Gyrotropic turbulence spectra

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Stationary gyrotropic turbulence spectra are investigated by means of the Wyld diagram technique. The spectra corresponding to a stationary helicity flux in the direct interaction model are found under the assumption of weak gyrotropy. The result is confirmed by dimensional estimates.

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1. It is well known (see Ref. 1, the review in Ref. 2, and the literature cited therein) that the generation of a magnetic field by random motions of a conducting liquid is possible only for gyrotropic turbulence. The existence of such generation frequently explains the origin of large-scale magnetic fields in astrophysical objects. In this connection, the problem of the spectra of gyrotropic turbulence, to which the present research is devoted, is of interest.

The spectral tensor of the velocity field of such turbulence

$$J_{\alpha\beta}(k)=J(k)\left(\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\right)+ie_{\alpha\beta\gamma}\frac{k_\gamma}{k}A(k) \qquad (1)$$

includes in it the scalar quantity $J(k)$ which characterizes the energy spectrum of the turbulence, and the pseudoscalar quantity $A(k)$, which is closely connected with the topological structure of the flow. It is known[3] that the topology of the flow of an ideal liquid can be characterized by the conserved quantity

$$I=\int(\mathbf{v},\operatorname{rot}\mathbf{v})\,d\mathbf{r}. \qquad (2)$$

The fact of the conservation of this integral is a direct consequence of Thompson's theorem, according to which the circulation vector along any "liquid" contour Γ conserved. In order to illustrate the foregoing, we consider, following Ref. 3, a velocity field consisting of two closed vortex lines:

$$\operatorname{rot}\mathbf{v}=\mathbf{n}_1\varkappa_1\delta(\mathbf{r}-\mathbf{l}_1(s_1))+\mathbf{n}_2\varkappa_2\delta(\mathbf{r}-\mathbf{l}_2(s_2)),$$

where $\mathbf{n}$ is a vector tangent to the vortex line. We calculate next the circulation of the velocity along the contours l_1 and l_2. Then

$$\oint(\mathbf{v}\,d\mathbf{l}_1)=m\varkappa_2,\qquad \oint(\mathbf{v}\,d\mathbf{l}_2)=m\varkappa_1,$$

where m is an integer, equal to zero if the vortex lines do not form a knot and different from zero if these two lines cnnot be uncoupled. Multiplying the first equality by κ, and the second by κ_2 and then adding these two expressions, we obtain the integral (2):

$$\int(\mathbf{v},\varkappa_1\,d\mathbf{l}_1+\varkappa_2\,d\mathbf{l}_2)=\int(\mathbf{v},\operatorname{rot}\mathbf{v})\,d\mathbf{r}=2m\varkappa_1\varkappa_2.$$

This formula can be generalized without difficulty to the cases of N vortices and then to a continuous distribution.

Thus the integral I characterizes the topplogy of the flow—the degree of its "knottedness." This is the so-called Hopf invariant.[4] In particular, this integral is therefore conserved also in a compressible liquid.

For homogeneous turbulence, the density of the integral (2) is explicitly expressed in terms of $A(k)$:

$$\langle\mathbf{v},\operatorname{rot}\mathbf{v}\rangle=\int kA(k)\,dk.$$

In other words, the quantity $A(k)$, with accuracy to within a factor k, represents the helicity density in k space. Just as in Kolmogorov spectra, which corresponds to a constant energy flux P_ϵ (a quantity conserved in the inertial interval), there should exist a helicity spectrum corresponding to a constant flux of helicity.

We determine this spectrum from considerations of dimensionality. We assume that the gyrotropy is weak and regards the second term in (1) as a perturbation to the Kolmogorov spectrum. Such an assumption corresponds to the real astrophysical situation, in which the gyrotropy is due to weak Coriolis forces (see, for example, Refs. 2 and 5).

It is obvious that in this case the characteristic time τ of the nonlinear process which leads to a change in the helicity will be of the order of the time of the energy scale redistribution in the inertial interval, i.e.,

$$\tau\sim k^{-2/3}(P_\epsilon/\rho)^{-1/3}.$$

Then the helicity flux P_s over the spectrum is estimated in the following fashion:

$$P_s\sim\frac{k^4A(k)}{\tau}\sim k^{14/3}A(k)\left(\frac{P_\epsilon}{\rho}\right)^{1/3}.$$

It then follows from the condition $P_s=\text{const}$ (cf. Ref. 6) that

$$A(k)\sim k^{-11/3}\left(\frac{P_\epsilon}{\rho}\right)^{2/3}\frac{1}{kL}\sim J(k)\frac{1}{kL},$$

where the pseudoscalar quantity $L=P_\epsilon/\rho P_s$ is the mean scale characterizing the total helicity of the system.

