and also gives rise to an anisotropy of this motion.

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Infrared divergence in field theory of a Bose system with a condensate

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Exact equations are obtained (in particular, for the anomalous self-energy part $\Sigma_{12}(0) = 0$ and for the scattering vertex of two phonons $\Gamma_4(p_i \to 0) = 0$), which indicate that in the general case the usual methods of summing the field perturbation-theory diagrams for Bose systems are not valid (the calculation must not be stopped when a converging result is obtained in the lower order in some small parameter). An investigation of the character of the infrared divergence of the field diagrams has yielded the region of applicability of the ordinary summation methods. It is shown that in the case $T>0$ or of a two-dimensional system at $T=0$ one must use a special regularization that calls for introducing the phonon vertices counterterms that, in contrast to the relativistic theories, contain no infinities and do not change the initial Hamiltonian at all. Examples of effective summation are presented for cases when the usual approach leads to an erroneous result [$\Sigma_{12}(p\to 0)$, $\Pi(p\to 0)$, and others]. The derivation of the asymptotic Gavoret and Nozieres formulas for the Green's functions and the susceptibilities is re-examined with account taken of the equality $\Sigma_{12}(0) = 0$.

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1. INTRODUCTION

The field theory developed in Refs. 1 and 2 has been successfully used for a qualitative analysis of the properties of superfluid He^4 (primarily the singularities of the spectrum; see, e.g., Refs. 3 and 4), as well as to verify, with the aid of simple models, the correctness of other microscopic approaches[5] and assump-

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tions of semiphenomenological character. An advantage of the field approach is the simplicity of the analytic form of the diagrams, which makes it possible to separate readily and to analyze the essential skeleton diagrams, to estimate the contribution of any diagram, and carry out partial summation.

Among the major successes of the field-theoretical approach were the investigations,[6,7] aimed at a microscopic corroboration of the acoustic character of the long-wave excitations and of the local-equilibrium connection between the speed of sound and the compressibility for a Bose system with an interaction of rather general form.

However, the field-theory diagrams constructed with the aid of exact particle lines are subject infrared divergence. In a large number of approximate calculations (see, e.g., Refs. 8–11), the diverging diagrams of higher order in the small parameter were simply discarded (the calculation was stopped when a convergent result was obtained in the lower orders in some small parameter).

In this paper (Sec. 1) we obtain for different vertices with zero external momenta (in particular, for the anomalous self-energy part $\Sigma_{12}(0)=0$, Ref. 12) exact equations that contradict the similar "naive" use of perturbation theory. For a low-density system of hard spheres, for example, the summation of the ladder diagrams with zero Green's functions G_0 (corresponding to the lowest power of the small parameter) yields

$$\Sigma_{12}(0)\approx\frac{4\pi a}{m}n_0,$$

where a is the scattering amplitude.[2] This example is of particular interest, since collective theories that have no infrared divergences[13-15] cannot be applied to this system (they call for a sufficiently rapid decrease of the Fourier transform of the potential, $V(q\to\infty)\to 0$).

We investigate in this paper the character of the infrared divergence of the diagrams and determine on the basis of the investigation the limits of applicability of the usual methods of the partial summation. We show that the divergences of diagrams without external lines, with not too small external momenta (and their sums), can be eliminated from some types of diagrams with arbitrarily small external momenta; these diagrams characterize, in particular, the energy of the ground state, the number of the particles in excess of the condensate, and the spectrum at momenta that are not too small. It is possible to use here a partial summation that is governed by the order of the diagrams with respect to the small parameter (for example, summation of ladder diagrams with nonzero Green's functions for a low-density system). The divergences in the indicated diagrams are eliminated already in the summation over the directions of the internal lines of the particles in the case $T=0$ (Sec. 2), and as a result of a special regularization that is somewhat similar to renormalizations in relativistic field theories in the cases of $T>0$ and of a two-dimensional Bose system (Sec. 4). In contrast to the relativistic field theories, the corresponding counterterms are finite and vanish in the sum. The vanishing of the counterterms in the Hamiltonian is due to the vanishing of the exact phonon vertices $\Gamma(p_i\to 0)$, and this in turn is due to the equality $\Sigma_{12}(0)=0$ (and makes the meaning of the latter simple).

In the general case of vertices with arbitrarily small external momenta (or their sums), the p_i divergences in the diagrams remain (as $p_i\to 0$); since the divergences cancel the small parameter, a correct calculation of these vertices calls for an effective summation that is determined by the character of the divergences; we present here examples of such a summation ($\Sigma_{12}(p\to 0)$, $\Pi(p\to 0)$, see Sec. 5). The situation recalls the "zero-charge" situation (or the situation with "asymptotic freedom") in relativistic field theories, where the diagrams continue to grow after the renormalizations in the limit of the large (small) external momenta (within the framework of the validity of perturbation theory, see, e.g., Ref. 16). In contrast to the relativistic approach, in our case we encounter no difficulties with violation of perturbation theory after the effective summation.

The analysis of the infrared divergence is based on the asymptotic formulas of Gavoret and Nozieres[7] for the Green's functions $G_{ik}(p\to 0)$. In the derivation of these formulas in Ref. 7 they used essentially the assumption $\Sigma_{12}(0)\neq 0$. In Sec. 6 the formulas are derived for $G_{ik}(p\to 0)$ with $\Sigma_{12}(0)=0$ taken into account. The confirmation of a simple connection between the sound velocity c and the ground-state energy E means that in the calculation of the first term in the expansion of the spectrum $\varepsilon(p)$ as $p\to 0$ we can use the ordinary field-theoretical summation methods; in particular, the use of "ladders" with G_0 for a rarefied system for the calculation of c (as well as of E) is permissible (whereas there is no justification for discarding the higher-order diverging diagrams from the formula that expresses the spectrum in terms of the current-current response function.[11])

2. EXACT EQUATIONS

1. Hugenholtz and Pines[8] represented the Belyaev field-theoretical diagram method,[2] which reduces to the operator substitutions a_0^+, $a_0\to n_0^{1/2}$ in the Hamiltonian $\hat{H}$, in a form convenient for investigations in the long-wave limit: namely, the diagrams are constructed for the effective Hamiltonian

$$\hat{H}'(n_0,\mu)=\hat{H}-\mu n'\quad\left(n'=\sum_{p\neq 0}a_p^+a_p\right),$$

in which the number of particles n_0 in the condensate and the chemical potential μ are regarded as independent parameters. The connection between n_0 and μ as well as the expression for the total number of particles n (the system volume is $V=1$) are defined by the formulas

$$\mu=\left(\frac{\partial E'}{\partial n_0}\right)_\mu, \tag{1}$$

$$n=n_0+n',\quad n'=-\left(\frac{\partial E'}{\partial\mu}\right)_{n_0}. \tag{2}$$

