Frequency redistribution and diffusion of radiation in resonance x-ray lines

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An analysis is made of the frequency redistribution in resonance lines due to the scattering of ions with a large residual charge Z when the hypothesis of a complete redistribution is violated. The problem of the evolution of the line profile in an infinite homogeneous medium is considered in detail. It is shown that in the Lorentz wings of a Voigt profile the line expands in accordance with the law $\Delta\nu \propto t^{1/4}$. Solutions for an infinite homogeneous medium are used to estimate the thermalization length τ_{th}. Asymptotic formulas found for τ_{th} are identical with those obtained on the assumption of a complete redistribution in the Doppler core of a Voigt profile and they differ considerably from those for the Lorentz wings. The thermalization length is estimated also for the case when the main role in the frequency redistribution is played by coherence violation due to the finite duration of the scattering process. in solution 78.70. – g

The discovery of x-ray emission lines in the spectra of cosmic sources[1] and attempts to explain them within the framework of the existing models[2,3] have made it necessary to review some of the results of the existing theory of radiative transfer in spectral lines. This theory is based on the assumption of a complete frequency redistribution in each resonance scattering event[4] and it is fully applicable to optical and ultraviolet resonance lines of ions with small values of Z. The approximation of a complete redistribution is usually justified by the fact that the width of an absorption profile $\sigma(\nu)$ is governed by the interaction with the surrounding plasma particles (linear Stark effect in hydrogen-like ions) whose spatial positions change in a time much shorter than the excited-state lifetime Γ^{-1}. However, the linear Stark effect decreases on increase of Z as Z^{-1} (it is assumed that the plasma consists mainly of hydrogen with small admixtures of heavy elements), the natural line width rises in accordance with the law $\Gamma \propto Z^4$, and in the range of temperatures in which a resonance line is formed the thermal velocity of particles rises only proportionally to Z. It is thus clear that the assumption of a complete redistribution should be violated at sufficiently high values of Z. For example, this assumption is totally inappropriate for resonant lines of hydrogen- and helium-like ions of iron at densities $N < 10^{22}$ cm^{-3} (Ref. 5).

Under these conditions the problem of radiative transfer in a spectral line should be solved for a real redistribution function which is obtained by a concrete analysis of the resonance scattering process. However, even in the simplest spatially inhomogeneous cases this problem causes enormous mathematical difficulties.[6] In view of this, the concept of the thermalization length τ_{th} (discussed in detail later) becomes particularly important in a qualitative analysis of the physical situation and in order-of-magnitude estimates. We shall find the thermalization length from the law of expansion of a line profile along the frequency axis in the case of an infinite homogeneous medium and situations in which the assumption of a complete redistribution is invalid.

1. KINETIC EQUATION

The kinetic equation describing the behavior of a line profile in an infinite homogeneous medium is

$$\frac{\partial n(t,x)}{N_z c\,\partial t}=\int\limits_{-\infty}^{+\infty}\left[n(t,x')\frac{d\sigma(x'\to x)}{dx}-n(t,x)\frac{d\sigma(x\to x')}{dx'}\right]dx', \tag{1}$$

where $x=(\nu-\nu_0)/\Delta\nu_D$ is the dimensionless frequency measured from the line center at $\nu=\nu_0$ and normalized to the Doppler width $\Delta\nu_D=\nu_0(2kT/Am_pc^2)^{1/2}$; $d\sigma(x\to x')/dx'$ is the differential (with respect to the frequency!) resonance scattering cross section; $n(t,x)dx$ is the number of photons per unit volume in the frequency interval $(x, x+dx)$; N_z is the number of ions (per unit volume) which scatter the investigated line.

Equation (1) is derived on the assumption that $|\nu-\nu_0| \ll \nu_0$ within the limits of the line profile and that factors of the $(\nu'/\nu)^2$ type differ from unity by a negligible amount. Within the framework of this approximation, the terms responsible for the stimulated scattering cancel out; a thermodynamic equilibrium is obtained for $n(t,x)=\text{const}$ and the principle of detailed equilibrium reduces to the condition of symmetry of the differential scattering cross section

$$\frac{d\sigma(x\to x')}{dx'}=\frac{d\sigma(x'\to x)}{dx}. \tag{2}$$

We shall find it more convenient to replace the differential cross section with the concept of a redistribution function

$$R(x,x')=\frac{1}{\Sigma}\frac{d\sigma(x\to x')}{dx'}, \tag{3}$$

where the quantity

$$\Sigma=\int\limits_{-\infty}^{+\infty}\sigma(x)\,dx=\int\limits_{-\infty}^{+\infty}dx\int\limits_{-\infty}^{+\infty}\frac{d\sigma(x\to x')}{dx'}dx'=\frac{\pi e^2}{m_e c}\frac{f_{12}}{\Delta\nu_D} \tag{4}$$

is governed by the oscillator strength f_{12} of a resonance transition $1\to 2$.