Thus the gyrotropic contribution to the spectrum is damped out within the inertial interval more rapidly than the energy spectrum.

2. We now show how this result can be obtained directly from the statistical equations in the direct interaction model.[7]

We consider the equation of ideal hydrodynamics in the k representation:

$$\frac{\partial v_k^\alpha}{\partial t}=\frac{1}{2}i\int\Gamma_k^\alpha\big|_{k_1k_2}^{\beta\gamma}v_{k_1}^{\beta*}v_{k_2}^{\gamma*}\delta_{k+k_1+k_2}\,dk_1\,dk_2. \qquad (3)$$

Here the vertex Γ is a homogeneous function of two

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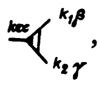
arguments k_i of degree 1;

$$\Gamma_k^{\alpha}\Big|_{k_1k_2}^{\beta\gamma}=[\Delta_k^{\alpha\alpha_1}k_{\beta_1}+\Delta_k^{\alpha\beta_1}k_{\alpha_1}]\Delta_{k_1}^{\alpha_1\beta}\Delta_{k_2}^{\beta_1\gamma}=$$  ,

$$\Delta_k^{\alpha\beta}=\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2},$$

satisfying the two identities

$$(\Gamma_k^{\alpha}\Big|_{k_1k_2}^{\beta\gamma}+\Gamma_{k_1}^{\beta}\Big|_{kk_2}^{\alpha\gamma}+\Gamma_{k_2}^{\gamma}\Big|_{k_1k}^{\beta\alpha})\delta_{k+k_1+k_2}=0,$$
$$(R_k^{\alpha}\Big|_{k_1k_2}^{\beta\gamma}+R_{k_1}^{\beta}\Big|_{kk_2}^{\alpha\gamma}+R_{k_2}^{\gamma}\Big|_{k_1k}^{\beta\alpha})\delta_{k+k_1+k_2}=0, \qquad (4)$$

where

$$R_k^{\alpha}\Big|_{k_1k_2}^{\beta\gamma}=\varepsilon_{\alpha\alpha_1\alpha_2}k_{\alpha_1}\Gamma_k^{\alpha_2}\Big|_{k_1k_2}^{\beta\gamma},$$

the first of which expresses the fact of energy conservation and the second, conservation of helicity.

The equations which describe the evolution of $M(k)$ and $A(k)$ follow directly from (3):

$$\frac{\partial J(k)}{\partial t}=-\frac{1}{2}\,\mathrm{Im}\int\Gamma_k^{\alpha}\Big|_{k_1k_2}^{\beta\gamma}J_{kk_1k_2}^{\alpha\beta\gamma}\delta_{k+k_1+k_2}dk_1\,dk_2, \qquad (5)$$

$$k\frac{\partial A(k)}{\partial t}=-\frac{1}{2}\,\mathrm{Re}\int R_k^{\alpha}\Big|_{k_1k_2}^{\beta\gamma}J_{kk_1k_2}^{\alpha\beta\gamma}\delta_{k+k_1+k_2}dk_1\,dk_2, \qquad (6)$$

where

$$\langle v_k^{\alpha}v_{k_1}^{\beta}v_{k_2}^{\gamma}\rangle=J_{kk_1k_2}^{\alpha\beta\gamma}\delta_{k+k_1+k_2}.$$

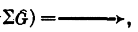
these equations obviously conserve the total energy $\int J(k)dk$ and the total helicity $\int kA(k)dk$. Thus the right sides of Eqs. (5) and (6) can be represented in the form of divergences with respect to k of the corresponding energy and helicity fluxes. A constant energy flux, as is well known, corresponds to a Kolmogorov spectrum. This result, as was shown in Ref. 8, can be obtained directly from the statistical equations in the direct interaction model. This approximation is most simply formulated by means of the Wylde diagram technique,[9] which operates with two quantities: $J_{k\omega}^{\alpha\beta}$—the Fourier component of the correlator of the velocity of stationary turbulence, and $G_{k\omega}^{\alpha\beta}$—the Green's function, which represents the linear response of the system to an infinitesimally small force. Allowance for the first diagrams corresponds to the direct interaction model. However, this approximation, which was first considered by Kraichnan,[7] overstates the role of long-wave pulsations[10] and leads to the spectrum $J_k\sim k^{-1/2}$, which does not agree with experiment.

