sigma^2+\tau^2)(1+\sigma^2)]^{-1/2}, 0, 0), \qquad (A.2)$$

where the matrix $\hat{G}_\sigma$ does not depend on the coordinates.

From (16) we get for $\check{G}_0$

$$\check{G}_0 = \check{G}(-\infty)\exp[-\check{G}_\sigma a(\pi/2 + \mathrm{arctg}\,\sigma)/l]. \qquad (A.3)$$

From (A.3), since $\check{G}(-\infty)$ and $\check{G}_\sigma$ anticommute, we obtain in analogy with (19)

$$\check{G}_\sigma = -\frac{l}{\pi a}\,\mathrm{Arsh}(2\check{G}_+\check{G}_-), \qquad (A.4)$$

where $\check{G}_\pm = (1/2[\check{G}_0(\sigma) \pm \check{G}_0(-\infty)]$ are equilibrium functions. The total current is expressed in terms of an integral of (A.2) over the cross section of the contact:

$$\check{G}_1 = \int_0^{\cos(\theta/2)} d\tau \int_0^{2\pi} d\varphi\, \check{G}_\sigma a^2 = 2\pi a^2 \cos\frac{\theta}{2}\,\check{G}_\sigma. \qquad (A.5)$$

Recognizing that the resistance of the hyperboloid is $R = \rho/2a\cos(\theta/2)$, we find from (A.4) and (A.5) that the total current through the contact is determined by the same expression (A.1) as in the case of a filament.

1) We note that the nonstationary Josephson effect in short bridges ($d < \xi(T)$) was analyzed by Golub[6] in the case of low voltages ($V < \tau_\varepsilon^{-1}$). However, his solution of the equation for the anomalous Green's function that determines the current is incorrect in the considered case. His results therefore differs substantially from ours.

2) A factor 2 is placed erroneously in Ref. 2 in front of $\sin\varphi_0$ in the last term in the curly brackets.

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