The "energy" $E'(n_0,\mu)=\langle\hat{H}'\rangle$ is given by the sum of all the connected vacuum diagrams.[1)] Comparison of the $\Sigma_{ik}(p\to 0)$ diagrams with the vacuum ones leads to the formula[6]

$$\Sigma_{11}(0)-\Sigma_{12}(0)=\left(\frac{\partial E'}{\partial n_0}\right)_\mu=\mu_0(n_0,\mu)=\mu, \tag{3}$$

from which it follows that there is no gap in the spectrum, $\varepsilon(p\to 0)\to 0$, (the subscripts 1 and 2 in Σ_{ik} correspond to external arrows directed to the right and left, respectively). Gavoret and Nozieres,[7] after expressing the coefficients of the expansion of $\Sigma_{ik}(p)$ in the momenta in terms of the derivative of $E'(n_0, \mu)$, obtained asymptotic formulas for the Green's function G_{ik} and for the response functions $F_{\mu\nu}$, corresponding to an acoustic character of the spectrum and to the usual connection between the speed of sound and the compressibility; in particular,

$$G_{11}(p\to 0)=-G_{12}(p\to 0)=\frac{n_0mc^2}{n(\varepsilon^2-c^2p^2+i\delta)}, \tag{4}$$

$$c^2=\frac{n}{m}\frac{d\mu}{dn}. \tag{5}$$

These formulas were derived in Ref. 7 by making use essentially of the assumption $\Sigma_{12}(0)\neq 0$, which, as we shall show, contradicts the exact equation for $\Sigma_{12}(p\to 0)$; these results remain valid also in the correct treatment, with $\Sigma_{12}(0)=0$ taken into account (see Sec. 6).

2. Using arguments similar to the derivation of formula (3) in Ref. 6, we can easily express, in terms of derivatives of $E'(n_0, \mu)$, the contribution of any exact many-line vertex $M(r_1, r_2, s)$ with zero external momenta, $p_i\to 0$ (i.e., the total assembly of diagrams irreducible in the particle lines, with fixed number and character of the external lines)[2]:

$$M(r_1,r_2,s)=n_0^{(r_2-r_1)/2}\left(-\frac{\partial}{\partial\mu}\right)^s_{n_0}\left(\frac{\partial}{\partial n_0}\right)^{r_2}_\mu\left[n_0^{r_1}\left(\frac{\partial}{\partial n_0}\right)^{r_1}_\mu\right]E'(n_0,\mu) \tag{6}$$

(r_1 is the number of incoming particle lines, r_2 the outgoing number, and s the number of external potential lines). In particular, for 2- and 3-line vertices we get (see Fig. 1; the potential line is shown dashed; the + and − mark directions of lines of Γ towards and away from the vertex, respectively)

$$\Pi(0)=\frac{\partial^2E'}{\partial\mu^2}=-\left(\frac{\partial n'}{\partial\mu}\right)_{n_0}, \tag{7}$$

$$\Lambda_1(0)=\Lambda_2(0)=n_0^{1/2}\left[1-\frac{\partial^2E'}{\partial\mu\,\partial n_0}\right]=n_0^{1/2}\left[1-\left(\frac{\partial\mu_0}{\partial\mu}\right)_{n_0}\right]$$
$$=n_0^{1/2}\left[1+\left(\frac{\partial n'}{\partial n_0}\right)_\mu\right], \tag{8}$$

$$\Sigma_{12}(0)=n_0\frac{\partial^2E'}{\partial n_0^2}=n_0\left(\frac{\partial\mu}{\partial n_0}\right)_\mu \tag{9}$$

[see (1)];

$$\Gamma_{+--}(0,0)=\Gamma_{-++}(0,0)=\frac{1}{n_0^{1/2}}\frac{\partial}{\partial n_0}\left(n_0\frac{\partial^2E'}{\partial n_0^2}\right),$$
$$\Gamma_{+++}(0,0)=\Gamma_{---}(0,0)=n_0^{3/2}\frac{\partial^3E'}{\partial n_0^3}, \tag{10}$$

$$g_{11}(0,0)=1-\frac{\partial^2}{\partial\mu\,\partial n_0}\left(n_0\frac{\partial E'}{\partial n_0}\right)=1-\left(\frac{\partial\mu_0}{\partial\mu}\right)_{n_0}-n_0\frac{\partial^3E'}{\partial\mu\,\partial n_0^2},$$
$$g_{12}(0,0)=-n_0\frac{\partial^3E'}{\partial\mu\,\partial n_0^2}. \tag{11}$$

3. We now write down the exact equations for π, Λ, and Σ, and separate in the skeleton diagrams the loops, i.e., the pairs of exact Green's functions adjacent to the potential line (Fig. 2; the broken line corresponds

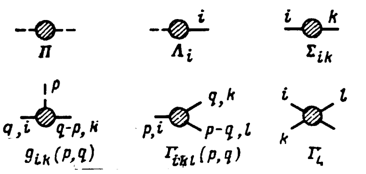

FIG. 1.

to a condensate particle). By virtue of (4), at zero external momentum, the loop introduces into the integrand a diverging factor $G^2(q)\sim 1/q^4$, which leads to a logarithmic divergence. Summing the diagrams over the directions of the ends of the Green's functions in the region of small q and taking (4) into account, we get

$$\Pi(0)\sim 2i\int G^2(q)\,[g_{12}(0,0)-g_{11}(0,0)]\,d^4q, \tag{12}$$

$$\Lambda_2(0),\ \frac{1}{n_0^{1/2}V_0}\Sigma_{12}(0)\sim 2i\int G^2(q)\,[\Gamma_{+--}(0,q)-\Gamma_{+++}(0,0)]\,d^4q. \tag{13}$$

In all cases the coefficient of $G^2(q)$ turns out to be connected with $\Sigma_{12}(0)$. For $\Lambda_i(0)$ and $\Sigma_{12}(0)$ this is seen directly from (9) and (10):

$$\Gamma_{+--}(0,0)-\Gamma_{+++}(0,0)=2n_0^{-1/2}\Sigma_{12}(0), \tag{14}$$

For $\pi(0)$ this is seen from (11) when account is taken of the connection between the derivatives $(\partial\mu_0/\partial\mu)_{n_0}$ and $(\partial\mu_0/\partial n_0)_\mu=\Sigma_{12}(0)/n_0$ [(1) and (9)]:

$$\frac{d\mu_0}{d\mu}=1=\left(\frac{\partial\mu_0}{\partial\mu}\right)_{n_0}+\left(\frac{\partial\mu_0}{\partial n_0}\right)_\mu\frac{dn_0}{d\mu}$$

i.e.,

$$1-\left(\frac{\partial\mu_0}{\partial\mu}\right)_{n_0}=\frac{1}{n_0}\Sigma_{12}(0)\frac{dn_0}{d\mu} \tag{15}$$

(the direct derivative correspond to physical changes of the parameters); hence

$$g_{11}(0,0)-g_{12}(0,0)=\frac{1}{n_0}\frac{dn_0}{d\mu}\Sigma_{12}(0). \tag{16}$$