As has already been shown in Ref. 11, this model can be improved by partial summation is carried out over the diagrams, which reduces to complete account of the effect of dragging of vortices of small scale by large vortices. The Kraichnan equations, improved in such fashion, already contain solutions with the Kolmogorov values of the indices:[11]

$$J(q)=\frac{1}{k^{13/3}}f\left(\frac{\omega}{k^{2/3}}\right),$$
$$\tilde{G}_q^{\alpha\beta}=\tilde{G}_q\Delta_k^{\alpha\beta},\quad \tilde{G}_q=\frac{1}{k^{2/3}}\sigma\left(\frac{\omega}{k^{2/3}}\right) \qquad (7)$$

where $J(q)=\langle\tilde{J}(k,\omega-\mathbf{k}\cdot\mathbf{v})\rangle_v$ and $G(q)=\langle\tilde{G}(k,\omega-\mathbf{k}\cdot\mathbf{v}\rangle_v$. Here the symbol $\langle\ldots\rangle$ denotes averaging over the ensemble of the rangom field velocity v at an arbitrary point r, t with help of the procedure of Wyld, and $q=(k,\omega)$.

A similar program can be carried out for the determination of the indices of the stationary spectrum of the helicity. For this, we linearize the Dyson equation and the Wyld equation for the Green's functions G and J (Ref. 9)

$$\hat{G}=\hat{G}_0(1+\Sigma\hat{G})=$$  ,

$$\hat{J}=\hat{G}\hat{\Phi}\hat{G}^{+}=$$

against the background of the Komogorov solution (7), setting

$$\delta G_q^{\alpha\beta}=i\varepsilon_{\alpha\beta\gamma}\frac{k_\gamma}{k}H_q=$$

$$\delta J_q^{\alpha\beta}=i\varepsilon_{\alpha\beta\gamma}\frac{k_\gamma}{k}A_q=$$

for the perturbations. Since the functions $\tilde{G}_q$ and $\tilde{J}_q$ are scale-invariant, then it is natural to seek a solution of the linearized equations in the form

$$H_q=\frac{1}{k^{s+2/3}}h\left(\frac{\omega}{k^{2/3}}\right),$$
$$A_q=\frac{1}{k^{t+2/3}}a\left(\frac{\omega}{k^{2/3}}\right).$$

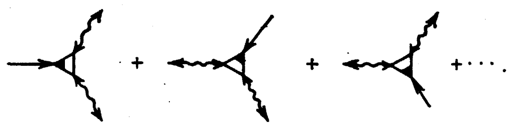
Then it follows from the similarity relations of the linearized equations that

$$t=s+{}^{11}/_3.$$

The second relation between the indices t and s determine from the stationary linearized equations (5), (6). For this we first express the correlation function $J_{kk_1k_2}$ in the form of a power series in $\hat{G}$ and $\hat{J}$. Graphically, this dependence is written in the form

$$J_{kk_1k_2}^{\alpha\beta\gamma}=\int\delta_{\omega+\omega_1+\omega_2}d\omega\,d\omega_1d\omega_2J_{qq_1q_2}^{\alpha\beta\gamma},$$

where

$J_{qq_1q_2}=$ 

The direct-interaction approximation corresponds in this series to account of the first three terms. It is seen that these terms go over into one another by rotation of the graph, which corresponds to cyclic permutation of the arguments; for example, the second graph transforms into the first after the substitutions

$$k\to k_1,\quad k_2\to k_1,\quad k_1\to k.$$

Carrying out linearization of $J_{kk_1k_2}$ and then substituting $\delta J_{kk_1k_2}$ in Eqs. (5) and (6), we convince ourselves, after simple transformations, that the right side of the first equation is identically equal to zero, while that of the second is different from zero:

$$0=\mathrm{Re}\int R_k^{\alpha}\Big|_{k_1k_2}^{\beta\gamma}\delta J_{kk_1k_2}^{\alpha\beta\gamma}\delta_{k+k_1+k_2}dk_1dk_2. \qquad (8)$$

