obtain the following criterion for the Langmuir turbulence:

$$W_e/nT<kr_d(m/M)^{1/2}.$$

This criterion implies also the absence of the collapse of the whole Langmuir wave spectrum.

[1)]If two or more high-frequency oscillation branches with similar frequencies have to be allowed for, the following substitution is required:

$$V_{kk_1k_2}\to V_{kk_1k_2}^{\lambda\lambda_1\lambda_2},\quad a_k\to a_k^\lambda,\quad \omega_k\to\omega_k^\lambda,\quad dk_1\to\sum_{\lambda_1}dk_1,\ \text{etc.},$$

where λ is the index representing each branch.

[2)]This example corresponds to a medium with a strong anisotropy, when the kernels of the kinetic equations are bihomogeneous functions of k_z and $k_\perp$, i.e., they permit independent elongations along and across a magnetic field.[8]

[3)]Here, as before, we are considering isotropic distributions. In particular, the averaging over the polarizations of the t waves is already carried out in this equation.

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Spectrum of bound roton-ion states in helium II

V. N. Bondarev

Physics Research Institute, Odessa State University

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It is shown that the formation of a bound roton-ion state in superfluid helium is possible for arbitrarily weak attraction. An equation for the energy spectrum of the bound states and its solutions for a zero total momentum of the compound quasiparticle are obtained on the basis of the roton-ion interaction potential found previously. The momentum dependence of the binding energy near the end points of various branches of the spectrum is found. The problem of experimental observation of the phenomena is discussed.

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The problem of the bound states of elementary excitations in various systems evokes appreciable interest. It was shown earlier[1,2] that an arbitrarily weak attraction is sufficient for the formation of bound states of two rotons in superfluid helium. The problem of the coupling of two elementary excitations in a crystal near the special points of the Brillouin zone was investigated in a recent work of Pitaevskiĭ.[3] At these points, the bound states develop at any nonzero interaction constant. Further investigation of the properties of two-roton states was undertaken in a paper by Pitaevskiĭ and Fomin,[4] in which such states were classified according to the value of the angular momentum of the system, and a dependence of the binding energy on the momentum was also found there.

In the present paper, we solve the problem of the binding of rotons with Newtonian particles—ions—in liquid helium. The problem of the dynamics of similar systems is nontrivial: the impossibility of complete separation of the motion of the center of mass and the relative motion makes the problem practically unsolvable in the general case. Here the smallness of the effective mass of the roton $\mu_0=0.16m_4$ in comparison with the mass of the ion $M\sim 50m_4$, where m_4 is the mass of the He^4 atom, does not mean that in the collision of the roton with the ion we can neglect the effect of the recoil of the latter (the characteristic momentum of the roton p_0 is close to the thermal momentum of the ion $\sim(MT)^{1/2}$ at $T\sim 1$ K).

1. It was shown in a previous paper of the author[5] that at distances that are large in comparison with the interatomic distances, forces of attraction of polarization origin operate between the ion and the roton. We write down the classical Hamiltonian of the ion-roton

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system with account of the central interaction U:

$$H=\frac{\mathbf{p}_i^2}{2M}+\Delta+\frac{(|\mathbf{p}|-p_0)^2}{2\mu_0}+U(|\mathbf{r}_i-\mathbf{r}|). \tag{1}$$

Here $\mathbf{r}_i, \mathbf{p}_i$ and $\mathbf{r}, \mathbf{p}$ are the radius vector and the momentum of the ion and the roton, respectively, and Δ is the roton gap. Introducing the conserved vector $\mathbf{q}=\mathbf{p}_i+\mathbf{p}$—the total momentum of the system—and the relative coordinate $\boldsymbol{\rho}=\mathbf{r}_i-\mathbf{r}$, we can, with the help of Hamilton's equations, express $\mathbf{p}_i$ and $\mathbf{p}$ in terms of these quantities (the dot indicates the time derivative):