The equality on Fig. 2c can be regarded as an exact equation from which it follows that

$$\Sigma_{12}(0)=0 \tag{17}$$

(this result is due to the divergence of the loop in the third term and the absence of divergences in the others). From (17) follow also exact equations (and a finite character) for $\pi(0)$ and $\Lambda_i(0)$:

$$\Lambda_i(0)=0 \tag{18}$$

(see (8) and (15))

$$\Pi(0)=-\frac{dn}{d\mu} \tag{19}$$

—see (7) and (15), with allowance for the equalities

$$\frac{dn}{d\mu}=\left[1+\left(\frac{\partial n'}{\partial n_0}\right)_\mu\right]\frac{dn_0}{d\mu}+\left(\frac{\partial n'}{\partial\mu}\right)_{n_0},$$
$$\left(\frac{\partial n'}{\partial n_0}\right)_\mu=-\frac{\partial^2E'}{\partial\mu\,\partial n_0}=-\left(\frac{\partial\mu_0}{\partial\mu}\right)_{n_0}.$$

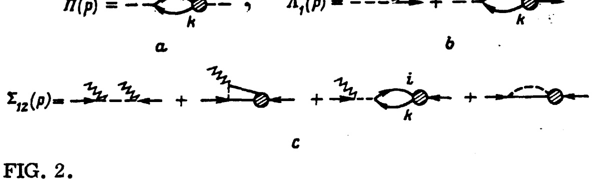

FIG. 2.

Using the identities

$$\Sigma_{12}(0)=\bar{\Sigma}_{12}(0)+V_0\bar{\Lambda}_1(0)V_0\Lambda_2(0),$$
$$\Lambda_i(0)=\bar{\Lambda}_i(0)/(1-V_0\Pi(0)),$$

we obtain also

$$\bar{\Sigma}_{12}(0)=\bar{\Lambda}_i(0)=0 \qquad (20)$$

(the bars over Σ, Λ, and π indicate irreducibility of the set of diagrams with respect to the potential line).

3. ANALYSIS OF DIVERGENCES. LIMITS OF APPLICABILITY OF ORDINARY SUMMATION METHODS

1. The disparity between the exact equations (17)–(20) and the results of field perturbation theory is easiest to trace with a model for which perturbation theory has been directly intended—a system with weak interaction $\xi = mp_0V \ll 1$ (V and p_0 are the characteristic values of the Fourier component of the potential and of the momentum transfer p, while ξ is the ratio of the average potential energy to the characteristic kinetic energy $(Vp_0^3/p_0^2m^{-1})$. Up to high density $n/p_0^3 \equiv \beta^2 \sim 1/\xi$, a simple diagram-selection rule holds: each integration with respect to the intermediate momentum introduces a small factor; the zeroth approximation is that of Bogolyubov $\Sigma_{12}^{B}(0)=n_0V_0$, $\Lambda^{B}(0)=n_0^{1/2}$, the first is given by diagrams with single integration

$$\Sigma_{12}^{(1)}=\bar{\Sigma}_{12}^{(1)}+n_0^2V_0^2\Pi_0, \quad \Lambda^{(1)}=\bar{\Lambda}^{(1)}$$

and so forth (see Fig. 3).

The exact equations (17), (18), and (20) signify that one must not trust the converging lowest-order approximations in ξ: the divergence in the remaining diagrams cancels out their formal smallness. Thus, for example, replacement of one of the zero-order vertices of the diverging sum of diagrams $\Sigma_{12}^{(0)}(0)$ by the exact one

$$\Sigma_{12}^{(1)}(0)\to\Sigma_{12}(0)-\Sigma_{12}^{B}(0)=-n_0V_0$$

with simultaneous elimination of the divergence eliminates likewise the small factor ξ. Even more unexpected are the cases $\Lambda(0)$ and $\bar{\Sigma}_{12}(0)$: although the loop $\Lambda^{(1)}(0)\sim\xi n_0^{1/2}$ converges, replacement of one of its zeroth vertices by the exact one $\Lambda^{(1)}(0)\to\Lambda'(0)$ also changes its order in $\xi(\Lambda'(0)=-\Lambda^{B}(0)=-n_0^{1/2})$; a similar replacement in the converging diagrams $\bar{\Sigma}_{12}^{(1)}(0)$ yields zero ($\bar{\Sigma}_{12}^{(1)}(0)\to\bar{\Sigma}_{12}(0)=0$), and in the converging diagrams $\bar{\Sigma}_{12}^{(2)}$ it changes the order with respect to $\xi(\bar{\Sigma}_{12}^{(2)}(0)\to-\bar{\Sigma}_{12}^{(1)}(0)$; the divergence of the diagrams $\Lambda(0)$ arises in second order, that in $\bar{\Sigma}_{12}(0)$ arises in the third ($\Lambda^{(2)}(0)$, $\bar{\Sigma}_{12}^{(3)}(0)$). The diagrams of the density–density reaction function, which are irreducible in the lines of the particles and of the potential, $\bar{\pi}(p)$, diverge even in the lowest (first) approximation $\Pi^{(1)}(p)\equiv\Pi_0(p)\sim(\xi/V_0)\ln(p/p_0)$ (as against the exact value

$$\Pi(0)=-n/m(c^2-c_B^2), \quad c_B^2=nV_0/m,$$

see (5) and (19)); the analogous irreducible diagrams of the current–current functions $\bar{F}_{33}(k)$, used in Ref. 11 to calculate the spectrum, diverge in third order in ξ (Fig. 3; in the zeroth vertices of the $\bar{F}_{33}(k)$ loop there are factors $(\mathbf{p}\cdot\mathbf{k}/k+k/2)$ on top of those in the $\pi_0(k)$ loop).

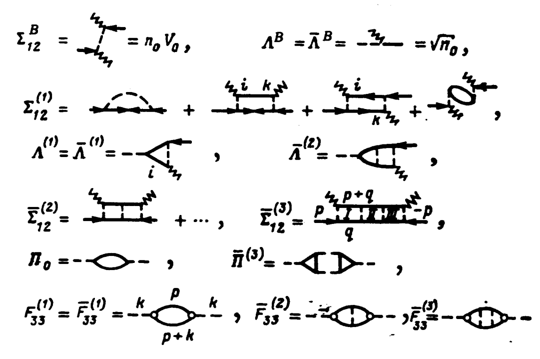

FIG. 3.

2. In the general case, the analysis of the character of the infrared divergence is best started with an examination of diagrams whose all external momenta tend to zero: $p_i\sim p\to 0$. Let us estimate in the diagram $M(r,s)$ (r is the number of external particle lines and s is the number of potential lines) the contribution of the region in which all the intermediate momenta are of the same order as the external ones ($\sim p$); setting each of the

$$[(3n_3+4n_4-r)/2]+s$$

internal lines $G(p)\sim 1/p_2$ in correspondence with each of the

$$[(3n_3+4n_4-r)/2]-(n_3+n_4-1)$$

integrations p^4, we get

$$M(r,s)\sim(p_0/p)^{n_3+r+2s-4} \qquad (21)$$

(n_3 and n_4 are respectively the numbers of 3- and 4-line vertices). The presented global calculation of the degree of divergence $x(M)$

$$x(M)=n_3+r+2s-4 \qquad (22)$$

would not be exact if there existed subdiagrams M' with negative degree of divergence $x'=n_3'+r'+2s'-4<0$—the contribution of such diagrams is small: $M'\sim p^{|x'|}$ for only small intermediate momenta, so that the true divergence of M would be larger than indicated in (22). No such subdiagrams exist, however: the cases $s'=0$ and $r'=1$ are not allowed; in the cases $s'=0$, $r'=3$ and $s'=1$, $r'=1$ we have automatically $n_3'>0$.

