oscillator in which $\psi$ corresponds to the coordinate and $\xi$ corresponds to time.


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# Collective oscillations and instability of a single-frequency state of parametrically excited waves


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Collective oscillations of parametrically excited waves are investigated with allowance for their scattering on one another and on random inhomogeneities of the medium. It is shown that in addition to oscillations investigated earlier in the $S$ theory (self-consistent field) approximation, there are also relatively low-frequency oscillations (according to the $S$ theory their frequency is zero). It is shown that distributions of parametrically excited waves with singular frequencies are unstable when their spectrum is broadened.



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Considerable attention is currently given to the phenomena which appear on parametric excitation of waves in ferromagnets,[1-3] antiferromagnets,[4,5] plasma,[6,7] ferroelectrics,[8] and other nonlinear media. In some important cases the wave dispersion law $\omega_{\mathbf{k}}$ is of the nondecaying type and an external field (pump wave) can be regarded as spatially homogeneous and monochromatic:

$$h(\mathbf{r},t)=h(t)=h\exp(-i\omega_p t). \quad (1)$$

A relatively simple theory, based on the self-consistent field approximation ($S$ theory), is developed for this case in Refs. 1 and 9. This theory is in good qualitative and quantitative agreement with many experimental observations on ferromagnets and antiferromagnets (for details see the review in Ref. 1 and also Refs. 4 and 5). However, recent experiments require interpretation which goes beyond this theory. For example, measurements have been made of the spectral density $N_\omega$ of parametrically excited waves,[10]

$$N_\omega=\int n_{\mathbf{k}\omega}\,d\mathbf{k}, \quad (2)$$

where $n_{\mathbf{k}\omega}$ is the Fourier component of the correlation function $n_{\mathbf{k}}(\tau)$ of the complex amplitudes $a_{\mathbf{k}}(t)$ of traveling waves:

$$n_{\mathbf{k}}(\tau)\delta(\mathbf{k}-\mathbf{k}')=\langle a_{\mathbf{k}}(t)a_{\mathbf{k}'}^*(t+\tau)\rangle. \quad (3)$$

The experimental results show that the $N_\omega$ line has a finite width $\Delta\omega$ which depends in a certain way on the pump amplitude and other experimental conditions. However, the $S$ theory, which describes correctly the integral characteristics of parametrically excited waves (PW's), predicts a singular distribution of PW's in the $\mathbf{k}$-$\omega$ space:

$$n_{\mathbf{k}\omega}\sim\delta(\tilde{\omega}_{\mathbf{k}}-\omega_p/2)\delta(\omega-\omega_p/2), \quad (4)$$

where $\tilde{\omega}_{\mathbf{k}}$ is the frequency $\omega_{\mathbf{k}}$ renormalized to the interaction [see Eq. (1.13) below].

L'vov[11] used the diagram technique to formulate integral equations for $n_{\mathbf{k}\omega}$ generalizing the $S$-theory equations by a systematic allowance for the Hamiltonian of the interaction of PW's given by Eq. (1.5). These equations give rise to a finite width of the distribution $n_{\mathbf{k}\omega}$ in respect of the modulus $k$:

$$\Delta\omega_{\mathbf{k}}\propto\nu,\qquad \nu^3\propto\gamma^2(TN)^2/kv. \quad (5)$$

Here, $\gamma$ is the logarithmic decrement of PW's; $N=\int n_{\mathbf{k}\omega}d\mathbf{k}\,d\omega$ is the total number of PW's; $k$ is the characteristic wave vector of PW's ($2\tilde{\omega}_{\mathbf{k}}=\omega_p$); $v$ is the group velocity.

The distribution of $N_\omega$ in respect of $\omega$ is more complex. The generalized equations have, like the $S$-theory equations, a "single-frequency" solution singular in $\omega$:

$$N_\omega=N\delta(\omega-\omega_p/2), \quad (6)$$

which is investigated in Ref. 11. However, as shown earlier,[10] this solution is not the only one. In addition to the central line of Eq. (6), the solution of $N_\omega$ may yield satellites at frequencies $\omega_p/2 \pm m\delta\omega$ ($m$ is an integer and $\delta\omega \propto \nu^2/\gamma$). The greatest interest lies in a single solution regular in $\omega$ whose line profile is

$$N_\omega = N\,\mathrm{ch}^{-1}\frac{2\omega-\omega_p}{2\eta}, \quad \eta \propto \frac{\nu^2}{\gamma}. \tag{7}$$

Experimental studies of the parametric excitation of spin waves in YIG single crystals[10] have established that $N_\omega$ is close to the theoretical profile (7). This raises the question of why the regular distribution (7) is observed in this case and whether it is possible to alter the external conditions (magnetic field, temperature, etc.) in such a way as to achieve the single-frequency solution (6). In addition to the purely theoretical interest, this question is also of practical importance. The point is this: the finite width of the $N_\omega$ line results, for example, in an increase in the noise temperature of a nondegenerate magnetostatic ferrite amplifier, in stray modulation of the amplitude at the output of a ferrite limiter, etc.

In § 4 we shall consider, by way of example, the problem of parametric excitation of waves in the case of axial or spherical symmetry (typical of experiments on ferromagnets and antiferromagnets) and we shall calculate the instability growth rate of a single-frequency state (6) in the case of broadening of the $N_\omega$ line in the frequency space [Eqs. (4.6) and (4.12)]. The instability range [Eqs. (4.7) and (4.13)] is of the order of line width in the regular solution (7) and is independent of the nature of the coefficients in the Hamiltonian. Moreover, we can say that the origin of the instability under discussion is not affected by our assumption of the symmetry of the problem and is of general (model-independent) nature. Therefore, the single-frequency state cannot be realized experimentally.

Another series of experiments whose interpretation requires going beyond the $S$-theory approximation is concerned with the parametric excitation of spin waves in ferromagnets and antiferromagnets that have random inhomogeneities.[12,2] These inhomogeneities—crystal defects, impurities, pores, etc.—result in additional scattering of PW's (known as the two-magnon scattering in magnetically ordered materials), characterized by the frequency $\gamma_{imp}$. The scattering does not alter the PW frequency and makes no contribution to the dissipation of the PW energy. Therefore, the expressions for the parametric excitation threshold, damping of collective oscillations, relaxation times of various types of perturbations in a PW system, etc., do not necessarily have to include $\gamma_{imp}$ as an additive term together with the contributions of the remaining damping mechanisms.

We are thus faced with the problem of the influence of inhomogeneities, present in all real crystals, on the behavior of a system of PW's. Their influence on the stationary state of PW's has been investigated[13,5] and, in particular, it has been shown that the scattering by inhomogeneities results in smearing of the angular distribution function of PW's:

$$\Delta\theta \propto [\gamma_{imp}/(\gamma+\gamma_{imp})]^{1/2}, \tag{8}$$

in violation of the phase correlation in pairs:

$$|\sigma_k|/n_k \propto [\gamma/(\gamma+\gamma_{imp})]^{1/2}, \tag{9}$$

and in an increase of the threshold of parametric excitation of waves:

$$h_{cr}V \propto [\gamma(\gamma+\gamma_{imp})]^{1/2}. \tag{10}$$

In § 3 we shall investigate the influence of inhomogeneities on collective excitations of a PW system. The spectrum of collective excitations $\Omega(\chi)$ of the single-frequency state has been investigated earlier in the $S$-theory approximation[1]: see Eq. (2.1) in § 2 for cubic ferromagnets. This spectrum has two notable features. Firstly, for a certain relationship between the interaction Hamiltonian coefficients $S_p$ and $T_p$ [namely for $S_p(2T_p+S_p)>0$] the collective oscillations of a mode of number $p$ ($\delta n_k \propto e^{-ip\varphi}$, where $\varphi$ is the azimuthal angle) are stable and their logarithmic decrement is equal to the corresponding decrement $\gamma$ of PW's deduced from the parametric excitation threshold: $h_{cr}V=\gamma$. Secondly, if $S_p(2T_p+S_p)<0$, the collective oscillations (spontaneous oscillations of PW's) become unstable for $h \geq h_{cr}$, i.e., the instability sets in immediately at the parametric excitation threshold. Our calculation of the spectrum of $\Omega_p$ for an inhomogeneous medium [see Eqs. (3.9)–(3.11) below] shows that the PW scattering by inhomogeneities alters considerably the behavior of the collective oscillations and particularly increases their stability. Thus, even weak scattering by inhomogeneities (i.e., the scattering corresponding to $\gamma_{imp} \ll \gamma$) suppresses spontaneous oscillations of an isotropic mode ($p=0$) in the range of supercriticalities $h<h_1$, where

$$\frac{h_1}{h_{cr}}-1=\frac{1}{4}\left(\frac{\gamma_{imp}}{2\gamma}\right)^{1/2}\left|\frac{S_0}{2T_0+S_0}\right|, \tag{11}$$

and spontaneous oscillations of anisotropic modes ($p \neq 0$) in the range

$$h/h_{cr}-1 \lesssim \gamma_{imp}/\gamma. \tag{12}$$

In the strong scattering case ($\gamma_{imp} \gg \gamma$), it follows from Eqs. (3.9)–(3.11) that spontaneous oscillations may not appear at all and the damping of the collective oscillations is of the order of $\gamma_{imp}$, which exceeds the value $(\gamma\gamma_{imp})^{1/2}$ that determines, in accordance with Eq. (10), the threshold of parametric excitation of waves.

In § 2 we shall consider the stability of a multifrequency stationary state in a spatially homogeneous situation and we shall use the complete Hamiltonian of the problem. The point is that in the $S$-theory approximation some of the collective motion modes are neutrally stable and inclusion of small frequency corrections associated with the scattering of PW's on one another may be important. We shall show that in the case of cubic ferromagnets

$$\Omega_p=-i\frac{\nu^2}{\gamma}\frac{S_p}{2T_p+S_p}, \tag{13}$$

so that for $S_p(2T_p+S_p)>0$, when the collective oscillations are stable in the $S$-theory [Eq. (2.1)], the mode described by Eq. (13) is also stable.

## § 1. PRINCIPAL EQUATIONS

In developing a new theory we can begin from the classical Hamiltonian equations of motion for complex amplitudes of traveling waves $a_k$:

$$i\,\partial a_k/\partial t=\delta H/\delta a_k^*. \tag{1.1}$$

The Hamiltonian of the problem[1]

$$H=\int \omega_k a_k a_k^*\,dk+H_p+H_{imp}+H_{int} \tag{1.2}$$

includes the interaction of waves with the pump field (1):

$$H_p=\frac{1}{2}\int [h(t)V_k a_k^* a_{-k}^*+\text{c.c.}]dk, \tag{1.3}$$

their interaction with static inhomogeneities (impurities, defects)[13]:

$$H_{imp}=\int g_{kk'}a_k a_{k'}^* b_{k''}\delta(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')\,d\mathbf{k}\,d\mathbf{k}'\,d\mathbf{k}'', \tag{1.4}$$

and their interaction with one another:

$$H_{int}=\frac{1}{2}\int T_{k_1k_2k_3k_4}a_{k_1}^* a_{k_2}^* a_{k_3}a_{k_4}\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\,d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3\,d\mathbf{k}_4. \tag{1.5}$$

The spectrum of the collective oscillations of a system of PW's can be found in the diagram technique framework[11] by summing the rows for the renormalization of the vertices describing the scattering accompanied by small momentum transfer. Another way, which makes it possible to restrict the treatment to the Dyson equations with unrenormalized vertices, involves allowance for spatially inhomogeneous and nonstationary states right from the beginning. This is done by dropping the assumption that in the Wyld diagramatic technique[14] the correlation function of a random force $\langle f_q f_{q'}^*\rangle$ is proportional to $\delta^4(q-q')$, and by introducing the green functions $G_{qq'}$ and $L_{qq'}$ as well as the pair averages $n_{qq'}$ and $\delta_{qq'}$ by means of the relationships

$$G_{qq'}=\left\langle\frac{\delta a_q}{\delta f_{q'}}\right\rangle,\quad L_{qq'}=\left\langle\frac{\delta a_q}{\delta f_{q'}^*}\right\rangle,\quad G^*_{qq'},\quad L^*_{qq'}; \tag{1.6}$$

$$n_{qq'}=\langle a_q a_{q'}^*\rangle=n^*_{q'q},\quad n^*_{qq'},\quad \sigma_{qq'}=\langle a_q a_{q'}\rangle,\quad \sigma^*_{qq'}. \tag{1.7}$$

Here, $q=(\mathbf{k},\omega)$. Summing, as usual, the reducible graphs, we obtain the Dyson system of equations for the quantities defined in Eqs. (1.6) and (1.7):

$$G_{qq'}=G_q^0[\delta^4(q-q')+(\Sigma G+\Pi L^*)_{qq'}],\qquad L_{qq'}^*=G_q^{0*}(\Pi^* G+\Sigma^* L^*)_{qq'}; \tag{1.8}$$

$$n_{qq'}=(G\Phi G^*+L\Psi^* G^*+L\Phi^* L^*+G\Psi L^*)_{qq'},\qquad \sigma_{qq'}=(G\Phi L+L\Psi^* L+L\Phi^* G+G\Psi G)_{qq'}. \tag{1.9}$$

Here, $\Sigma_{qq'}$, $\Pi_{qq'}$, $\Phi_{qq'}$, $\Psi_{qq'}$ are the sums of the irreducible graphs: $\Sigma$ and $\Phi$ are the normal graphs (i.e., those which conserve the arrow directions); $\Pi$ and $\Psi$ are the anomalous irreducible graphs. The $\Phi$ and $\Psi$ graphs can, in contrast to the graphs for $\Sigma$ and $\Pi$, be cut into two parts only along the wavy lines. Integration has to be carried out over the internal indices in Eqs. (1.8) and (1.9). For example,

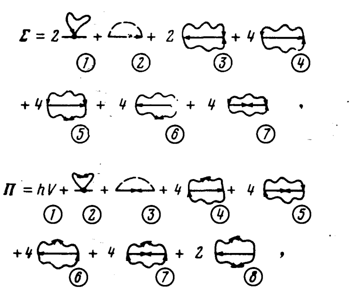
$$(\Sigma G)_{qq'}=\int \Sigma_{qq_1}G_{q_1q'}\,dq_1,$$

$$(G\Phi L)_{qq'}=\int G_{qq_1}\Phi_{q_1q_2}L_{q'q_2}\,dq_1\,dq_2.$$

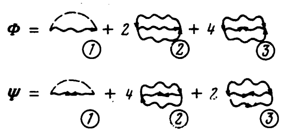
As shown earlier,[11] for moderately large supercriticalities

$$h/h_{cr}<(kv/\gamma)^{1/2},$$

the graphs for $\Sigma,\Pi,\Phi$, and $\Psi$ can be simplified by ignoring the vertex renormalization. Then,

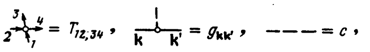


(1.10)



(1.11)