$$\dot{\boldsymbol{\rho}}=\frac{\mathbf{q}}{M}-\left(\frac{p}{M}+\frac{p-p_0}{\mu_0}\right)\frac{\mathbf{p}}{p},$$

$$p\equiv|\mathbf{p}|=\mu\left[\frac{p_0}{\mu_0}\pm\left(\dot{\rho}^2+\frac{q^2}{M^2}-2\frac{\dot{\boldsymbol{\rho}}\mathbf{q}}{M}\right)^{1/2}\right], \quad \frac{1}{\mu}=\frac{1}{\mu_0}+\frac{1}{M}, \quad q\equiv|\mathbf{q}|. \tag{2}$$

The two signs before the radical in Eq. (2) and elsewhere follow from the non-unique dependence of the momentum of the roton on its energy.

It is convenient to transform to the Lagrangian function in accord with the standard formula

$$L=\mathbf{p}_i\dot{\mathbf{r}}_i+\mathbf{p}\dot{\mathbf{r}}-H=-\Delta-\frac{p_0^2}{2(M+\mu_0)}+\frac{q^2}{2(M+\mu_0)}+\frac{\mu\dot{\rho}^2}{2}$$
$$\pm\frac{\mu p_0}{\mu_0}\frac{\dot{\rho}^2-\dot{\boldsymbol{\rho}}\mathbf{q}/M}{(\dot{\rho}^2-2\dot{\boldsymbol{\rho}}\mathbf{q}/M+q^2/M^2)^{1/2}}-U|\boldsymbol{\rho}|). \tag{3}$$

Conservation of energy of the system in the variables $\boldsymbol{\rho}, \dot{\boldsymbol{\rho}}$ has the form

$$E=\Delta+\frac{p_0^2}{2(M+\mu_0)}+\frac{q^2}{2(M+\mu_0)}+\frac{\mu\dot{\rho}^2}{2}$$
$$\pm\frac{\mu p_0}{\mu_0}\frac{\dot{\boldsymbol{\rho}}\mathbf{q}/M-q^2/M^2}{(\dot{\rho}^2-2\dot{\boldsymbol{\rho}}\mathbf{q}/M+q^2/M^2)^{1/2}}+U(|\boldsymbol{\rho}|). \tag{4}$$

The transformation (2) actually brings about the transformation to the center-of-mass system, but it follows from the form of the Lagrangian function that the relative motion depends essentially on the total momentum $\mathbf{q}$ and this dependence makes the relative motion generally nonplanar. However, we can still point out one integral of the motion. Introducing a set of coordinates with the polar axis along $\mathbf{q}$, we obtain the Lagrangian function in the form ($\rho\equiv|\boldsymbol{\rho}|$, ϑ and φ are the angles)

$$L=-\Delta-\frac{p_0^2}{2(M+\mu_0)}+\frac{q^2}{2(M+\mu_0)}+\frac{\mu}{2}(\dot{\rho}^2+\rho^2\dot{\vartheta}^2+\rho^2\dot{\varphi}^2\sin^2\vartheta)$$
$$\pm\frac{\mu p_0}{\mu_0}\left[\dot{\rho}^2+\rho^2\dot{\vartheta}^2+\rho^2\dot{\varphi}^2\sin^2\vartheta-\frac{q}{M}(\dot{\rho}\cos\vartheta-\rho\dot{\vartheta}\sin\vartheta)\right]$$
$$\times\left[\dot{\rho}^2+\rho^2\dot{\vartheta}^2+\rho^2\varphi^2\sin^2\vartheta-2\frac{q^2}{M}(\dot{\rho}\cos\vartheta-\rho\dot{\vartheta}\sin\vartheta)+\frac{q^2}{M}\right]^{-1/2}-U(\rho). \tag{5}$$