2. REDISTRIBUTION FUNCTION

If the influence of the surrounding plasma particles can be ignored, then the frequency redistribution func-

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tion of a system in which the scattering ion is at rest has the form

$$R(x,x')=\frac{1}{\pi}\frac{a}{x^2+a^2}\delta(x-x'). \tag{5}$$

We have introduced here a dimensionless natural line width $a=\Gamma/4\pi\Delta\nu_D$. Going over to the laboratory coordinate system and averaging over the Maxwellian distribution of the velocities of the scattering ions,[7] we obtain the following form of the redistribution function from Eq. (5):

$$R(x,x')=\pi^{-3/2}\int_0^\infty\left(\operatorname{arctg}\frac{y-s}{a}+\operatorname{arctg}\frac{y+s}{a}\right)\exp[-(y+|u|)^2]dy, \tag{6}$$

where $s=(x+x')/2$ and $u=(x-x')/2$. The absorption profile corresponding to Eq. (6) is described by the Voigt function $U(a,x)$:

$$\sigma(x)=\Sigma U(a,x)=\Sigma\frac{a}{\pi^{3/2}}\int_{-\infty}^{+\infty}\frac{e^{-y^2}}{a^2+(x-y)^2}dy. \tag{7}$$

To perform our task we have to, strictly speaking, solve Eq. (1) using the redistribution function (6). Since this cannot be done analytically, we shall consider only the asymptotic behavior in regions of interest to us. We shall begin by distinguishing two cases of a wide Doppler core ($a\ll 1$) and a narrow Doppler core ($a\gg 1$). Since the $a\ll 1$ case is the one of practical interest,[5] we shall concentrate on it. Only a summary of the main results will be given for the opposite case.

If $a\ll 1$, the absorption profile (7) can be divided into two regions: the Doppler core,

$$\sigma(x)\approx\Sigma\pi^{-1/2}e^{-x^2},\quad |x|<x_D, \tag{8}$$

and the Lorentz wings

$$\sigma(x)\approx\frac{\Sigma}{\pi}\frac{a}{x^2},\quad |x|>x_D. \tag{9}$$

The value $x=x_D$ separating these two regions is found from

$$x_D^2\exp(-x_D^2)=a/\sqrt{\pi}. \tag{10}$$

The function $x_D(a)$ varies extremely slowly: $x_D(0.1)=2.08$ and $x_D(0.01)=2.67$.

The function in the integrand of Eq. (6)

$$\psi(y,s)=\operatorname{arctg}\frac{y-s}{a}+\operatorname{arctg}\frac{y+s}{a}=\operatorname{Arctg}\frac{2ya}{a^2+s^2-y^2} \tag{11}$$

can be approximated in the $a\ll 1$ case by

$$\psi(y,s)\approx\begin{cases}2ya/s^2, & 0<y<|s|.\\ \pi, & |s|<y\end{cases} \tag{12}$$

The relative error in the integral (6) is then of the order of s^{-2} for $|s|\gg 1$. According to Eq. (12), the redistribution function (6) splits into two terms, $R_D(x, x')$, and $R_a(x, x')$ where

$$R_D(x,x')=\frac{1}{\pi^{1/2}}\int_{\max(|x|,|x'|)}^{\infty}\exp(-y^2)dy, \tag{13}$$

$$R_a(x,x')=\frac{2a}{\pi^{3/2}s^2}\int_0^{|s|}y\exp[-(y+|u|)^2]dy. \tag{14}$$

The function $R_D(x, x')$ is independent of the parameter a and represents the limit of (6) when $a\to 0$. Its contribution to Eq. (6) is much greater than the contribution of $R_a(x, x')$ for $|x|\sim|x'|\sim 1$ and becomes comparable with the contribution of $R_a(x, x')$ for $|x|\sim|x'|\sim x_D$, falling exponentially for $|x|\gg x_D$ or $|x'|\gg x_D$. On the other hand, the function $R_a(x, x')$ decreases only as x^{-2} when $|x-x'|\sim 1$ and $|x|\to\infty$. On the basis of these properties we shall assume that a frequency redistribution in the Doppler core, $|x|<x_D$, occurs in accordance with the function $R_D(x, x')$ and outside this core, $|x|>x_D$, this happens in accordance with the function $R_a(x, x')$, which we shall approximate—using the fact that $x_D\gtrsim 2$—by

$$R_a(x,x')\approx\frac{2a}{\pi^{3/2}s^2}\int_0^\infty y\exp[-(y+|u|)^2]dy=\frac{a}{\pi^{3/2}s^2}\{e^{-u^2}-\pi^{1/2}|u|[1-\operatorname{erf}(|u|)]\}. \tag{15}$$

Each of the functions $R_D(x, x')$ and $R_a(x, x')$ satisfies separately the symmetry condition (2).