Here,




where $c$ is the impurity concentration. The expressions (1.10) and (1.11) represent a closed system of integral equations for the quantities $G_{qq'}$, $L_{qq'}$, $n_{qq'}$, $\sigma_{qq'}$. This system has a partial solution of the type

$$\begin{aligned}&G_{qq'}=G_q\delta^4(q-q'),\quad L_{qq'}=L_q\delta^4(q-\bar q'),\\&n_{qq'}=n_q\delta^4(q-q'),\quad \sigma_{qq'}=\sigma_q\delta^4(q-\bar q'),\\&\Sigma_{qq'}=\Sigma_q\delta^4(q-q'),\quad \Pi_{qq'}=\Pi_q\delta^4(q-\bar q'),\\&\Phi_{qq'}=\Phi_q\delta^4(q-q'),\quad \Psi_{qq'}=\Psi_q\delta^4(q-\bar q'),\\&\bar q'=(-\mathbf{k}',\ \omega_p-\omega'),\end{aligned} \tag{1.12}$$

corresponding to a stationary and spatially homogeneous state of PW's which, for brevity, we shall call the ground state. In particular, the Green functions in Eq. (1.12) are[11]

$$\begin{aligned}&G_q=(\omega_p-\omega-\tilde\omega_q-i\Gamma_q)\Delta_q^{-1},\quad L_q^*=\Pi_q^*\Delta_q^{-1},\\&\Delta_q=(\omega_p-\omega-\tilde\omega_q-i\Gamma_q)(\omega-\tilde\omega_q+i\Gamma_q)-|\Pi_q|^2,\\&\tilde\omega_q=\omega_k+\operatorname{Re}\Sigma_q,\quad \Gamma_q=-\operatorname{Im}\Sigma_q.\end{aligned} \tag{1.13}$$

Linearizing the system (1.8)–(1.9), we obtain the following system of linear integral equations for the quantities $\tilde G$, $\tilde L$, $\tilde n$, and $\tilde\sigma$, representing the deviations of $G$, $L$, $n$, and $\sigma$ from their values (1.12) in the ground state:

$$\begin{aligned}&\tilde G_{qq'}=\tilde A_{qq'}G_{q'}+\tilde B_{qq'}L_{q'},\quad \tilde L_{qq'}=\tilde A_{qq'}L_{q'}+\tilde B_{qq'}G_{q'}^*,\\&\tilde n_{qq'}=(\tilde A_{qq'}n_{q'}+\tilde B_{qq'}\sigma_{q'}^*)+(\tilde A^*_{q'q}n_q+\tilde B^*_{q'q}\sigma_q)+(\tilde C_{qq'}G_{q'}^*+\tilde D_{qq'}L_{q'}^*),\\&\tilde\sigma_{qq'}=(\tilde A_{qq'}\sigma_{q'}+\tilde B_{qq'}n_{q'})+(\tilde A_{q'q}\sigma_q+\tilde B_{q'q}n_q)\\&\qquad+(\tilde C_{qq'}L_{q'}+\tilde D_{qq'}G_{q'});\end{aligned} \tag{1.14}$$

$$\begin{aligned}&\tilde A_{qq'}=G_q\tilde\Sigma_{qq'}+L_q\tilde\Pi^*_{\bar q q'},\quad \tilde B_{qq'}=G_q\tilde\Pi_{qq'}+L_q\tilde\Sigma^*_{\bar q\bar q'},\\&\tilde C_{qq'}=G_q\tilde\Phi_{qq'}+L_q\tilde\Psi^*_{\bar q q'},\quad \tilde D_{qq'}=G_q\tilde\Psi_{qq'}+L_q\tilde\Phi^*_{\bar q\bar q'}.\end{aligned} \tag{1.15}$$

The quantities $\tilde A$, $\tilde B$, $\tilde C$, and $\tilde D$ can be expressed in terms of $\tilde\Sigma$, $\tilde\Pi$, $\tilde\Phi$, and $\Psi$, which are deviations of the mass operators $\Sigma,\Pi,\Phi$, and $\Psi$ from their values in the ground state (1.12). The expressions for them are obtained by linearization of the graphs (1.10) and (1.11):

$$\Sigma_{qq'}=2\int T_{\mathbf{k}\mathbf{k}_1\mathbf{k}'\mathbf{k}_2}\dot{\tilde{n}}_{q_1q_2}\delta(q+q_1-q'-q_2)\,dq_1\,dq_2$$
$$+c\int g_{\mathbf{k}'\mathbf{k}_1}g_{\mathbf{k}_2\mathbf{k}}\tilde{G}_{\mathbf{k}_1\omega,\mathbf{k}_2\omega'}\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}'-\mathbf{k}_2)\,d\mathbf{k}_1\,d\mathbf{k}_2+\text{graphs 3–7},$$
$$\Pi_{qq'}=2\int T_{\mathbf{k}\mathbf{k}_1,\mathbf{k}'\mathbf{k}_2}\tilde{\sigma}_{q_1q_2}\delta(q+q_1-q'-q_2)\,dq_1\,dq_2 \qquad (1.16)$$
$$+c\int g_{\mathbf{k}\mathbf{k}_1}g^*_{\mathbf{k}_2\mathbf{k}'}\tilde{L}_{\mathbf{k}_1\omega,\mathbf{k}_2\omega'}\delta(\mathbf{k}+\mathbf{k}'-\mathbf{k}_1-\mathbf{k}_2)\,d\mathbf{k}_1\,d\mathbf{k}_2+\text{graphs 4–8},$$
$$\tilde{\Phi}_{qq'}=c\int g_{\mathbf{k}\mathbf{k}_1}g^*_{\mathbf{k}'\mathbf{k}_2}\tilde{n}_{\mathbf{k}_1\omega,\mathbf{k}_2\omega'}\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}'+\mathbf{k}_2)\,d\mathbf{k}_1\,d\mathbf{k}_2+\text{graphs, 2, 3,}$$
$$\Psi_{qq'}=c\int g_{\mathbf{k}\mathbf{k}_1}g^*_{\mathbf{k}_2\mathbf{k}'}\tilde{\sigma}_{\mathbf{k}_1\omega,\mathbf{k}_2\omega'}\delta(\mathbf{k}+\mathbf{k}'-\mathbf{k}_1'-\mathbf{k}_2)\,d\mathbf{k}_1\,d\mathbf{k}_2+\text{graphs 2, 3}. \qquad (1.17)$$

Assuming that $\mathbf{k}'-\mathbf{k}=\varkappa$, $\omega'-\omega=\Omega$ in the quantities $\tilde{G}_{qq'}$ and $\tilde{n}_{qq'}$ and that $\mathbf{k}'+\mathbf{k}=\varkappa$, $\omega'+\omega-\omega_p=\Omega$ in $\tilde{L}_{qq'}$ and $\tilde{\sigma}_{qq'}$, we can obtain the condition for solubility of the linear system of equations (1.14)–(1.15) in the form of the dependence of $\Omega$ on $\varkappa$, i.e., the spectrum of the collective oscillations of a system of PW's relative to the ground state. The stability of the ground state is ensured if $\mathrm{IM}\,\Omega(\varkappa)<0$ for all values of $\varkappa$.