We note that the function $\delta J_{kk_1k_2}$ is a homogeneous function function of k, of degree $y=-s-7$. For the determination of this degree, exactly as in Ref. 8, we carry out conformal transformations[12,13] that keep the region of integration unchanged. For this it is convenient to introduce, in an arbitrary plane specified by the external vector k, the complex quantity

$$w=k_x+ik_y,$$

in terms of which this transformation is written in the form

$$w=w'\frac{w}{w'},\quad w_1=w\frac{w}{w'},\quad w_2=w''\frac{w}{w'}.$$

As a result of this, "rotation" of the vertex $R_k|_{k_1k_2}:R_k|_{k_1k_2}\to R_{k_1}|_{kk_2}$ takes place in (7) and the factor $(k/k_1)^{y+8}$ arises in the integrand.

Carrying out the "rotation" $k_2\to k_1$, $k\to k_2$, $k_1\to k_1$ in analogous fashion, and adding all three expressions (the expression (8) plus the two "rotated" ones), we obtain the equation

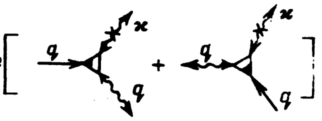
$$\operatorname{Re}\int\left[R_k{}^{\alpha}|^{\beta\gamma}_{k_1k_2}+\left(\frac{k}{k_1}\right)^{y+8}R_{k_1}{}^{\beta}|^{\gamma\alpha}_{k_2k}+\left(\frac{k}{k_2}\right)^{y+8}R^{\gamma}_{k_2}|^{\alpha\beta}_{kk_1}\right]\delta J^{\alpha\beta\gamma}_{kk_1k_2}\delta_{k+k_1+k_2}dk_1dk_2=0.$$

By virtue of the identity (3), this expression vanishes at $y+8=0$. The result $t=11/3+1$ then follows, in complete accord with the dimensional estimates.

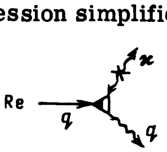
3. The solutions obtained above have meaning only under conditions of the convergence of the integrals in Eq. (8). The requirement that the integrals converge is equivalent to localization of the spectra, since Eqs. (5) and (6) describe the interaction between vortices of different scales. In this sense, the linearized equation (8) plays the same role as the kinetic equations of weak turbulence for the drift spectra.[13]

To verify the convergence of the integrals in (8), it is necessary to know the asymptotic form of $\delta J_{kk_1k_2}$ when one of the momenta k_1 or k_2 is either large or small in comparison with k. Simple study of the first diagrams shows that the integrals converge at the upper limit. Verification of the convergence in regions of small momenta, however, requires analysis.

It is clear that the greatest divergence is connected with the diagrams in $\delta J_{kk_1k_2}$ for which the terms portional to δJ_κ have small momentum κ, i.e.,

$\operatorname{Re}\delta J_{qq\varkappa}\approx 2\operatorname{Re}$ 

$(q\gg\kappa)$. This expression simplifies to the form

$\operatorname{Re}\delta J_{qq\varkappa}\approx -2\operatorname{Re}$  (9)

by virtue of the properties of the vertex (3) and $G^+_{-q}=-G_q$. Thus, an additional small factor proportional to $\varkappa$ because the vertex appears in this expression.

Substituting now (9) in (8), we obtain the integral

$$\int\frac{d\varkappa}{\varkappa k}R_k{}^{\alpha}|^{\beta\gamma}_{k\varkappa}R_\varkappa{}^{\gamma}|^{\alpha\beta}_{kk}\varkappa^{-14/3}k^{-11/3}.$$

Simple calculation of the powers in this integral gives a divergence at zero. However, using the explicit form of Γ we can show that the convolution $R\cdot R$ contained here vanishes identically.

The product of the two vertices R thus varies at small momenta at least as $\varkappa^3$ and therefore the integral converges like $\varkappa^{1/2}$ at the lower limit.

It should be noted that the integrals for the Kolmogorov spectrum converge in this approximation in similar fashion.[8]

All this provides a basis for expecting convergence of the integrals in the general case through cancellations due to the properties of the vertex Γ. This possibly leads to a rigorous solution of the Kolmogorov problem of strong turbulence.

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