Since the redistribution function (6) decays very rapidly in the wings, it follows that after a sufficiently long time interval the frequency redistribution is dominated by the processes violating the resonance scattering coherence in the system in which the scattering ion is at rest and these processes are characterized by a much weaker (proportional to x^{-2}) decrease in the wings. For example, these processes may be collisions with charged particles, photoionization of the ground state, etc. (for details see Ref. 5), which restrict the duration of the scattering process to the time γ^{-1}. We shall allow for the influence in the limit $\gamma\ll\Gamma$, when instead of Eq. (5) in the system in which the scattering ion is at rest we have to use the redistribution function[5]

$$R(x,x')=\frac{1}{2\pi^2}\frac{ab}{b^2+(x-x')^2}\left(\frac{1}{x^2+a^2}+\frac{1}{x'^2+a^2}\right), \tag{16}$$

where $b=\gamma/2\pi\Delta\nu_D\ll a$. The function (16) is obtained as the limit for $\gamma\ll\Gamma$ from the redistribution law in the case of scattering in subordinate lines[8] and, in contrast to the formula used by Hummer,[9] satisfies the symmetry condition (2).

To obtain the redistribution function allowing for the thermal motion it is necessary to go over to the laboratory system in Eq. (16) and to average over the Maxwellian distribution. However, if $b\ll a$ there is really no need for this because for $|x-x'|\lesssim 1$ the redistribution law obtained in this way is practically identical with $R(x, x')$ of Eq. (6) and in the wings where $|x-x'|\gg 1$ and Eq. (6) decreases as $\exp[-(x-x')^2]$, it is practically identical with Eq. (16). Thus, for our purpose it is sufficient to consider radiative transfer using the function (6) and then to allow for the influence of the wings by Eq. (16).

3. EVOLUTION OF A LINE PROFILE IN AN INFINITE HOMOGENEOUS MEDIUM

Within the limits of the Doppler core $|x|<x_D$, where we can assume that $R(x, x')=R_D(x, x')$, the solution of Eq. (1) can be obtained analytically.[10] Subject to the initial condition $n(0, x)=\delta(x)$, it has the form

$$n(t,x)=t\int_{|x|}^{\infty}\exp(-\xi^2-te^{-\xi^2})d\xi+e^{-t}\delta(x). \tag{17}$$

Here, t is the dimensionless time which is measured in units of $\sqrt{\pi}/\Sigma N_Z c$ and which indicates the number of

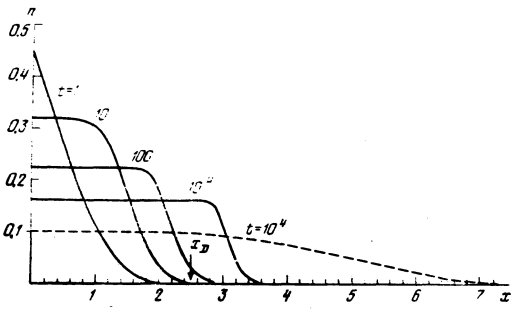

FIG. 1. Line profile $n(t,x)$ at various moments of the dimensionless time t, indicating the number of scattering events at the line center. The continuous curve is obtained for the frequency redistribution in accordance with the function $R_D(x,x')$; the dashed curve corresponds to the assumption that the redistribution obeys the function $R_a(x,x')$, where $a=0.02$.

scattering events at the line center $x=0$. A graph of the function (17) for several values of t is represented by continuous curves in Fig. 1. The asymptotic behavior of $n(t,0)$ for $t\gg 1$ is described by

$$n(t,0)={}^1/_2(\ln t)^{-1/2}. \tag{18}$$

In the Lorentz wings $|x|>x_D$, where the contribution of $R_D(x,x')$ can be ignored and we have $R(x,x')\approx R_a(x,x')$, Eq. (1) can be solved in the diffusion approximation. This approximation is justified by the very fast [see Eq. (15)] fall of $R_a(x,x')$ on both sides of the point $x=x'$. Expanding $n(t,x')=n(t,x+\xi)$ as a series in ξ and applying Eq. (15), we obtain

$$\frac{\partial n}{\partial t}=\frac{a}{2\sqrt{\pi}}\frac{\partial}{\partial x}\left(\frac{1}{x^2}\frac{\partial n}{\partial x}\right). \tag{19}$$

The solution of this equation satisfying the initial condition $n(0,x)=\delta(x)$ is

$$n(t,x)=\frac{2^{3/4}\pi^{1/8}}{\Gamma(1/4)(at)^{1/4}}\exp\left(-\frac{\pi^{1/2}}{8}\frac{x^4}{at}\right). \tag{20}$$

A graph of the function (20) for $a=0.02$ and $t=10^4$ is represented by the dashed curve in Fig. 1.