## § 2. STABILITY CONDITION FOR A MULTIFREQUENCY STATE OF PARAMETRICALLY EXCITED WAVES

In this section we shall consider the collective excitations of PW's in media without random inhomogeneities and we shall do this by analyzing, as an example, the parametric excitation of spin waves in cubic ferromagnets which has been subjected to quite thorough experimental investigations (for details see Ref. 1). For M|| [100] and M|| [111] the problem is axially symmetric and can be considered in detail. In this case, PW's are localized in the k space near the equator of the resonance surface $\tilde{\omega}_\mathbf{k}=\omega_p/2$. The collective excitations of PW's have been analyzed[1,7] in terms of the $S$ theory (i.e., allowing only for self-consistent graphs in $\Sigma$ and $\Pi$: $|\sigma_\mathbf{k}|=n_\mathbf{k}$, $|\Pi_\mathbf{k}|=\gamma_\mathbf{k}$). The frequencies of spatially homogeneous ($\varkappa=0$) collective excitations $\Omega_p^\pm$ are found to be given by the simple expression

$$\Omega_p^\pm=-i\gamma\pm[-\gamma^2+4S_p(2T_p+S_p)N^2]^{1/2}. \qquad (2.1)$$

Here, $p$ is the number of a collective oscillation mode $\tilde{n}_\mathbf{k}\propto e^{-ip\varphi}$, $\varphi$ is the azimuthal angle,

$$T_p=\int_{-\pi}^{\pi}T(\varphi-\varphi')\exp[-ip(\varphi-\varphi')]\,d(\varphi-\varphi'),$$

$$S_p=\int_{-\pi}^{\pi}S(\varphi-\varphi')\exp[-i(p-2)(\varphi-\varphi')]\,d(\varphi-\varphi'),$$

and $N$ is the integral amplitude of PW's. This result, like the initial equations of the $S$ theory, is insensitive to details of the distribution of PW's in the $\mathbf{k}$-$\omega$ space and it applied to multifrequency and single-frequency states of PW's. The higher graphs in Eqs. (1.10) and (1.11), associated with the scattering of PW's by one another, introduce in Eq. (2.1) small corrections $\Delta\Omega\propto\gamma[(TN)^2\gamma^{-1}(kv)^{-1}]^{2/3}$ and are of no interest here. The oscillations described by Eq. (2.1) are stable ($\mathrm{Im}\,\Omega_p^\pm<0$) if

$$S_p(2T_p+S_p)>0. \qquad (2.2)$$

However, it should be pointed out that the linearized system (1.14) is of high order and, in the $S$-theory approximation, it has not only the eigenvalues $\Omega_p^\pm$ of Eq. (2.1) but also the trivial eigenvalue $\Omega_p=0$. The oscillations corresponding to the latter eigenvalue are the most interesting because a weak interaction of PW's ($\propto T^2$) should destroy their neutral stability.

We shall consider the following integral quantities in the frequency range $\omega,\omega'\ll\gamma$:

$$\tilde{N}_\varphi=\int[\tilde{n}_{\mathbf{k}\omega}(\Omega)+\tilde{n}_{\mathbf{k}\omega}{}^+(\Omega)]k^2\,dk\,d\omega,$$
$$\tilde{K}_\varphi=\frac{i}{\gamma}\int[\Pi_\mathbf{k}{}^*\tilde{\sigma}_{\mathbf{k}\omega}(\Omega)-\Pi_\mathbf{k}\tilde{\sigma}_{\mathbf{k}\omega}{}^+(\Omega)]k^2\,dk\,d\omega, \qquad (2.3)$$
$$\tilde{M}_\varphi=\frac{1}{\gamma}\int[\Pi_\mathbf{k}{}^*\tilde{\sigma}_{\mathbf{k}\omega}(\Omega)+\Pi_\mathbf{k}\tilde{\sigma}_{\mathbf{k}\omega}{}^+(\Omega)]k^2\,dk\,d\omega;$$

where the index "+" denotes the substitution $\mathbf{k}\to-\mathbf{k}$, $\omega\to\omega_p-\omega$ and complex conjugacy. The equations for $\tilde{N}_\varphi$, $\tilde{K}_\varphi$, and $\tilde{M}_\varphi$ are of the form:

$$\tilde{N}_\varphi=\frac{1}{\pi}\int\left[\frac{(\nu_+'+\nu_-'+\nu_-)N_{\omega'}}{\nu_-(\nu_+'+\nu_-)(\nu_-'+\nu_-)}+\frac{(\nu_++\nu_-+\nu_+')N_\omega}{\nu_+'(\nu_++\nu_+')(\nu_-+\nu_+')}\right]$$
$$\times[-i\gamma(\tilde{\Sigma}_\varphi-\tilde{\Sigma}_\varphi{}^+)+\Pi_\varphi{}^*\tilde{\Pi}_\varphi+\Pi_\varphi\tilde{\Pi}_\varphi{}^+]\,d\omega+$$
$$+\frac{\pi k^2}{v}\int\frac{\gamma}{\nu-\nu_+'(\nu_-+\nu_+')}[\gamma(\tilde{\Phi}_\varphi+\tilde{\Phi}_\varphi{}^+)+i(\Pi_\varphi\tilde{\Psi}_\varphi-\Pi_\varphi{}^*\tilde{\Psi}_\varphi)]\,d\omega,$$
$$\tilde{K}_\varphi=\tilde{N}_\varphi, \qquad (2.4)$$
$$\tilde{M}=\frac{1}{2\pi}\int\left[\frac{(\nu_+'+\nu_-'+\nu_-)N_{\omega'}}{\nu_-(\nu_+'+\nu_-)(\nu_-'+\nu_-)}+\frac{(\nu_++\nu_-+\nu_+')N_\omega}{\nu_+'(\nu_++\nu_+')(\nu_-+\nu_+')}\right]$$
$$\times\left[i\Omega(\tilde{\Sigma}_\varphi+\tilde{\Sigma}_\varphi{}^+)-\frac{1}{\gamma}\left(\frac{\nu^2}{\gamma}-i\Omega\right)(\Pi_\varphi{}^*\tilde{\Pi}_\varphi-\Pi_\varphi\tilde{\Pi}_\varphi{}^+)\right]d\omega;$$
$$\nu_\pm=(\nu^2\pm2i\gamma\omega)^{1/2};\quad \nu_\pm'=(\nu^2\pm2i\gamma\omega')^{1/2};$$

where $\nu$ is the width of the distribution of PW's in the $\mathbf{k}$ space given by Eq. (5).

Introducing new variables given by Eq. (2.3), we automatically limit the investigated class of oscillations to the motion that alters the integral amplitude of PW's. In addition to them, a system of PW's includes oscillations of the distribution $n_{\mathbf{k}\omega}$ which do not alter the integral characteristics. Such oscillations will not be considered here.

If $SN\gg\nu$, an analysis of the system (2.4) shows that we can confine ourselves to the self-consistent graphs in $\tilde{\Sigma}$ and $\tilde{\Pi}$ and thus obtain $\Phi=\Psi=0$. In this approximation the system (2.4) is closed and, since the equation for $\tilde{N}_\varphi$ has a small denominator, this system can be reduced to $\tilde{M}_\varphi=0$ or

$$[i\Omega 2T_p-(\nu^2/\gamma-i\Omega)S_p]\tilde{N}_p=0, \qquad (2.5)$$

where

$$\tilde{N}_p=\int_{-\pi}^{\pi}\tilde{N}_\varphi e^{-ip\varphi}\,d\varphi,\quad \tilde{M}_p=\int_{-\pi}^{\pi}\tilde{M}_\varphi e^{-ip\varphi}\,d\varphi.$$

Equation (2.5) gives finally the expression (13) for $\Omega_p$. The amplitudes $n_\mathbf{k}$ and $|\sigma_\mathbf{k}|$ with a fixed phase of $\sigma_\mathbf{k}$ participate in these low-frequency, compared with Eq. (2.1), oscillations. The difference between $\Omega_p$ and zero is due to the difference between $|\Pi_\mathbf{k}|$ and $\gamma_\mathbf{k}$ and between $|\sigma_\mathbf{k}|$ and $n_\mathbf{k}$. It should be stressed that the condition of stability of low-frequency oscillations (2.6) is identical with the stability condition (2.2) for a high-frequency mode.