In the latter expression, the coordinate φ is cyclical and this gives us still another conservation law:

$$\mu\rho^2\dot{\varphi}\sin^2\vartheta\left\{1\pm\frac{p_0}{\mu_0}\left[\dot{\rho}^2+\rho^2\dot{\vartheta}^2+\rho^2\dot{\varphi}^2\sin^2\vartheta-3\frac{q}{M}(\dot{\rho}\cos\vartheta-\rho\dot{\vartheta}\sin\vartheta)\right.\right.$$
$$\left.\left.+2\frac{q^2}{M^2}\right]\left[\dot{\rho}^2+\rho^2\dot{\vartheta}^2+\rho^2\dot{\varphi}^2\sin^2\vartheta-2\frac{q}{M}(\dot{\rho}\cos\vartheta-\rho\dot{\vartheta}\sin\vartheta)+\frac{q^2}{M^2}\right]^{-3/2}\right\}=\mathfrak{M}. \tag{6}$$

The conserved quantity $\mathfrak{M}$ represents the projection of the angular momentum on the direction of the total momentum of the system.

Unfortunately, the presence of two integrals of the motion, E and $\mathfrak{M}$, leaves the problem very complicated and it is not possible to obtain a solution in the general case. However, it is easy to see that the value $\mathbf{q}=0$ is unique, and the equations of motion are completely integrated in this case. At $\mathbf{q}=0$, the preferred direction associated with $\mathbf{q}$ disappears and the relative motion will be planar. Formally, we must set $\vartheta=\pi/2$, $\mathbf{q}=0$ in (4)–(6), and we obtain

$$\mu\rho^2\dot{\varphi}\pm\frac{\mu p_0}{\mu_0}\frac{\rho^2\dot{\varphi}}{(\dot{\rho}^2+\rho^2\dot{\varphi}^2)^{1/2}}=\mathfrak{M}, \tag{7}$$

$$\Delta+\frac{p_0^2}{2(M+\mu_0)}+\frac{\mu}{2}(\dot{\rho}^2+\rho^2\dot{\varphi}^2)+U(\rho)=E. \tag{8}$$

It follows from the last formula that finite motion is possible only at $U(\rho)<0$ and the energy $E<\Delta+p_0^2/2(M+\mu_0)$ corresponds to it. With the aid of (7) and (8), we can obtain the equation of the trajectory

$$\varphi=\mathfrak{M}\int\frac{d\rho}{\rho^2}\left\{\left[\left(2\mu\left[E-\Delta+\frac{p_0^2}{2(M+\mu_0)}-U(\rho)\right]\right)^{1/2}\pm\frac{\mu}{\mu_0}p_0\right]^2-\frac{\mathfrak{M}^2}{\rho^2}\right\}^{-1/2}. \tag{9}$$

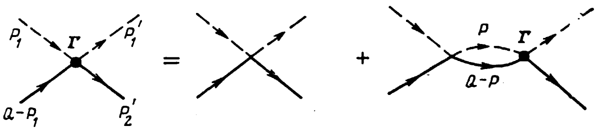
We shall not concern ourselves with the specific form of the $\varphi(\rho)$ dependence in the field $U(\rho)\sim-\rho^{-n}$. In the only case having physical meaning $n=4$ (see Ref. 5), integration in formula (9) can not be carried out to the end. We note only that we can explain all the singularities of the finite and infinite motion of similar systems at $q=0$ in terms of the exactly solvable models $n=1$ and $n=2$.

2. It is convenient to carry out the study of the spectrum of the bound ion-roton states by using many-particle methods (see, for example, Ref. 6). We shall first show that, just as in the case of two-roton states, the binding of the roton with the ion takes place at any weak attraction, at least in the case of zero total momentum $\mathbf{q}=0$. At $\mathbf{q}=0$, the mass operator is a loop containing one roton and one ion line,

$$\Sigma(E)\sim\lambda^2\int d\mathbf{p}\bigg/\left(\Delta-E+\frac{(p-p_0)^2}{2\mu_0}+\frac{p^2}{2M}\right)\sim\lambda^2\bigg/\left(\Delta-E+\frac{p_0^2}{2(M+\mu_0)}\right)^{1/2} \tag{10}$$

has a square-root singularity at $E\approx\Delta+p_0^2/2M$, and the corresponding divergence is capable of cancelling the possible smallness of λ—the roton-ion coupling constant. At non-vanishing $\mathbf{q}$, the singularity disappears, the bound state is broken up until it vanishes completely at some q_{max}. The point q_{max} is the end point of the spectrum of the bound state.