The line profiles (17) and (20) have a specific form: $n(t,x)$ is practically constant for $|x|<x_m(t)$ and decreases exponentially for $|x|>x_m(t)$. A characteristic half-width of the profile $x_m(t)$ is given by the expressions

$$x_m(t)=(\ln t)^{1/2},\quad t<\pi^{1/2}x_D^2/a, \tag{21a}$$

$$x_m(t)=2^{-5/4}\pi^{-1/8}\Gamma(1/4)(at)^{1/4}=1.321(at)^{1/4},\quad t>x_D^4/a. \tag{21b}$$

We can summarize these results by saying that for $t<\pi^{1/2}x_D^2/a$ the profile of a line emitted at the center $x=0$ of an absorption contour at a moment $t=0$ expands on the frequency scale in accordance with the solution (17), whereas for $t>x_D^4/a$ this happens in accordance with the solution (20).

The ranges of validity of the asymptotic formulas (21a) and (21b) do not merge. This is due to the fact that in the expansion of the profile beyond the Doppler core $|x|>x_D$ the transition from the asymptote (18) to the asymptote (20) takes a longer time than that needed to reach the point $x_m(t)=x_D$.

The case of a narrow Doppler core, when $a\gg 1$, is basically similar to the case of the Lorentz wings. For $a\gg 1$, we can replace Eq. (12) with

$$\varphi(y,s)\approx\frac{2ua}{a^2+s^2}, \tag{22}$$

so that Eq. (1) considered in the diffusion approximation becomes

$$\frac{\partial n}{\partial t}=\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{1+x^2/a^2}\frac{\partial n}{\partial x}\right). \tag{23}$$

Here, t is the dimensionless time normalized to one scattering event at the line center, i.e., to $\pi a/\Sigma N_Z c$. The solution of Eq. (23) for $|x|\gg a$ is analogous to Eq. (20):

$$n(t,x)=\frac{2^{1/2}}{\Gamma(1/4)(a^2t)^{1/4}}\exp\left(-\frac{x^4}{8a^2t}\right). \tag{24}$$

We shall allow for the spreading of the line profile under the action of the wings of the function (16) in the following way. We shall assume that $n(t,x)=n(t)$ for $|x|<x_m$ and $n(t,x)=0$ for $|x|>x_m$. This approximation is justified by the solutions (17) and (20) obtained earlier. Next, we shall use Eq. (1) to estimate the rate of rise $n(t,x)$ for $x\gtrsim x_m+1$, when the difference between the wings of the function (16) and the wings (6) becomes important. We thus find that

$$\frac{\partial n(t,x)}{\partial t}=n(t)\int_{-x_m}^{+x_m}\pi^{1/2}R(x',x)\,dx'\approx\frac{b}{2\pi^{1/2}}\frac{n(t)}{x^2}. \tag{25}$$

Here, as in the $a\ll 1$ case, t denotes the dimensionless time normalized to $\pi^{1/2}/\Sigma N_Z c$. It should be noted that the main rise of $n(t,x)$ in Eq. (25) occurs because of scattering from the region $|x'|\lesssim a$. The law $x_m(t)$ can be found from the law of conservation of the total number of photons

$$\int_{x_m}^{\infty}\frac{\partial n(t,x)}{\partial t}dx=n(t)\frac{dx_m}{dt}, \tag{26}$$

which gives

$$x_m(t)=\pi^{-1/4}(bt)^{1/2}. \tag{27}$$

We can now readily write down the approximate line profile spreading under the action of the wings of the redistribution function (16):

$$n(t,x)\approx\begin{cases}\pi^{1/4}/4(bt)^{1/2}, & |x|<x_m\\ \pi^{-1/4}(bt)^{1/2}/4x^2, & |x|>x_m\end{cases}. \tag{28}$$

It should be noted that the law $n(t,0)$ obtained in this way is practically identical with the asymptote $n(t,0)$ derived assuming a complete frequency redistribution. In fact, the solution of Eq. (1) with the redistribution function

$$R(x,x')=\frac{1}{\pi^2}\frac{b}{x^2+b^2}\frac{b}{x'^2+b^2} \tag{29}$$

is easily obtained by the method of Ivanov[11] and has the form

$$n(t,x)=\frac{\pi^{-1/2}}{x^2+b^2}\int_0^t I_0\left(\frac{t'}{2b\pi^{1/2}}\right)\exp\left[-\frac{b(t-t')}{\pi^{1/2}(x^2+b^2)}-\frac{t'}{2b\pi^{1/2}}\right]dt'+\exp\left(-\frac{t}{b\pi^{1/2}}\right)\delta(x), \tag{30}$$

and hence for $x=0$, $t\gg 1$ we have

$$n(t,0)=\pi^{-1/4}(bt)^{-1/2}. \tag{31}$$

Here, $I_0(x)$ is a modified Bessel function.