## §3. COLLECTIVE OSCILLATIONS OF PARAMETRICALLY EXCITED WAVES IN MEDIA WITH RANDOM INHOMOGENEITIES

### 1. Principal equations

In this section we shall study the influence of inhomogeneities on the collective oscillations of a system of PW's. As in Ref. 13, we shall allow only for the self-consistent interaction and the scattering by random inhomogeneities in the lowest order in $H_{\text{imp}}$:

$$\begin{aligned}
\Sigma_k &= 2\int T_{\mathbf{kk}'}n_{\mathbf{k}'}\,d\mathbf{k}' + c\int |g_{\mathbf{kk}'}|^2 G_{\mathbf{k}'}\,d\mathbf{k}',\\
\Pi_k &= hV_{\mathbf{k}} + \int S_{\mathbf{kk}'}\sigma_{\mathbf{k}'}\,d\mathbf{k}' + c\int g_{\mathbf{k}-\mathbf{k}'}g_{-\mathbf{kk}'}L_{\mathbf{k}'}d\mathbf{k}',\\
\Phi_k &= c\int |g_{\mathbf{kk}'}|^2 n_{\mathbf{k}'}d\mathbf{k}',\\
\Psi_k &= c\int g_{\mathbf{k},\mathbf{k}'}g_{-\mathbf{k},-\mathbf{k}'}\sigma_{\mathbf{k}'}d\mathbf{k}'.
\end{aligned} \tag{3.1}$$

Moreover, we shall confine ourselves to oscillations of not too low frequency $\Omega > \eta$, for which the multifrequency nature of the parametric turbulence is unimportant. This restriction allows us to regard the frequency distribution of PW's to be singular and to represent, in accordance with Eqs. (1.14)–(1.17), the quantities $\tilde{n}_{qq'}$ and $\tilde{\sigma}_{qq'}$ as sums of two terms:

$$\begin{aligned}
\tilde{n}_{qq'} &= \tilde{n}_{\mathbf{k}q'}\delta(\omega-\omega_p/2)+\tilde{n}_{q\mathbf{k}'}\delta(\omega'-\omega_p/2),\\
\tilde{\sigma}_{qq'} &= \tilde{\sigma}_{\mathbf{k}q'}\delta(\omega-\omega_p/2)+\tilde{\sigma}_{q\mathbf{k}'}\delta(\omega'+\omega_p/2),
\end{aligned} \tag{3.2}$$

where Eqs. (1.14) and (1.17) representing terms of the first and second kinds can be decoupled.

In qualitative investigations we can consider only the axially symmetric case and made the following model assumptions:

$$\begin{aligned}
&\gamma_{\mathbf{k}}=\gamma,\ g_{\mathbf{kk}'}=g,\ V_{\mathbf{k}}=Vf(\cos\theta)e^{2i\varphi},\\
&S_{\mathbf{kk}'}=S(\varphi-\varphi')f(\cos\theta)f(\cos\theta'),\\
&T_{\mathbf{kk}'}=T(\varphi-\varphi')f(\cos\theta)f(\cos\theta'),\\
&\int_{-1}^{1} f^2(x)\,dx=1.
\end{aligned} \tag{3.3}$$

This makes it possible to reduce the initial integral equations (1.14) to a system of six algebraic equations for integral quantities:

$$\begin{aligned}
X_p &= \int k^2\,dk\,d\cos\theta\,d\varphi\,e^{-ip\varphi}\frac{\tilde{n}_{\mathbf{k}}+\tilde{n}_{\mathbf{k}}^+}{2N} \equiv \left\langle\!\!\left\langle \frac{\tilde{n}_{\mathbf{k}}+\tilde{n}_{\mathbf{k}}^+}{2N}\right\rangle\!\!\right\rangle_p,\\
Y_p &= \left\langle\!\!\left\langle \frac{\tilde{n}_{\mathbf{k}}-\tilde{n}_{\mathbf{k}}^+}{2iN}\right\rangle\!\!\right\rangle_p,\quad U_p = \left\langle\!\!\left\langle \frac{\Pi_{\mathbf{k}}^*\tilde{\sigma}_{\mathbf{k}}-\Pi_{\mathbf{k}}\tilde{\sigma}_{\mathbf{k}}^+}{2i\Gamma N}\right\rangle\!\!\right\rangle_p,\\
Z_p &= \left\langle\!\!\left\langle \frac{\Pi_{\mathbf{k}}^*\sigma_{\mathbf{k}}+\Pi_{\mathbf{k}}\tilde{\sigma}_{\mathbf{k}}^+}{2\Gamma N}\right\rangle\!\!\right\rangle_p,\\
F_p &= \left\langle\!\!\left\langle \frac{\tilde{G}_{\mathbf{k}}-\tilde{G}_{\mathbf{k}}^+}{2i\pi k^2/v}\right\rangle\!\!\right\rangle_p\quad Q_p = \left\langle\!\!\left\langle \frac{\tilde{G}_{\mathbf{k}}+\tilde{G}_{\mathbf{k}}^+}{2\pi k^2/v}\right\rangle\!\!\right\rangle_p
\end{aligned} \tag{3.4}$$

for each axial harmonic $p=0$, $p=\pm 1$, etc. This gives:

$$\begin{aligned}
X_p &= \frac{1}{\mu(\nu+\mu)^2}\left\{\left[(\Gamma-i\Omega)\left(2\nu+\mu-\frac{\nu^3}{\Gamma^2}\right)+\frac{|\Pi|^2}{\Gamma}(2\nu+\mu)\right]\right.\\
&\left.\times\gamma_{imp}\Delta(p)F_p+\left[(2\Gamma-i\Omega)(2\nu+\mu)+\frac{\nu^2}{\Gamma}(\mu-\nu)\right]2S_pNZ_p\right\}\\
&+\frac{\Gamma(\Gamma-i\Omega)+\nu\mu+|\Pi|^2}{\mu(\nu+\mu)\Gamma^2}\nu^2\Delta(p)X_p,\\
Y_p &= \frac{1}{\mu(\nu+\mu)^2}\left\{\left[(\Gamma-i\Omega)\left(2\nu+\mu-\frac{\nu^3}{\Gamma^2}\right)-\frac{|\Pi|^2}{\Gamma}(2\nu+\mu)\right]\right.\\
&\left.\times[\gamma_{imp}\Delta(p)Q_p+4T_pNX_p]-\left[-i\Omega(2\nu+\mu)+\frac{\nu^2}{\Gamma}(\nu+\mu)\right]2S_pNU_p\right\}\\
&+\frac{\Gamma(\Gamma-i\Omega)+\nu\mu-|\Pi|^2}{\mu(\nu+\mu)\Gamma^2}\nu^2\Delta(p)Y_p,\\
U_p &= \frac{1}{\Gamma\mu(\nu+\mu)^2}\left\{-|\Pi|^2\left[i\frac{\Omega}{\Gamma}(2\nu+\mu)+\frac{\nu^2}{\Gamma}(\mu-\nu)\right]\right.\\
&\left.\times\gamma_{imp}\Delta(p)F_p-\left[(\Gamma-i\Omega)\left(2\nu+\mu-\frac{\nu^3}{\Gamma^2}\right)+\frac{|\Pi|^2}{\Gamma}(2\nu+\mu)\right]2S_pN\Gamma Z_p\right\}\\
&-\frac{|\Pi|^2(2\Gamma-i\Omega)\nu^2}{\mu(\nu+\mu)\Gamma^3}\Delta(p)X_p,\\
Z_p &= \frac{1}{\Gamma\mu(\nu+\mu)^2}\left\{|\Pi|^2\left[i\frac{\Omega}{\Gamma}(2\nu+\mu)+\frac{\nu^2}{\Gamma}(\mu-\nu)\right]\right.\\
&\times[\gamma_{imp}\Delta(p)Q_p+4T_pNX_p]+\left[(\Gamma-i\Omega)\left(2\nu+\mu-\frac{\nu^3}{\Gamma^2}\right)\right.\\
&\left.\left.-\frac{|\Pi|^2}{\Gamma}(2\nu+\mu)\right]2S_pN\Gamma U_p+\frac{i\Omega|\Pi|^2\nu^2}{\Gamma^3\mu(\nu+\mu)}\Delta(p)Y_p,\right.\\
F_p &= -\frac{1}{\nu\mu(\nu+\mu)}\{[\Gamma(\Gamma-i\Omega)-\nu\mu+|\Pi|^2]\gamma_{imp}\Delta(p)F_p+(2\Gamma-i\Omega)2\Gamma S_pNZ_p\},\\
Q_p &= -\frac{1}{\nu\mu(\nu+\mu)}\{[\Gamma(\Gamma-i\Omega)-\nu\mu-|\Pi|^2]\\
&\times[\gamma_{imp}\Delta(p)Q_p+4T_pNX_p]+i\Omega 2S_pN\Gamma U_p\},
\end{aligned} \tag{3.5}$$