To calculate the degree of divergence of a diagram N with vertex momenta that are not small, we separate in the diagram the subdiagrams M_k (which are not connected with one another) in which all the momenta, including the external ones, can be arbitrarily small; taking into account the integration over the external momenta of the diagrams M_k, we get

$$x(N)=\sum_k[x(M_k)+2r_k-4(r_k+s_k-1)]=\sum_k(n_3^{(k)}-r_k-2s_k). \qquad (23)$$

Account was taken here of the fact that the external momenta of each subdiagram M_k are connected with the external momenta of the diagram N only by the summary conservation law, and not by any other equations—such equations would not allow them to be arbitrarily small.

A special case is when the external momenta of the diagram R, which are not small, add up to a sum that

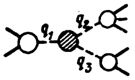

FIG. 4.

tends to zero, $q_j \to 0$ ($j=1,\ldots m$); The number of independent external momenta of M_k (and hence the number of integrations) can be smaller than (r_k+s_k-1). Diagrams of this kind have a maximum divergence when the momenta $q_j \to 0$ "go over" to the potential lines (Fig. 4):

$$\max x(R) = n_3 + 2m - 4. \tag{24}$$

3. A qualitative change in the character of the divergence of the diagrams is caused by summation over the directions of the internal particle lines. We demonstrate this first within the framework of the global approach. Summation over the directions causes one of the lines of each 3-line vertex (excluding the vertices of the type of Fig. 5a) to correspond to the sum $G_{11}+G_{12} \sim 1/\Delta\Sigma(p) \sim 1/\ln p$ (see (42)) instead of the quadratic divergence $G_{ik} \sim 1/p^2$; to be sure, such a line may turn out to be common to two 3-line vertices. Thus, each of the $n_3' = (n_3 - \tilde{n}_3)$ three-line vertices must be assigned, in addition to (21)–(24), at least a factor p ($\tilde{n}_3$ is the number of 3-line vertices of the type 5a), i.e.,

$$\begin{aligned} &\max x(M) = \tilde{n}_3 + r + 2s - 4,\\ &x(N) < 0,\\ &\max x(R) = 2m - 4. \end{aligned} \tag{25}$$

In the derivation of (25), however, no account was taken of the possible unevenness of the distribution, in the integrals, of the small factors that arise in the summation over the directions—in some integrals the divergence may be cancelled out with a margin to spare, and in others it may still remain. Thus, despite (25), a logarithmic divergence (mentioned in Item 1 of the present section) is contained in $\Sigma_{12}^{(3)}$ and $\bar{\Lambda}^{(2)}$ (Fig. 3). Even the addition of a small factor in each zero vertex (e.g., in the model with $V_{p\to 0} \sim p^{\alpha}$, $\alpha \geq 1$) would not eliminate the divergences, and, in particular, a divergence $\propto \ln p$ would remain as before in the central link II of $\bar{\Sigma}_{12}^{(13)}(p)$ (Fig. 3); this divergence not integrable in links I and III.[3)]

On the face of it, the difficulty with the uneven distribution of the small factors makes the global calculation of max x entirely valueless. This is not so: the only source of the difficulty—the subdiagrams with negative divergence ($x' < 0$)—can be effectively excluded. There were no subdiagrams with $x' < 0$ in the derivation of (22)–(24); after summing over the directions, two possibilities arise: $s'=0$, $r'=3$ ($\Gamma_3^{(1)}$) and $s'=1$, $r'=1(\Lambda^{(1)})$, for which now $x=-1$ (Fig. 6). The "global" calculation of max x is justified if the diagrams $\Gamma_3^{(n)}$ and $\tilde{\Gamma}_3^{(n)}$ (Fig. 6) are regarded as single

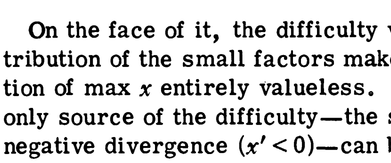

FIG. 5.

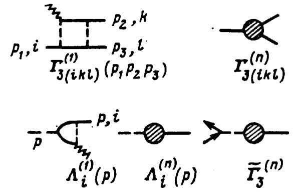

FIG. 6.

effective 3-line vertices. Let ascertain whether such vertices introduce, like the zero vertices, a small factor $\sim p$. As a result of summation over the directions, the product of the three Green's functions $G(p_i) \sim 1/p_i$ corresponding to the lines of the diagram $\Gamma_3^{(n)}$ will enter with a coefficient

$$\Gamma_3^{(n)} = \sum_{i,k,l=\pm 1} ikl\Gamma_{3(ikl)}^{(n)}(p_1p_2p_3).$$

The terms in this sum can be grouped in such a way that vertices $\Gamma_{3(ikl)}^{(n)}$ with opposite directions of all the lines are added pairwise:

$$\Gamma_3^{(n)} = \frac{1}{2}\sum ikl[\Gamma_{3(ikl)}^{(n)} - \Gamma_{3(-i,-k,-l)}^{(n)}],$$

and in this sum each of the diagrams from $\Gamma_{3(ikl')}^{(n)}$ is set in correspondence with a diagram $\Gamma_{3(-i,-k,-l)}^{(n)}$ with the directions of all the internal lines reversed. Thus, the coefficient $\Gamma_3^{(n)}$ is equal to the difference of two integrals whose integrands differ only in the sign of the arguments of the Green's functions:

$$\Gamma_3^{(n)} = F\{G(p_i+q)\} - F\{G(-p_i-q)\};$$

this means that $\Gamma_3^{(n)}$ contains only terms with odd powers of ε_i ($p_i \equiv \mathbf{p}_i, \varepsilon_i$), multiplied by expressions having the same character of divergence as $\Gamma_{3(ikl)}^{(n)}$, (as will be shown later, the divergence is not higher than logarithmic). A similar structure is possessed by the vertex $\Gamma_3^{(n)}$ after summing over the particle-line directions in $\Lambda^{(n)}$:

$$\Gamma_3^{(n)} \sim V_p(\Lambda^{(n)}(p) - \Lambda^{(n)}(-p)).$$

Thus, each internal effective 3-line vertex introduces at the very least a factor $p(\ln p)^m$; this justifies the result (25), with the only refinement that the number $\tilde{n}_3$ must include any external vertex (as an effective 3-line vertex—Figs. 5a and 5b), excluding only the cases of Figs. 5c and 5d (a 3- or 4-line vertex is connected to a diagram that is irreducible in the two horizontal lines of the particles GG). The foregoing explains, in particular, why $\bar{\Sigma}_{12}^{(1),(2)}$ and $\bar{\Lambda}^{(1)}$ have no divergences but $\bar{\Sigma}_{12}^{(n\geq 2)}$ and $\bar{\Lambda}^{(n\geq 2)}$ have them.