A comparison of Eqs. (27) and (21b) shows that if the resonance scattering process is limited, on the average, to the time interval $\gamma^{-1} \gg \Gamma^{-1}$, then the effect just discussed begins to dominate the line profile spreading from the moment

$$t^* \approx \frac{\pi^{1/2}\Gamma^4(1/4)}{9^5}\frac{a}{b^2} = 9.6\frac{a}{b^2} = 30\frac{\Gamma\Delta\nu_D}{\gamma^2}. \qquad (32)$$

4. THERMALIZATION LENGTH

In those cases when in addition to the scattering in the line there are processes resulting in photon annihilation, we can introduce the concept of a thermalization length τ_{th}, which is the average distance between the point of creation and the point of annihilation of a photon, expressed in terms of the range at the line center. This concept makes it much easier to analyze specific physical situations. For example, if the optical thickness of a cloud at the line center is $\tau \ll \tau_{th}$, then practically all the line photons emitted in this cloud escape outside. In this case the line radiation may be a powerful mechanism of plasma cooling of the cloud. However, if $\tau \gg \tau_{th}$, then only the photons created in a thin surface layer $0 < \tau \lesssim \tau_{th}$ can escape; in this case the line mechanism of cooling of the inner parts of the cloud can be ignored.

We shall estimate the values of τ_{th} using the solutions obtained for an infinite homogeneous medium. We shall define τ_{th} as the average optical thickness $\bar{\tau}$ (t_{th}) traversed by a photon emitted at the line center from the point of its creation up to a moment t_{th}, when the probability of its absorption rises to $(1 - e^{-1})$. We can then distinguish two cases:

a) in addition to resonance absorption described by $\sigma(x)$, there is also weak absorption in the continuum whose cross section σ_a is frequency-independent;

b) the albedo of single scattering is $\lambda < 1$, i.e., the absorption cross section is $\sigma_a(x) = (1 - \lambda)\sigma(x)$.

In case a) the moment in question is given by $t_{th} = \sigma(0)/\sigma_a \equiv \sigma_s/\sigma_a$, whereas in case b) it is found from the condition $\bar{\nu}(t_{th}) = (1 - \lambda)^{-1}$, where $\bar{\nu}(t)$ is the average number of collisions experienced by a photon in a time interval t. In other words, we can estimate τ_{th} by finding the average distance traveled by a photon $\bar{\tau}(t)$ in a time interval t and the average number of photon collisions $\bar{\nu}(t)$. All the estimates given below are asymptotic and valid only if $t \gg 1$, $\sigma_s \gg \sigma_a$, $1 - \lambda \ll 1$.

This definition of τ_{th} is physically identical with that adopted by Ivanov[4] for a steady-state situation and, for exact solutions of the same problems, should give values differing from those of Ivanov[4] by amounts not greater than a numerical factor of the order of unity. This is easily illustrated by considering the thoroughly investigated example of monochromatic scattering, when the scattering cross section σ_s is completely frequency-independent. In this case the problem of propagation of radiation is easily solved in the diffusion approximation:

$$\frac{\partial n(t,\tau)}{\partial t} = \frac{1}{3}\frac{1}{\tau^2}\frac{\partial}{\partial \tau}\left(\tau^2\frac{\partial n}{\partial \tau}\right). \qquad (33)$$

The thermalization length

$$\tau_{th} = \left(\frac{3}{4\pi t_{th}}\right)^{1/2}\int_0^\infty 4\pi\tau^3 \exp\left(-\frac{3\tau^2}{4t_{th}}\right)d\tau = \left(\frac{16}{3\pi}\frac{\sigma_s}{\sigma_a}\right)^{1/2}, \qquad (34)$$

calculated in accordance with the above definition, differs from that used by Ivanov[4] only by the factor $(8/\pi)^{1/2} = 1.6$.

In more complex cases discussed below the problem of spatial propagation of radiation cannot be solved analytically. Instead, we shall propose a procedure for an approximate estimate of $\bar{\tau}(t)$ which we shall illustrate in detail by the example of the Doppler core of a Voigt profile in the $a \ll 1$ case, when the frequency redistribution function is $R(x, x') \approx R_D(x, s')$.