where $\Delta(0)=1$, $\Delta(p)=0$ for $p\neq 0$; $\nu^2=\Gamma^2-|\Pi|^2$; $\mu^2=\nu^2-2i\Gamma\Omega-\Omega^2$. Here, the quantity $\nu$ is governed by the scattering on inhomogeneities[13]:

$$\nu^2\approx\gamma_{imp}(\gamma_{imp}+2\gamma).$$

The system (3.5) is different for isotropic ($p=0$) and anisotropic ($p\neq 0$) oscillation modes. The scattering of PW's by inhomogeneities makes the distribution function of $n_{\mathbf{k}}$ isotropic and, therefore, it has the greatest influence on anisotropic modes by contributing additional damping.

### 2. High-frequency collective oscillations

When the scattering by inhomogeneities is weak ($\gamma_{\text{imp}}\ll\gamma$), the high-frequency branches of collective oscillations are not greatly affected and $\Omega_p^{\pm}$ are given by Eq. (2.1) to within $\gamma_{\text{imp}}$. In the case of small supercriticalities $\Omega_p^{\pm}$ we find from Eq. (2.1) that

$$\Omega_p^+=-2iS_p(2T_p+S_p)N^2/\gamma,\qquad \Omega_p^-=-2i\gamma. \tag{3.6}$$

In the case of anisotropic modes the influence of inhomogeneities becomes important for

$$(h-h_{cr})/h_{cr}\sim\gamma_{imp}/\gamma,$$

when $\Omega_p^{\pm}$ becomes comparable with $\gamma_{\text{imp}}$. In the case of smaller supercriticalities, we find from Eq. (3.5) that instead of Eq. (3.6) we now have

$$\Omega_p^+=-i\gamma_{imp},\quad \Omega_p^-=-2i\gamma$$

[see Eq. (3.10) for $\Delta_p<\nu$]. When

$$S_p(2T_p+S_p)<0, \tag{3.7}$$

we find from Eq. (3.6) that the collective oscillations become unstable. It is clear from Eq. (3.10) that inhomogeneities suppress this instability in the region (12). In high-frequency motion ($\Omega_p\gg\gamma_{\text{imp}}$) the correlation functions of $\tilde{n}_{\mathbf{k}}$ and $\tilde{\sigma}_{\mathbf{k}}$ play the dominant role and the eigenvectors of these modes agree to within $\gamma_{imp}/\gamma \ll 1$ with those calculated from the $S$ theory. At low frequencies ($\Omega_p \sim \gamma_{imp}$) the Green functions $\tilde{G}$ and $\tilde{L}$ contribute also to the oscillations. In particular, their damping influences results in the instability suppression mentioned above.

For an isotropic mode ($p=0$) for $\gamma_{imp} \ll \gamma$ we can obtain from the system (3.5) the expression (3.9) which generalizes Eq. (2.1).

When $S_0(2T_0+S_0)<0$, the collective oscillations become unstable and the threshold is given by Eq. (11). Strong scattering by inhomogeneities ($\gamma_{imp} \gg \gamma$) modifies greatly the collective oscillation pattern. A study of high-frequency modes ($\Omega \gtrsim \gamma_{imp}$) can be carried out using the inequalities $\mu^2 \gtrsim \nu^2 \gg |\Pi|^2$ simplifying the system (3.5) to

$$(\nu+\mu)U_p+2S_pNZ_p=0,\qquad (\nu+\mu)Z_p-2S_pNU_p=0. \tag{3.8}$$

Hence, the following formulas are obtained for the frequency $\Omega_p^{\pm}$: if $\gamma_{imp} \ll \gamma$, then

$$\Omega_0^{\pm}=-i\gamma\pm(-\gamma^2+\nu\gamma+\Delta_0^2)^{1/2}, \tag{3.9}$$

$$\Omega_p^{\pm}=-i\gamma\pm(-\gamma^2+\nu^2+\Delta_p^2)^{1/2},\quad p\neq 0,\quad \Delta_p=4S_p(2T_p+S_p)N^2, \tag{3.10}$$

whereas for $\gamma_{imp} \gg \gamma$, we obtain

$$\Omega_p^{\pm}=-2i\gamma_{imp}\pm 2S_pN. \tag{3.11}$$

Here, $\nu^2=2\gamma\gamma_{imp}$. The formula (3.10) provides interpolation in the region of $\nu \sim \Delta_p$.

It is clear from the system (3.8) that only the anomalous correlation functions of $\sigma_k$ and $\sigma_k^*$ participate in the motion at these frequencies; the quantities $n_k$ do not oscillate.

### 3. Low-frequency collective oscillations

We shall consider the oscillation modes whose frequencies vanish in the $S$-theory approximation. It is shown in § 2 that allowance for the scattering of PW's by one another gives rise to a finite oscillation frequency of these modes. We shall show here that the same effect is produced by the scattering of PW's by inhomogeneities. In the case of low-frequency modes a characteristic feature is a large amplitude of oscillations of the sum $\tilde{n}_p+\tilde{n}_p^*$ and a relatively small amplitude of oscillations of $\Pi^*\sigma_p-\Pi\tilde{\sigma}_p^*$. In the approximation of Eq. (3.3), the condition $\Pi^*\tilde{\sigma}_p+\Pi\tilde{\sigma}_p^*=0$ gives for $p\neq 0$ the following dispersion equations applicable to different limiting cases:

$$\mu(\tilde{n}_p+\tilde{n}_p^+)=0 \tag{3.12}$$

for $(h-h_{cr})/h_{cr}<\gamma_{imp}/\gamma$, $\gamma_{imp}\ll\gamma$,

$$\mu^2+2\nu\mu+\nu^2\left(\frac{4S_p}{2T_p+S_p}-3\right)=0 \text{ for } \frac{h}{h_{cr}}-1>\frac{\gamma_{imp}}{\gamma},\quad \gamma_{imp}\ll\gamma, \tag{3.13}$$