The refined criterion (25) points to the following cases of complete cancellation of the divergences: diagrams without external lines (e.g., $\langle H_{\rm int}\rangle - \tilde{n}_3 = r = s = 0$, $x = -4$); diagrams N with external momenta (or their sum) that are not small ($p_i \gg p_1$; p_1 is the momentum at which the divergence cancels out the small parameter); diagrams M with small external momenta, with $\tilde{n}_3 = 0$, $r = 2, 3$ ($\tilde{n}_3 = 0$ denotes irreducibility in GG) or $\tilde{n}_3 = r = 0$, $s = 1$ (n', see (2)).

It must be added to the result (25) that in the diagrams with $x=0$ the divergences are not quite identical: $\ln p$ can enter raised to different powers (e.g., $\Pi_0 \sim \ln p$, $\Pi^{(3)} \sim (\ln p)^2$); this makes it possible to eliminate the divergences by effective summation.

The importance of the remarks made concerning formulas (25) is emphasized by the fact that without these remarks Eqs. (25) would contradict the exact equations: independence of (25) of the internal structure of the diagrams would mean covergence of the diagrams $\bar{\Sigma}_{12}(0)$ and $\bar{\Lambda}(0)$ and divergence of $\Pi(p)\sim\ln p$:

$$\Sigma_{12}^{(n)}\sim\xi^{n-1}\Sigma_{12}^{(1)},\quad \bar{\Lambda}^{(n)}\sim\xi^{n-1}\bar{\Lambda}^{(1)},\quad \Pi^{(n)}\sim\xi^{n-1}\Pi_0(p)\sim\xi^n\ln p,$$

in contradiction to (17)–(20)

$$\Sigma_{12}(0)=\bar{\Lambda}(0)=0,\quad \Pi(0)=-n/m\left(c^2-\frac{nV_0}{m}\right);$$

it would then follow from the skeleton equation

$$\Sigma_{12}(p)=\bar{\Sigma}_{12}+\bar{\Lambda}_1 V(1-V\bar{\Pi})^{-1}\bar{\Lambda}_2$$

that $\Sigma_{12}(0)$ also contains only converging diagrams ($\Sigma_{12}(0)=\bar{\Sigma}_{12}(0)$), the skeleton equation on Fig. 2c notwithstanding. It can be shown that in the case $T>0$ (or a two-dimensional Bose system with $T=0$), where x includes $3n_3/2+n_4$ (rather than n_3—see Sec. 4), the result (25) without the indicated remarks would formally be proved ($V_{p\to 0}\to V_p(1-V_p\Pi(p))^{-1}\sim-\Pi(p)^{-1}\sim-\Pi_0(p)^{-1}\sim p$), leading thereby to contradictions to the exact equations.

4. It may appear that a ladder made up of exact Green's functions and leading to a nonintegrable logarithmic divergence in diagrams with small external momenta (e.g., $\bar{\Sigma}_{12}^{(3)}(p\to 0)$ makes a diverging contribution (actually, infinity) in the diagrams $\langle H_{\rm int}\rangle$, n', and N (Fig. 7). This is not the case—the refinement of (25) eliminates completely the difficulty with the nonuniformity of the small factors; the diagrams of Fig. 7 are forbidden in a technique with exact Green's functions—would lead to a redundant calculation. The convergence of the diagrams $\langle H_{\rm int}\rangle$ can also be verified by the representation of $\langle H_{\rm int}\rangle$ in Fig. 8a.

The absence of divergences not only permits calculation of $\langle H_{\rm int}\rangle$, n', and other quantities in arbitrary order in ξ for the model with $\xi\ll 1$ and $\beta^2\leqslant 1/\xi$, but also the partial summation that is dictated by the character of the small parameter (e.g., a ladder with the functions G_0 for a low-density system $\beta\ll 1$—Fig. 8b; summation of the ladders corresponds to replacement of V_0 of the lowest-order approximation in $\xi\ll 1$ by $4\pi a/m$, where a is the scattering amplitude in vacuum). At low dispersion, calculation of the speed of sound with the aid of $\Sigma_{ik}(p\gg p_1)$ coincides with a calculation by formula (5).

4. FIELD THEORY FOR TWO-DIMENSIONAL BOSE SYSTEM AND FOR A THREE-DIMENSIONAL ONE AT $T>0$.

For a two-dimensional Bose system, the integration with respect to the momenta and frequency, in the global calculation of the degree of divergence is introduced by the factor $\sim p^3$. We obtain accordingly in

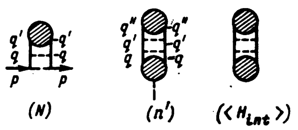

FIG. 7.

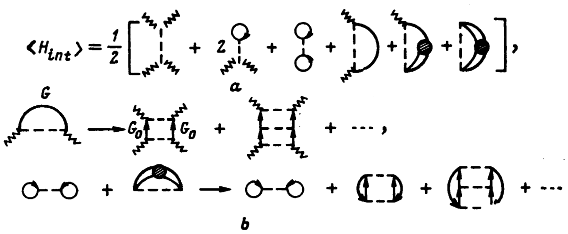

FIG. 8.

place of (22)–(24)

$$x(M)={}^3/{}_2 n_3+n_4+{}^1/{}_2 r+2s-3,$$

$$x(N)=\sum_k[x(M_k)+2r_k-3(r_k+s_k-1)]=\sum_k({}^3/{}_2 n_3+n_4-{}^1/{}_2 r-s),\qquad(26)$$

$$x(R)={}^3/{}_2 n_3+n_4+2m-3.$$

The same formulas are valid if the 3- and 4-line vertices are effective (contain an internal structure) but are finite at zero momenta of the lines. To cause the growing divergences to vanish, we carry out a regularization procedure analogous to some degree to the elimination of the ultraviolet divergence in renormalizable relativistic field theories. We describe this theory first without taking into account the contribution of the three-line vertices to the divergence. We supplement the Hamiltonian $\hat{H}$ with the term $\hat{H}_1$:

$$\hat{H}'=\hat{H}+\hat{H}_1,$$

$$\hat{H}_1=(-\Gamma)\sum_{p_i<P}(a_{p_1}^+a_{p_2}^+a_{p_3}^+a_{p_4}^+ + a_{p_1}^+a_{p_2}^+a_{p_3}^+a_{-p_4}+\ldots+a_{-p_1}a_{-p_2}a_{-p_3}a_{-p_4}),\qquad(27)$$

in which Γ is the sum of all the 4-line vertex diagrams with nonzero external momenta, averaged over the line directions:

$$\Gamma=\sum_{n=0}^{\infty}\Gamma_4^{(n)}/2^4\cdot 4!,\quad \Gamma_4^{(n)}\sim\xi^n$$