We shall first estimate the distance traveled by a photon from the point of its creation by direct flight. We shall do this by averaging the mean distance

$$\bar{\tau}_f(t,x) = \frac{\sigma(x)}{\sigma_s}\int_0^t \tau \exp\left[-\tau\frac{\sigma(x)}{\sigma_s}\right]d\tau \Big/ \left\{1 - \exp\left[-t\frac{\sigma(x)}{\sigma_s}\right]\right\}, \qquad (35)$$

which is traversed in a straight line by a photon of frequency x and we shall use the distribution (17) obtained by solving Eq. (1):

$$\bar{\tau}_f(t) = \int_{-\infty}^{+\infty} \bar{\tau}_f(t,x)\, n(t,x)\, dx = t/2\ln t. \qquad (36)$$

In estimating the diffusion displacement of a photon of frequency x we shall use the simple formula

$$\bar{\tau}_d(t,x) = \begin{cases} [t\sigma_s/\sigma(x)]^{1/2}, & t\sigma(x) > \sigma_s \\ t, & t\sigma(x) < \sigma_s \end{cases}, \qquad (37)$$

which is free of any numerical factors. Averaging Eq. (37) over the distribution (17), we obtain

$$\bar{\tau}_d(t) = \int_{-\infty}^{+\infty} \bar{\tau}_d(t,x)\, n(t,x)\, dx \approx t/\ln t, \qquad (38)$$

which, apart from a factor of the order of 1/2, is identical with Eq. (36). Since both integrals (36) and (38) are calculated within the range $|x| \sim (\ln t)^{1/2}$ [where Eq. (37) is already considerably in error] and since they are identical within the limits of precision of our estimates, it follows that photons travel from the points of their creation mainly by direct flight. This result was derived earlier by Rybicki and Hummer[12] on the assumption of a complete frequency redistribution. In contrast to $\bar{\tau}(t)$, the average number of collisions $\bar{\nu}(t)$ experienced by a photon in a time t can be estimated rigorously from the self-evident formula

$$\bar{\nu}(t) = \int_0^t dt' \int_{-\infty}^{\infty} n(t',x)\frac{\sigma(x)}{\sigma_s}dx = \frac{\pi^{1/2}}{2}\frac{t}{(\ln t)^{1/2}}. \qquad (39)$$

Substituting the required values of t_{th} in Eq. (36), we obtain expressions for the thermalization length (46a) and (47a) (discussed below). They differ only by the factor $\pi^2/8$ from the values deduced by solving the steady-state problem on the assumption of a complete frequency redistribution.[4] Such a good agreement can naturally be expected if we bear in mind the proven[11] identity of the asymptotic expressions for $n(t, x)$ obtained using the exact redistribution function $R_D(x, x')$ and assuming a complete redistribution over a Doppler profile. This agreement can be regarded as the justification of the procedure adopted for estimating $\bar{\tau}(t)$. The procedure can also be checked easily in the case of a complete redistribution over a Lorentz profile, when the solution of Eq. (1) is written in the form (30). In this case the expressions for the thermalization lengths differ from those given by Ivanov[4] by the factor

$$\frac{18}{\pi^{5/4}}\int_0^\infty \frac{1}{y^2}\frac{e^y-y-1}{e^y-1}e^{-y}\int_0^{\sqrt{y}} e^{x^2}dx\,dy = 1.07.$$

Applying the above method to the Lorentz wings of a Voigt profile, where the scattering cross section has the form (9), we immediately find that in this frequency range the spatial motion of photons is due to diffusion and not direct flight. In fact, substituting Eq. (9) into Eq. (35) and Eq. (20) into Eq. (36), we obtain

$$\bar{\tau}_f(t)=\frac{4\pi^{5/4}}{\Gamma^2(1/4)}\left(\frac{t}{a}\right)^{1/2}, \tag{40}$$

whereas the diffusion displacement is

$$\bar{\tau}_d(t)=\frac{2^{3/4}\pi^{5/8}}{\Gamma(1/4)}\frac{t^{3/4}}{a^{1/4}}=0.949\frac{t^{3/4}}{a^{1/4}}\gg\bar{\tau}_f(t). \tag{41}$$

The average number of collisions is

$$\bar{\nu}(t)=\frac{2^{9/4}\pi^{5/8}}{3\Gamma(1/4)}\frac{t^{3/4}}{a^{1/4}}=0.894\frac{t^{3/4}}{a^{1/4}}. \tag{42}$$