$$\mu(\tilde{n}_p+\tilde{n}_p^+)=0 \tag{3.14}$$

for $\gamma_{imp}\gg\gamma$. The corresponding frequencies of collective oscillations are: for $p=0$ subject to $\gamma_{imp}\ll\gamma$,

$$\Omega_0=-8i\nu(h/h_{cr}-1),\qquad (S_0N)^2<\nu\gamma, \tag{3.15}$$

$$\Omega_0=-4i\gamma_{imp}(K_0^{1/2}-K_0),\qquad (S_0N)^2>\nu\gamma, \tag{3.16}$$

whereas in the range $\gamma_{imp}\gg\gamma$,

$$\Omega_0=-4i\gamma(1-h_{cr}^2/h^2). \tag{3.17}$$

For $\gamma\neq 0$ in the range $\gamma_{imp}\ll\gamma$, we have

$$\Omega_p=-i\gamma_{imp},\qquad S_0N<\nu, \tag{3.18}$$

$$\Omega_p=-4i\gamma_{imp}(K_p^{1/2}-K_p), \tag{3.19}$$

whereas in the range $\gamma_{imp}\gg\gamma$, we obtain

$$\Omega_p=-i\gamma_{imp}. \tag{3.20}$$

Here,

$$K_p=2T_p/(2T_p+S_p),\qquad \nu^2=2\gamma\gamma_{imp}.$$

It should be noted that oscillations of the frequency given by Eq. (3.19) are unstable if the following two inequalities are satisfied:

$$S_p(2T_p+S_p)<0,\quad h/h_{cr}-1>\gamma_{imp}/\gamma,$$

which agrees [see Eq. (12)] with the condition of instability of high-frequency mode (3.10). Thus, isotropization of the distribution of $n_k$ occurs in a time $\gamma_{imp}^{-1}$ [naturally, if the stability conditions (2.2) and (12) are satisfied].

If $p=0$, then the isotropization mechanism of the distribution of $n_k$ is no longer applicable and, moreover, the Green functions $\tilde{G}_0$ participate, in addition to $\tilde{n}_0+\tilde{n}_0^*$ in the collective oscillations. Calculations carried out bearing these points in mind give the frequencies of isotropic oscillations [see Eqs. (3.15)–(3.18)]. The instability range of this mode

$$S_0(2T_0+S_0)<0,\quad h/h_{cr}-1>(\gamma_{imp}/\gamma)^{1/2}$$

again is identical with the instability range of the corresponding high-frequency mode (3.12). The fact that the inequality $S_p(2T_p+S_p)<0$ is the instability condition for low-frequency oscillations in homogeneous and randomly inhomogeneous ferromagnets [Eqs. (3.16) and (3.19)] is due to the identical nature of oscillations in the corresponding ranges of the parameters.

We shall conclude by noting that the qualitative nature of the results obtained in the present section applies also when $\Omega \lesssim \eta$.

## § 4. INSTABILITY OF SINGLE-FREQUENCY TURBULENCE OF PARAMETRICALLY EXCITED WAVES

As pointed out earlier, integral equations describing a stationary state of a PW system have a particular solution of the type

$$n_{k\omega},\ \sigma_{k\omega}\sim\delta(\omega-\omega_p/2).$$

In the present section we shall show that this "single-frequency state" is unstable in the case of broadening of the spectrum of $n_{k\omega}$ along the frequency scale.

In the initial equations (1.14)–(1.17) this instability corresponds to

$$\tilde{n}_{k,\omega+\Omega/2;\,k,\omega-\Omega/2}=\tilde{n}_{k\omega}(\Omega),\qquad \tilde{\sigma}_{k,\omega+\Omega/2;\,-k,\omega_p-\omega-\Omega/2}=\tilde{\sigma}_{k\omega}(\Omega), \tag{4.1}$$

which are quantities describing the exponential rise of the initial perturbations $n_{k\omega}$ and $\sigma_{k\omega}$ in the range of frequencies $\omega$ satisfying the condition $\mathrm{Im}\,\Omega(\omega)>0$. For perturbations of the (4.1) type the frequencies $\omega_1$ and $\omega_2$ are not equal to $\omega_p/2$, which simplifies greatly the system (1.14) so that it consists only of the terms containing $\tilde{\Phi}$ and $\tilde{\Psi}$, i.e., two equations for $\tilde{n}_{q_1q_2}$ and $\tilde{\sigma}_{q_1q_2}$ are obtained. Analysis of these equations shows that, to within terms of the order of $(\eta/\Gamma)\ll 1$, we have

$$\Gamma_k\overset{+}{\tilde{\sigma}}_{k\omega}(\Omega)-i\Pi_k^*\tilde{n}_{k\omega}(\Omega)=0, \tag{4.2}$$

which allows us to eliminate $\tilde{\sigma}$. Integrating the equation for $\tilde{n}_{k\omega}(\Omega)$ along the normal to the resonance surface, we obtain

$$\tilde{n}_0+\tilde{n}_0{}^+=\frac{\pi k^2}{v}$$
$$\times\frac{[(\Gamma_0-i\omega_1)(\Gamma_0+i\omega_2)+\nu_+\nu_-+|\Pi_0|^2](\tilde{\Phi}_0+\tilde{\Phi}_0{}^+)+2i\Gamma_0(\Pi_0\tilde{\Psi}_0{}^+-\Pi_0^*\tilde{\Psi}_0)}{\nu_+\nu_-(\nu_++\nu_-)}. \tag{4.3}$$

In contrast to § 2, we now have

$$\nu_\pm{}^2=\nu_0{}^2\mp 2i\Gamma\omega_\pm-\omega_\pm{}^2, \quad \omega_\pm=\omega\pm\Omega/2. \tag{4.4}$$

1. We shall consider first an axially symmetric case which is encountered in cubic ferromagnets when M|| [100] or M|| [111]. For small supercriticalities and in the absence of random inhomogeneities, parametric waves are excited only on the resonance surface equator. Then,

$$n_k=\frac{N}{2\pi}\delta\left(\theta-\frac{\pi}{2}\right), \quad \tilde{n}_k\sim\delta\left(\theta-\frac{\pi}{2}\right),$$

where $\theta$ is the polar angle and the quantities $\Gamma=\gamma$, $\nu$, and $\nu_\pm$ are independent of the azimuthal angle $\varphi$. In the expressions for $\tilde{\Phi}$ and $\tilde{\Psi}$ we have to use the relationship (4.2) and bear in mind that in our geometry the quantities $\mathbf{k}$, $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$ in these integrals lie in the same plane and, to within terms of the order of $\nu/kv\ll 1$ have the same lengths. Consequently, either $\mathbf{k}+\mathbf{k}_1\approx\mathbf{k}_2+\mathbf{k}_3\approx 0$ or $\mathbf{k}\approx\mathbf{k}_2$, $\mathbf{k}_1\approx\mathbf{k}_3$ or $\mathbf{k}\approx\mathbf{k}_3$, $\mathbf{k}_1\approx\mathbf{k}_2$.