(ξ is a dimensionless interaction parameter and is the formal field diagram expansion for any system); we assume the limiting momentum P to be much less than the characteristic momentum transfer p_0, but large enough to keep the effects of infrared growth of the diagrams from setting in ($\xi p_0\ll P\ll p_0$). The diagram series for H' differs from the series for $\hat{H}$ in that in its diagrams one subtracts from all the zeroth 4-line vertices with $p_i\leqslant P$ one subtracts the values of these vertices at $p_0=0$ (the contribution of $\hat{H}_1$ of zeroth order in ξ; this ensures convergence of the first-order 4-line vertices; from all the 4-line vertices of first order one subtracts their values at $p_i=0$—the contribution of H_1 of first order; this results in convergence of the second-order vertices, and so forth. As a result, all the effective 4-line vertices of the diagrams no longer contribute to the degree of divergence (26). It is obvious that in the definition of $\hat{H}_1$ it is implied that the coefficients $\Gamma_4^{(n>0)}$ are assumed to be regularized (i.e., in the vertices of lower order with $p_i<P$ a subtraction was carried out of their values at $p_i=0$); thus, in contrast to the relativistic theories, the counterterms are finite. Even more peculiar to this case is that actually $\hat{H}_1=0$, i.e., we are dealing not with a change of the initial Hamiltonian, but only with a refinement of the meaning of the expansion in ξ. In fact, the coefficient in the sum of the counterterms of $\hat{H}_1$ is propor-

tional to the total vertex for the scattering of two phonons: $\Gamma_4(p_i\to 0)$ (Fig. 1), for which it is easy to prove, in analogy with the procedure used in Sec. 2, that

$$\Gamma_4(p_i\to 0)=12n_0^{-1}\Sigma_{12}(0);$$

on the other hand, the sum of the regularized diagrams $\Sigma_{12}(p)$ satisfies the skeleton equation on Fig. 2c, from which it follows that $\Sigma_{12}(0)=0$.

We now take into account the 3-line vertices. In the two-dimensional case, interest attaches both to the factor

$$\Gamma_3^{(n)}=\sum_{i,k,l=\pm 1} ikl\Gamma_{3(ikl)}^{(n)}$$

(the coefficient of the product of the pole terms of three Green's functions $G(p_1)G(p_2)G(p_3)$, and to the factor

$$\Gamma_{(1)3}^{(n)}=2\sum_{k,l} kl\Gamma_{3(1kl)}^{(n)}$$

(the coefficient of the product

$$\Delta G(p_1)G(p_2)G(p_3),\quad G_{ik}(p)=(-1)^{i+k}G(p)+\Delta G,$$

see (42)). We verify first that

$$\Gamma_3^{(n)}={}^1/_2[\Gamma_{(1)3}^{(n)}-\Gamma_{(-1)3}^{(n)}]$$

does not contain terms of either zeroth or first order in the momenta. This follows from the equations

$$\Gamma_{(1)3}^{(n)}(0,p,-p)=\Gamma_{(-1)3}^{(n)}(0,p,-p)=n_0^{1/2}\frac{\partial}{\partial n_0}[\Sigma_{11}^{(n)}(p)$$
$$+\Sigma_{11}^{(n)}(-p)-2\Sigma_{12}^{(n)}(p)]\approx 2n_0^{-1/2}\Sigma_{12}^{(n)}(0)+O(p^2),\qquad (28)$$

if it is recognized that the function $\Gamma_3^{(n)}(p_1p_2p_3)$ is symmetrical in the arguments. It is clear therefore that in terms of the type $\Gamma_3^{(n)}G(p_1)G(p_2)G(p_3)$ the contribution of $\Gamma_3^{(n)}$ to the degree of divergence has been completely eliminated. We consider now terms of the type

$$\Gamma_{(1)3}^{(n)}\Delta G(p_1)G(p_2)G(p_3)\Gamma_{(1)3}^{(n')}G(p_4)G(p_5)$$

(Fig. 9; $\Sigma_{ik}G_{ik}(p_1)=4\Delta G(p_1)$, see (42)); if $\Gamma_{(1)3}^{(n)}(0)\neq 0$, then the additional factor $\sim p_1^2$, due to replacement of G by ΔG, is insufficient to eliminate the contribution of two 3-line vertices to the degree of divergence (26); we therefore apply a regularization procedure similar to the preceding one, and supplement the Hamiltonian with a three-operator expression

$$H_2=-\Gamma_{(1)}\sum_{p_i<P}(a_{p_1}^+a_{p_2}^+a_{p_3}^+ + a_{p_1}^+a_{p_2}^+a_{-p_3}+\ldots+a_{-p_1}a_{-p_2}a_{-p_3}),$$

where

$$\Gamma_{(1)}=\sum_n \Gamma_{(1)3}^{(n)}/2^3\cdot 3!.$$

Just as in the case (27), this does not mean a change in the initial Hamiltonian, since

$$\Gamma_{(1)}=\frac{2n_0^{-1/2}}{2^3\cdot 3!}\Sigma_{12}(0)=0$$

(see (28)). This solves the problem of eliminating in the two-dimensional case the infrared divergence that increase with the order of the approximation.

We note that the skeleton equation for $\Sigma_{12}(p)$ does not

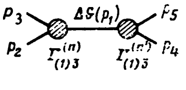

FIG. 9.

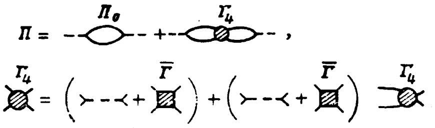

FIG. 10.

change when counterterms are introduced—the vertices used here are exact, and consequently their correction should be effected by an exact sum of the contributions of the counterterms, which vanishes precisely as a result of the skeleton equation.

The described approach is valid in the case of a three-dimensional Bose system at $T>0$ (the integration with respect to frequency is accompanied by an additional factor

$$n(\varepsilon)=1/(e^{\varepsilon/T}-1)\sim T/\varepsilon\sim T/cp\quad(\varepsilon\sim cp\ll T)).$$

5. EFFECT OF SUMMATION OF DIAGRAMS WITH INFRARED DIVERGENCES

The region of validity of field perturbation theory can be greatly expanded by using the effective summation dictated by the character of the divergence (in particular, the exact equalities of Sec. 2); it thus becomes possible to calculate quantities for which a naive application of the field theory is impossible or leads to an error. With the aid of (19) and (5) we obtain[4]

$$\Pi(0)=-\frac{n}{mc^2},\quad \Pi(0)=-\frac{n}{m(c^2-c_B^2)}\quad\left(c_B^2=\frac{nV_0}{m}\right).\qquad(29)$$

Assuming that the function $\Gamma(p_i\to 0)$ is finite (Fig. 10) and accordingly replacing the integrals of Γ with pairs of Green's functions by the product of $\Gamma(0)$ and logarithmically diverging polarized loops, we get

$$\Pi(p\to 0)\approx\frac{\Pi_0(p\to 0)}{1-[V_p-\Gamma(0)]\Pi_0(p\to 0)}\to-\frac{1}{V_0+\Gamma(0)},$$

whence, taking (29) into account,

$$\Gamma(p_i=0)=-V_0-\frac{1}{\Pi(0)}=\frac{m}{n}(c^2-c_B^2),\qquad(30)$$

$$\Pi(p\to 0)\approx\Pi_0(p)\Big/\left(1-\frac{mc^2}{n}\Pi_0(p)\right)\qquad(31)$$

(Γ is the 4-line vertex summed over the direction of the lines, reducible with respect to the potential line and with respect to one (G) and two (GG) lines of particles in the horizontal direction:

$$\Gamma=\sum_{i,k,l,m=\pm 1} iklm\Gamma_{iklm},$$

where i, k, l, and m are indices of the line directions).