We carry out the further investigation in the spirit of Ref. 4. The spectrum of the bound states should be sought as the pole of the four-point vertex part Γ, the diagram equation for which has the form



(11)

i.e., it corresponds to the dangerous loop mentioned above. The solid lines in Eq. (11) denote the Green's functions of the ion, and the dashed lines, those of the roton, $Q=(\mathbf{q}, E)$, $P=(\mathbf{p}, \omega)$ and so on; The point designates the matrix element of the roton-ion interactions

$$\Gamma(|\mathbf{p}_1-\mathbf{p}|)=\int e^{i(\mathbf{p}_1-\mathbf{p})\rho/\hbar}U(|\rho|)d\rho.$$

In seeking the pole of Γ we can discard the first term on the right hand side of Eq. (11). Moreover, the quantities P_1' and $P_2'=Q-P_1'$ actually enter only as parameters. Further simplification is possible if we note that, thanks to the gap character of the roton spectrum and the fact that the most important values in the integral over d^4P are $\omega\approx\Delta$ and $|p|\approx p_0$, we can assume the corresponding arguments in Γ to be equal to these values. A similar conclusion can also be drawn for the relation $V(|\mathbf{p}_1-\mathbf{p}|)\approx V(2p_0\sin\vartheta/2)$, where ϑ is the angle between $\mathbf{p}$ and $\mathbf{p}_1$. Thus, the desired vertex $\Gamma(P,Q-P;P_1',P_2')$ essentially depends only on $q\equiv|\mathbf{q}|$, E, and the polar and azimuthal angles between q and p. With account of this, we can complete the integration in (11) over ω and $p\equiv|\mathbf{p}|$, substituting $p^2dp\approx p_0^2dp$ as usual in the integration over the momentum of the roton.

As a result, we have

$$\Gamma(q,E;x_1,\varphi_1)=-\frac{\pi p_0^2(2\mu)^{1/2}}{(2\pi\hbar)^3}\int_0^{2\pi}d\varphi'\int_{-1}^{1}dx'V\left(2p_0\sin\frac{\vartheta}{2}\right)\Gamma(q,E;x',\varphi')$$
$$\times\left\{\Delta-E+\frac{p_0^2}{2(M+\mu_0)}+\frac{q^2}{2M}-\frac{\mu q^2x'^2}{2M^2}-\frac{\mu p_0qx'}{\mu_0M}\right\}^{-1/2},\quad(12)$$

where x' and x_1 are the cosines of the angles formed by the vector q and the corresponding vectors $\mathbf{p}$, $\mathbf{p}_1$, φ' and φ_1 are the azimuthal angles in the system of coordinates with polar axis along q (the second axis can be drawn, for example, in the plane of the vectors $\mathbf{p}_1',\mathbf{p}_2'$). We can neglect $\mu q^2x'^2/(2M^2)$ in the denominator of the integrand of (12) in comparison with $q^2/2M$, and also everywhere we can write $\mu\approx\mu_0$, $(M+\mu_0)^{-1}\approx M^{-1}$.