We can improve the estimate (41) and obtain the "correct" numerical coefficient by solving the diffusion equation

$$\frac{\partial n(t,x,\tau)}{\partial t}=\frac{a}{2\pi^{1/2}}\frac{\partial}{\partial x}\left(\frac{1}{x^2}\frac{\partial n}{\partial x}\right)+\frac{\pi^{1/2}}{3}\frac{x^2}{a}\frac{1}{\tau^2}\frac{\partial}{\partial\tau}\left(\tau^2\frac{\partial n}{\partial\tau}\right), \tag{43}$$

which describes simultaneous frequency and spatial diffusion. However, since it is not possible to solve Eq. (43) analytically, we shall give the following arguments in support of Eq. (41). It follows from Eq. (43) that

$$n(t,x)=4\pi\int_0^\infty \tau^2 n(t,x,\tau)\,d\tau$$

satisfies Eq. (19) so that at a moment t the fraction of photons of frequency $|x|\gg(at)^{1/4}$ is exponentially small [see Eq. (20)]. On the other hand, according to Eqs. (37) and (9) the greatest contribution to the spatial motion is made by photons with the maximum possible value of x^2, i.e., with $|x|\sim(at)^{1/4}$, and this leads directly to Eq. (41). It should be noted that the differential operator on the right-hand side of Eq. (43) does not admit stationary solutions in an infinite medium.

For $t>t^*$, when the finite duration of the resonance scattering is the dominant effect, the spatial motion of the line photons by direct free flight is

$$\bar{\tau}_f(t)=t\left(\frac{b}{a}\right)^{1/2}\frac{1}{4}\int_0^\infty\frac{1-(1+y)e^{-y}}{1-e^{-y}}\frac{dy}{y^{3/2}}=0.647t\left(\frac{b}{a}\right)^{1/2}. \tag{44}$$

An estimate of the diffusion spreading gives a value of the same order of magnitude. The average number of collisions is

$$\bar{\nu}(t)={}^1/_2\pi^{3/4}(t/b)^{1/2}. \tag{45}$$

Combining the above results for $\overline{\tau}(t)$ and $\overline{\nu}(t)$, we find that in case a) the thermalization length should be estimated using the formulas

$$\tau_{th}=\frac{\sigma_s}{2\sigma_a}\left(\ln\frac{\sigma_s}{\sigma_a}\right)^{-1},\quad 1\ll\frac{\sigma_s}{\sigma_a}<\frac{\pi^{1/2}}{a}x_D^2 \tag{46a}$$

$$\tau_{th}=a^{-1/4}\left(\frac{\sigma_s}{\sigma_a}\right)^{3/4},\quad \frac{x_D^4}{a}<\frac{\sigma_s}{\sigma_a}\ll t^*, \tag{46b}$$

$$\tau_{th}=0,65\left(\frac{b}{a}\right)^{1/2}\frac{\sigma_s}{\sigma_a},\quad t^*<\frac{\sigma_s}{\sigma_a} \tag{46c}$$

[we recall that $\sigma_s\equiv\sigma(0)=\Sigma/\pi^{1/2}$ for $a\ll1$], whereas in case b) we should use the expressions

$$\tau_{th}=\frac{1}{\pi^{1/2}}\frac{1}{1-\lambda}\left(\ln\frac{1}{1-\lambda}\right)^{-1/2},\quad 1\ll\frac{1}{1-\lambda}<\frac{\pi x_D}{2a}, \tag{47a}$$

$$\tau_{th}=\frac{1}{1-\lambda},\quad \frac{x_D^3}{a}<\frac{1}{1-\lambda}\ll\frac{t^{*3/4}}{a^{1/4}}, \tag{47b}$$

$$\tau_{th}=\frac{2.6\,b^{1/2}}{\pi^{3/2}a^{1/2}}\frac{1}{(1-\lambda)^2},\quad \frac{t^{*3/4}}{a^{1/4}}<\frac{1}{1-\lambda}. \tag{47c}$$

If the Doppler core can be ignored and $a\gg1$, then instead of Eqs. (46) and (47), we have to use correspondingly

$$\tau_{th}=a^{-1/2}\left(\frac{\sigma_s}{\sigma_a}\right)^{3/4},\quad a^2<\frac{\sigma_s}{\sigma_a}\ll\frac{t^*}{a}, \tag{48a}$$

$$\tau_{th}=0.65\left(\frac{b}{a}\right)^{1/2}\frac{\sigma_s}{\sigma_a},\quad \frac{t^*}{a}<\frac{\sigma_s}{\sigma_a} \tag{48b}$$

and

$$\tau_{th}=\frac{1}{a}\frac{1}{1-\lambda},\quad a^{1/4}<\frac{1}{1-\lambda}\ll\frac{t^{*3/4}}{a^{1/4}}, \tag{49a}$$

$$\tau_{th}=\frac{2.6}{\pi^2}\left(\frac{b}{a}\right)^{3/2}\frac{1}{(1-\lambda)^2}\quad \frac{t^{*3/4}}{a^{1/4}}<\frac{1}{1-\lambda}. \tag{49b}$$

Here, $\sigma_s\equiv\sigma(0)=\Sigma/\pi a$.