To calculate $\Sigma_{12}(p\to 0)$ we use the identity

$$\gamma(p,0)=2n_0^{-1/2}\Sigma_{12}(p),\qquad(32)$$

which can be easily obtained by the method used in Sec. 2; $\gamma=\Sigma_{ik}ik\Gamma_{ik1}$, and Γ_{ik1} is a 3-line vertex irreducible in G.

We break up the aggregate of the diagrams γ into irreducible ones γ_0 and reducible ones in the pairs of the Green's functions GG (Fig. 11a). The diagrams γ_0 have no divergences: $\bar{n}_3=0$, $r=3$, $s=0$; $x=-1$ (see Sec. 3). From among the diagrams with pairs of Green's functions (A, B, C), only C has a singularity

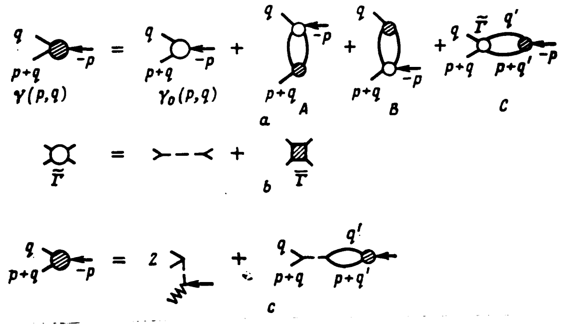

FIG. 11.

at small momenta in the integrand of $\int GG\Gamma_3\Gamma_4$, since summation over the directions of all the lines of Γ_3 eliminates the singularity in the diagrams A and B (the vertex Γ_4 in diagrams A and B is characterized by a divergence not higher than logarithmic: $\bar{n}_3=0$, since a line with momentum $(-p)$ can not be added with the aid of a zeroth 3-line vertex of the type of Fig. 5a and 5b—the corresponding diagrams are already accounted for in C).

If the vertex $\tilde{\Gamma}$ (Figs. 11b) is approximated by its value at zero momenta $\tilde{\Gamma}=-1/\Pi(0)=mc^2/n$, then the equation on Fig. 11a

$$\gamma=(\gamma_0+A+B)+\int \Gamma GG\gamma \tag{33}$$

is transformed into a closed integral equation for the calculation of the function γ, using field perturbation theory without divergences (for γ_0, A, B, and c^2), and in particular by using partial summation (ladders for a low-density system).

For the model with $\xi\ll 1$ and $\beta\sim\xi^{-1/2}$, in the lowest perturbation theory order, the equation for γ takes the form shown in Fig. 11c; the right-hand side of the equation is independent of q, so that the integral equation reduces to an algebraic one. We obtain

$$\gamma(p,q)=\gamma(p,0)=2n_0^{1/2}V_0+V_0\Pi_0(p)\gamma(p,0), \tag{34}$$

i.e.,

$$\gamma(p,q)=\gamma(p,0)=2n_0^{1/2}V_0/(1-V_0\Pi_0)); \tag{35}$$

with (32) taken into account, we get

$$\Sigma_{12}(p)=n_0V_0/(1-V_0\Pi_0(p)),$$

$$V_0\Pi_0(p)\sim\begin{cases}\xi\ln\left(\dfrac{p}{p_0}\right) & (T=0),\\ -\xi\left(\dfrac{p_0}{p}\right) & (T>0 \text{ or a two-dimensional system at } T=0)\end{cases} \tag{36}$$

In the general case, assuming a weak dependence of the right-hand side of the equation on q as $q\to 0$, we obtain by replacing the integral equation (33) by an algebraic one

$$\gamma(p,q)\approx\gamma(p,0)\approx\frac{\gamma_0+A+B}{1-(mc^2/n)\Pi_0(p)}\approx\frac{2}{n_0^{1/2}}\Sigma_{12}(p).$$

6. ASYMPTOTIC FORMULAS FOR THE GREEN'S FUNCTIONS

1. Proceeding to the derivation of the formulas for G_{ik}, $F_{\mu\nu}$, and c^2 with allowance for (17), we note first that (17) does not mean that the sound velocity vanish, as might appear from a comparison of (4) with (37):

$$G_{11}(p\to 0)=-G_{12}(p\to 0)=\frac{\Sigma_{12}(0)}{B(\varepsilon^2-c^2p^2+i\delta)} \tag{37}$$

(see Ref. 17), since $B=0$, i.e., (37) is indeterminate). Actually, using the equality[7]

$$\frac{\partial\Sigma_{11}(0)}{\partial\varepsilon}=-\left(\frac{\partial n'}{\partial n_0}\right)_\mu,$$

as well as (1), (2), (15), and (17), we get

$$\frac{\partial\Sigma_{11}(0)}{\partial\varepsilon}=\frac{\partial^2E'}{\partial\mu\,\partial n_0}=\left(\frac{\partial\mu_0}{\partial\mu}\right)_{n_0}=1-\frac{1}{n_0}\Sigma_{12}(0)\frac{dn_0}{d\mu}=1. \tag{38}$$

Substitution of (38) in the definition of B (see Ref. 17) yields $B=0$.

An important role in the expansions of $\Sigma_{ik}(p)$ in the 4-momentum p, which are needed for the derivation of the formulas for $G_{ik}(p\to 0)$, is played (because of the vanishing of the zeroth term of $\Sigma_{12}(p)$) by the nonanlytic terms of $\Delta\Sigma_{ik}$; by virtue of the relations of Ref. 7, we have for $\Sigma_{11}(p)-\Sigma_{12}(p)$

$$\Delta\Sigma_{11}=\Delta\Sigma_{12}(=\Delta\Sigma(p)) \tag{39}$$

(for the explicit form of $\Delta\Sigma(p)\approx\Sigma_{12}(p)$ see Sec. 5). Substituting in Belyaev's general formulas[2]

$$G_{11}=\varepsilon+\varepsilon_p{}^0-\mu+\Sigma_{11}(-p)/Z,$$
$$G_{12}=-\Sigma_{12}(p)/Z, \tag{40}$$
$$Z=\left(\varepsilon-\frac{\Sigma_{11}(p)-\Sigma_{11}(-p)}{2}\right)^2-\left(\varepsilon_p{}^0-\mu+\frac{\Sigma_{11}(p)+\Sigma_{11}(-p)}{2}\right)^2+\Sigma_{12}{}^2(p)$$

the expansion of $\Sigma_{ik}(p)$ with (17), (38), and (39) taken into account

$$\Sigma_{11}(p)=\mu+\varepsilon+\Delta\Sigma(p)+a\varepsilon^2+bp^2+\ldots,$$
$$\Sigma_{12}(p)=\Delta\Sigma(p)+a_1\varepsilon^2+b_1p^2+\ldots \tag{41}$$

and recognizing that $\Delta\Sigma(p)\gg\varepsilon^2$ and $\Delta\Sigma(p)\gg p^2$ [see (36)], as well as that[7]

$$a-a_1=-\frac{1}{2n_0}\left(\frac{\partial n'}{\partial\mu}\right)_{n_0},\quad b-b_1=\frac{n'}{2n_0m},$$

we get

$$G_{ik}(p\to 0)=(-1)^{i+k}\frac{n_0mc^2}{n(\varepsilon^2-c^2p^2+i\delta)}-\frac{1}{4\Delta\Sigma(p)} \tag{42}$$