We finally obtain

$$\tilde{N}=6\nu^2\tilde{N}/\nu_+\nu_-(\nu_++\nu_-), \quad \tilde{N}=\int_{-\pi}^{\pi}\tilde{N}_\varphi\,d\varphi. \tag{4.5}$$

The above equation is derived using Eq. (32) of Ref. 11 for $\nu^3$ expressed in terms of $T_{12,34}$ and $N^2$, which follows from

$$N_0=\frac{\pi k_0{}^2}{v_0}\frac{\gamma_0}{\nu_0{}^3}(\gamma_0\Phi_0+\operatorname{Im}\Pi_0^*\Psi_0).$$

Equations (4.4) and (4.5) yield the dispersion relationship which gives the frequency

$$[(2\omega-\omega_p)^2+(2\eta-i\Omega)^2][2\eta-i\Omega+((2\omega-\omega_p)^2+(2\eta-i\Omega)^2)^{1/2}]=48\eta^3.$$

Here, the characteristic frequency is $\eta=\nu^2/2\gamma$. We can thus see that the most stable waves are those with $\omega\to\omega_p/2$:

$$\Omega(\omega_p/2)=2i(3^{2/3}-1)\eta\approx 2.08i\eta. \tag{4.6}$$

The instability range $[\operatorname{Im}\Omega(\omega)>0]$ is given by

$$|\omega-\omega_p/2|<2.10\eta \tag{4.7}$$

and is of the same order of magnitude as the width of a regular (in respect of the frequency) stationary state (7).

2. We shall now consider a medium with random inhomogeneities and we shall confine our attention to the most interesting limiting case when $\gamma_{imp}\gg\gamma$. As in Ref. 10, we shall analyze Eq. (4.3) including not only higher terms but also those of the first order in $\gamma/\gamma_{imp}\ll 1$ and we shall assume that $\omega$ and $\Omega$ are smaller than $\Gamma$. We must also bear in mind that for $\gamma_{imp}\gg\gamma$ the distribution of $n_k$ on the resonance surface is isotropic. Consequently, the quantity

$$\tilde{N}=\frac{1}{2}\int(\tilde{n}_0+\tilde{n}_0{}^+)\,do$$

is described by the equation

$$\tilde{N}_\omega(\Omega)=\frac{\Gamma_1}{(\omega-\omega_p/2)^2+\eta^2-i\Gamma_1\Omega}\,\frac{\tilde{\Phi}_{int}+\overset{+}{\tilde{\Phi}}_{int}}{2},$$
$$\tilde{\Phi}_{int}+\overset{+}{\tilde{\Phi}}_{int}=6\frac{\langle T^2\rangle}{kv}\tilde{N}, \tag{4.8}$$

where $\Gamma_1\approx\gamma_{imp}^2/\gamma$ and $\langle T^2\rangle$ is the average value of the square of the modulus of the matrix element $T_{12,34}$ (for details see Ref. 10). These averages occur in the equation which describes the structure of the distribution in a stationary state:

$$N_\omega=\frac{\Gamma_1\Phi_{int}(\omega)}{(\omega-\omega_p/2)^2+\eta_0{}^2}, \quad \Phi_{int}(\omega)=2\frac{\langle T^2\rangle}{kv}$$
$$\times\int N_{\omega_1}N_{\omega_2}N_{\omega_3}\delta(\omega+\omega_1-\omega_2-\omega_3)\,d\omega_1\,d\omega_2\,d\omega_3. \tag{4.9}$$

The coefficient $\eta_0$ which occurs in Eq. (4.8) is found from Eq. (4.9) on substitution of the single-frequency solution (6):

$$\eta_0{}^2=\Gamma_1 2\langle T^2\rangle N^2/(kv). \tag{4.10}$$

Equations (4.8) and (4.10) lead to the dispersion equation

$$i\Gamma_1\Omega=(\omega-\omega_p/2)^2-2\eta_0{}^2. \tag{4.11}$$

As in the case of a homogeneous ferromagnet, the maximum instability increment corresponds to waves with $\omega\to\omega_p/2$:

$$\Omega(\omega_p/2)=\frac{2i\eta_0{}^2}{\Gamma_1}=4i\frac{\langle T^2\rangle N^2}{kv}\sim i\frac{\gamma_{imp}^2}{kv}\left(\frac{h^2}{h_{cr}{}^2}-1\right). \tag{4.12}$$

The instability range where $\operatorname{Im}\Omega>0$, is now

$$|\omega-\omega_p/2|<2^{1/2}\eta_0. \tag{4.13}$$

The limit of the instability range $\Delta\omega=2^{1/2}\eta_0$ is identical with the position of the first satellite of a stationary state in the range $(E-E_{min})<E_{min}$ [a more detailed description of stationary solutions of Eq. (4.9) can be found in Ref. 10] and is of the same order of magnitude as the width of the regular solution (7) of Eq. (4.9).

The instability of the single-frequency solution (6) can be deduced from Eq. (4.9). We shall do this by finding the linear response of a stationary state to thermal noise, which gives rise to an additive correction $\delta\Phi_t$ to $\Phi$:

$$\Phi=\Phi_{int}+\delta\Phi_t,$$

which is independent of $\omega$. Assuming that

$$N_\omega=N\delta(\omega-\omega_p/2)+\delta N_\omega,$$

we obtain from Eq. (4.9) in the linear approximation

$$\delta N_\omega = \delta\Phi_t[(\omega-\omega_p/2)^2 - 2\eta_0^2],$$

and hence it follows that $\delta N_\omega$ is not small when $\Delta\omega < 2^{1/2}\eta_0$ and, consequently, the single-frequency solution is unstable. Moreover, we can show that even the multisatellite solutions of Eq. (4.9) obtained in Ref. 10 are also unstable. In fact, Eq. (4.9) linearized against the background of the multisatellite solution transforms into a system of linear algebraic equations and has neutrally stable solutions, corresponding to a small change in the parameter $E$ (see Ref. 10). The determinant of the linearized system vanishes at the points $\omega_i$ where the satellites are located. Since this determinant changes sign near the first satellite for $E = E_{min}$ (i.e., against the background of the single-frequency solution), we may expect it to change the sign also near satellites for any value of $E$. It thus follows that there is a range of frequencies $\omega$ in which the determinant is negative and perturbations grow.

It follows from the above analysis that the instability of single-frequency and multisatellite states is related to the nonlinear nature of the interaction of PW's: $\Phi_{int} \propto N^\alpha, \alpha > 1$ and, in spite of the fact that we have proved this only for specific situations, it is generally true. Therefore, the only stable (in the case of frequency broadening) state of the system is a multifrequency turbulence of PW's with a continuous frequency distribution.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# Fluctuations in a nonstationary nonequilibrium system near its instability threshold

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The example of electrostatic oscillations is used in considering the growth of fluctuations in a nonequilibrium system in which transition from a stable state (characterized by a small-perturbation logarithmic decrement $\gamma_{-\infty}$) to a stable state occurs in a finite time $\tau$. In contrast to a stationary system, fluctuations at the instability threshold are bounded even in the linear approximation. If the eigenfrequency of weakly damped fluctuations is a simple root of the permittivity (or a root of the corresponding generalized susceptibility in the case of other fluctuations), the ratio of the intensity of fluctuations at the instability threshold to the intensity in the stable region is $(\pi\gamma_{-\infty}\tau)^{1/2}$ before the onset of the transition.

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## 1. INTRODUCTION

It is well known that when a system is not in thermodynamic equilibrium and when the small-perturbation logarithmic decrement $\gamma$ tends to zero as the characteristic parameter of the system $a$ approaches a certain value $a_0$, the level of fluctuations considered in the linear approximation can rise without limit for $a \to a_0$. In this case the fluctuation level is restricted only by nonlinear effects. This situation occurs, in particular, on approach to an instability threshold where $\gamma$ changes its sign. However, strictly speaking, this result applies to the case when the system is stationary. In reality, the system becomes unstable only after a finite time (and it then cannot exist in a stationary state). Allowance for the nonstationary state results in limitation of fluctuations even in the linear approximation[1)] and, as shown below, there is a simple relationship between

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