We can summarize the above analysis as follows. Under the conditions such that the criteria of validity of the hypothesis of a complete frequency redistribution in resonance scattering is violated and it is necessary to use the real redistribution function [given by Eq. (6) for the isotropic case], we can nevertheless use the results obtained on the basis of this hypothesis when the redistribution plays an important role only within the Doppler core $|x|\leq x_D$. However, if the line profile expands in the available time beyond this core, where $|x|>x_D$, the hypothesis of a complete redistribution becomes totally unacceptable. In order-of-magnitude estimates and qualitative analysis of the situation one can then use the expressions for the thermalization length given by Eqs. (46)–(49); in a more rigorous analysis it is necessary to solve the problem of radiative transfer with the appropriate redistribution function.

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Investigation of the mechanism of energy dissipation in the front of a turbulent electrostatic shock wave

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The nature of turbulent processes and the mechanism of collisionless energy dissipation in the front of an electrostatic turbulent shock wave due to them are studied experimentally. It is shown that the development of instability of opposing ion beams results in a high level of turbulent noise, $W/nT_e \lesssim 0.2$ in the shock wave. The scattering of incident plasma ions by this noise is a mechanism which ensures beam energy dissipation over a distance of twenty Debye lengths. A phenomenological shock wave model is proposed which is based on two-stream electrostatic ion instability.

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I. INTRODUCTION

A turbulent electrostatic shock wave with Mach number $M=u/c_s\approx 2-3$ was discovered in experiments[1,2] on the interaction of the flow of a rarefield nonisothermal plasma ($T_e \gg T_i$) with a magnetic "barrier" (u is the velocity of plasma flow, c_s is the ion sound velocity). The width of the shock front amounted to several centimeters, which is 2–3 orders smaller than the free path length of the particles of the plasma relative to pair collisions. The formation of a collisionless shock front was attributed to instability development brought about by the part of the flow reflected from the barrier.

The aim of the present work is the study of the nature of the turbulent processes and the character of the dissipation mechanism due to them in the front of an electrostatic turbulent shock wave. The study of the mechanism of dissipation in shock waves of such a type is of general physical and applied interest, since similar effects can play a decisive role in such phenomena as an earth surface shock wave,[3] isomagnetic discontinuities in the form shock wave front in magnetized plasma,[4] the interaction of ion beams with a plasma target,[5] and so on.

The study of processes in the front of an electrostatic turbulent wave has been carried out in two directions: 1) the study of macroscopic density distributions and distribution of potential and flow velocity of the ions, and also the spectrum of random electrostatic oscillations, time and amplitude characteristics, which makes it possible to establish the nature of the turbulent processes; 2) the study of the distribution function of the ions and ahead and behind the front, which enables us to decide on the character of the dissipation mechanism.

II. APPARATUS AND METHOD OF DIAGNOSTICS

1. The experiments were carried out on the "SOMB" apparatus.[1,2] The diagram of the apparatus is shown in Fig. 1. The plasma flow was created in a metallic vacuum chamber having the shape of a cylinder of length (L) 200 cm, diameter (D) 60 cm. A pulse of the working gas (argon) was let into the volume, which has been pumped down to a pressure of $\lesssim 5\times10^{-6}$ Torr, from one of the ends, making the pressure in the region of the ionizer $\approx 10^{-4}$ Torr. The expanding cloud was ionized by the current of electrons accelerated from the heated cathode. The plasma that was formed spread out in the vacuum at a velocity $u\approx(5\times10^5-10^6)$ cm/sec along the x axis of the volume, much greater than the velocity of motion of the boundary of the neutral gas, $\lesssim 10^4$ cm/sec. Within a time of 10^{-4} sec, a quasistationary flow of plasma was established in the region of the magnetic "barrier," located at a distance of 100 cm from the ion-

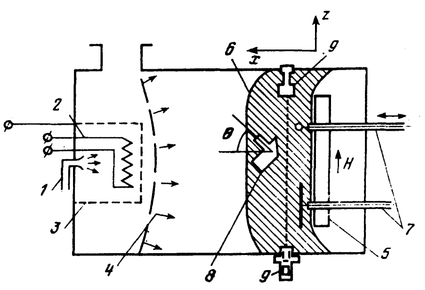

FIG. 1. Diagram of the experimental setup: 1—pulsed valve for letting in the gas; 2—heated cathode; 3—wire-gauze anode; 4—front of the neutral gas; 5—solenoid; 6—front of the shock wave; 7—Langmuir probes; 8—Hughes-Rojansky analyzer; 9—apparatus for probing the plasma by a beam of ions.

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