(cf. (4)), with

$$c^2=n/m\left(\frac{\partial n'}{\partial\mu}\right)_{n_0}. \tag{43}$$

Comparison of (7) and (19) yields

$$\left(\frac{\partial n'}{\partial\mu}\right)_{n_0}=\frac{dn}{d\mu},$$

so that (43) agrees with (5). It is remarkable that the statistical susceptibility $F_{44}(\varepsilon=0,\ p\to 0)=-dn/d\mu$ coincides with the analogous characteristic of a system of particles in excess of the condensate at a fixed number of particles in the condensate.

In analogy with the derivation of (43), we can verify the formulas[7] for $F_{\mu\nu}$. It is interesting to note that the function $G_{11}(p)-G_{12}(p)$ (the pole term of the functions G_{ik}) can be expressed exactly (without cancelling out $\Sigma_{12}(0)$) in a form from which it is clear that it is independent of $\Sigma_{12}(0)$ in the lowest order in p and ε:

$$[G_{11}(p)-G_{12}(p)]^{-1}=\varepsilon-\varepsilon_p{}^0+\mu-[\Sigma_{11}(p)-\Sigma_{12}(p)]-\frac{\Sigma_{12}(p)\{2\varepsilon-\Sigma_{11}(p)+\Sigma_{11}(-p)\}}{\varepsilon+\varepsilon_p{}^0-\mu+\Sigma_{11}(-p)+\Sigma_{12}(p)}$$
$$\approx\frac{n}{2n_0mc^2}(\varepsilon^2-c^2p^2)-\{\Sigma_{12}(0)[(dn_0/d\mu)^2(\varepsilon/n_0)^2\times[\Sigma_{12}(p)-\Sigma_{12}(0)]+O(\varepsilon^3,\varepsilon p^2)]\}[\Sigma_{12}(p)+O(\varepsilon^2,p^2)]^{-1}.$$

This explains why the results of a calculation essentially based on the erroneous assumption $\Sigma_{12}(0) \neq 0$ agrees with the correct results.

2. All the exact results given in Sec. 2 were based on allowance for the divergence of an integral of two exact Green's functions, in contact with the line of a potential with zero momentum. We indicate also another (more lucid) derivation of these relations: from the formula

$$\frac{dn}{d\mu} - \frac{n}{mc^2} = \frac{(1+\partial n'/\partial n_0)^2}{\partial^2 E'/\partial n_0^2} + \frac{\partial n'}{\partial \mu} = \frac{1}{n_0}\Sigma_{12}(0)\left(\frac{dn}{d\mu}\right)^2 - \Pi(0)$$

(see Ref. 7 with allowance for (8), (15), and (9)), it follows that

$$\Pi(0) = -\left(\frac{\partial n'}{\partial \mu}\right)$$

is finite. We now recognize that

$$n' = \int \frac{d^3p}{(2\pi)^3} \frac{A(p, \varepsilon_p)}{2\varepsilon_p} + D,$$

where D is the contribution made to

$$n' = i\int \frac{d^4p}{(2\pi)^4} G_{11}(p) e^{i\varepsilon\delta}$$

by the nonpole singularities of $G_{11}(p)$. Finite $(\partial n'/\partial \mu)_{n_0}$ means that an infinitesimally small change of μ at a fixed n_0 can not produce in the spectrum of ε_p a gap $\Delta(\varepsilon_p = (\Delta^2 + c^2p^2)^{1/2})$ such that $(\partial \Delta^2/\partial \mu)_{n_0, \Delta=0} \equiv K \neq 0$; otherwise $(\partial n'/\partial \mu)_{n_0}$ would contain the logarithmically diverging term

$$\int \frac{d^3p}{(2\pi)^3}\left(\frac{\partial}{\partial \mu}\frac{1}{\varepsilon_p}\right)\frac{A}{2} = -\int \frac{d^3p}{(2\pi)^3}\left(\frac{KA}{4\varepsilon_p^3}\right)_{\Delta=0}.$$

But

$$\Delta^2 \sim [\Sigma_{11}(0) - \Sigma_{12}(0)] - \mu,$$

so that the fact that $-\Pi(0) = (\partial n'\partial \mu)_{n_0}$ is finite means validity of the relation

$$\left(\frac{\partial}{\partial \mu}[\Sigma_{11}(0) - \Sigma_{12}(0)]\right)_{n_0} = \left(\frac{\partial \mu_0}{\partial \mu}\right)_{n_0} = 1,$$

and hence also of (17)–(20) [see (15)].

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[1)] If $E'(n_0, \mu)$ is regarded as the thermodynamic potential of two subsystems, in one of which (condensate) the number of particles n_0 is fixed, and in the other (the system of particles in excess of the condensate) the chemical potential μ is fixed, then Eq. (1) means that equality of the chemical potentials of the two subsystems, due to the possibility of particle exchange: $\mu = \mu_0$, $\mu_0(n_0, \mu) \equiv (\partial E'/\partial n_0)_\mu$ is the chemical potential of the condensate.

[2)] In the derivation of (3) and (6) it is recognized that the limit of $M(p_i \to 0)$ (if finite) does not depend on the relation between the external momenta when they tend to zero (the difference from a Fermi system is due to the different distribution of the poles of the integrand).

[3)] This example demonstrates also the mechanism that makes possible the change of the order of magnitude even of a converging diagram if the zeroth vertex is replaced by the effective one (see Item 1 of the present section): the vertex made up of links II and III introduces a divergence as $p \to 0$, but this is not due to power-law singularity $\gtrsim 1/q^2$, but to the nonintegrable factor $\ln p$.

[4)] It is of interest to note that for a system with $\S \ll 1$, at not too high a density we have $c^2 < c_B^2$, i.e., $1 - V_0\bar{\Pi}(0) < 0$, (overscreening: $V_{eff}(p \to 0) = V_p/(1 - V_p\bar{\Pi}(p)) < 0$ at $V_{p-0} > 0$). This, however, does not lead to difficulties similar to the case of non-Abelian gauge theories: in the total line of the interaction $V_p[1 + V_pF_{44}(p)](F_{44}(p) = np^2/m(\epsilon^2 - c^2p^2)$ is the density–density response function[7]) the overscreened functions V_{eff}, jointly with the "tachyon" propagators

$$\bar{G}_{ik}(p) = (-1)^{i+k} mc_1^2/n(\varepsilon^2 - c_1^2p^2), \quad c_1^2 = c^2 - c_B^2 < 0$$

make up after summation a quantity that does not contain anomalies.

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