Expanding Γ in a series in the spherical functions

$$\Gamma(q,E;x_1,\varphi_1)=\sum_{l,m}\Gamma_{lm}(q,E)N_l^mP_l^m(x_1)e^{im\varphi_1},\quad(13)$$

where

$$N_l^m=\left[\left(l+\frac{1}{2}\right)\frac{(l-|m|)!}{(l+|m|)!}\right]^{1/2},$$

and applying the addition theorem for Legendre polynomials in the expansion

$$V\left(2p_0\sin\frac{\vartheta}{2}\right)=\sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)V_lP_l(\cos\vartheta),\quad(14)$$

we finally obtain the following homogeneous system for the determination of the unknown Γ_{lm}:

$$\sum_l(V_{l'}B_{l'm}^{lm}+\delta_{ll'})\Gamma_{lm}=0.\quad(15)$$

Here $\delta_{ll'}$ is the Kronecker delta and the notation

$$B_{l'm}^{lm}=\frac{(2\mu_0)^{1/2}p_0^2}{4\pi\hbar^3}N_{l'}^mN_l^m\int_{-1}^{1}P_{l'}^m(x)P_l^m(x)\left(\Delta-E+\frac{p_0^2}{2M}+\frac{q^2}{2M}-\frac{p_0qx}{M}\right)^{-1/2}dx\quad(16)$$

has been introduced. Summation in (15) is carried out over all l—even and odd, in contrast with the two-roton bound state,[4] in which only the even l appear as a consequence of the quantum-mechanical indistinguishability of the rotons.

The spectrum of the bound roton-ion states is determined from the secular equation—the vanishing of the determinant

$$\mathrm{Det}(V_{l'}B_{l'm}^{lm}+\delta_{ll'})=0.\quad(17)$$

At $q=0$, we can easily find the quantities $B_{l'm}^{lm}\sim\delta_{ll'}$ and the roots of Eq. (17). The total number of different roots will be equal to the number of negative V_l in the expansion (14) and, moreover, each root will be $(2l+1)$-fold degenerate in m.

We find the explicit expressions for the coefficients V_l, starting out from the energy of the ion-roton interaction at distances that are large in comparison with the interatomic:

$$U(\rho)=-\frac{\tau}{2\rho^4},\quad\tau=\left|\left(\frac{\partial\ln\Delta}{\partial\ln\rho_0}\right)_0\right|\frac{\Delta\alpha e^2}{m_4c^2}.\quad(18)$$

In the last formula, ρ_4 is the density of liquid helium, α is the atomic polarizability of helium, c is the speed of first sound in He II, e is the electron charge. The experimental value of $(\partial\ln\Delta/\partial\ln\rho_4)_0x-0.57.$[7] The interaction (18) does not occur at small distances, and it is necessary to introduce a cutoff radius—the size of the ion R. Since the parameter $\varkappa\equiv p_0R/\hbar\sim10$, the relative motion of the ion-roton system is always quasiclassical, because the angular momentum values $l>\varkappa$ are real. As to the states with sufficiently large l (the total angular momentum of the system), the regions of quasiclassical motion corresponding to it can exceed the characteristic dimensions of the problem; here the detailed path of the potential at small distances will be unimportant and the cutoff radius will generally not enter into the answer. Thus the spectrum of weakly bound states can be found without bringing in any model considerations on the structure of the ion.

We write down at once the formula for the l-th term in the expansion (14);

$$V_l=-\pi\tau\int_R^{\infty}\frac{d\rho}{\rho^2}\int_0^{\pi}\sin\vartheta'd\vartheta'\int_0^{\pi}\exp\left[i\frac{p_0}{\hbar}\rho\sin\left(\frac{\vartheta}{2}\right)\cos\vartheta'\right]$$
$$\times P_l(\cos\vartheta)\sin\vartheta\,d\vartheta=-\frac{8\pi\hbar\tau}{p_0}\int_R^{\infty}\frac{d\rho}{\rho^3}\int_0^1\sin\left(\frac{p_0}{\hbar}\rho z\right)P_l(1-2z^2)dz.\quad(19)$$

The second integral in Eq. (19) is computed exactly (see Ref. 8, p. 91 of Russ. translation, formula (1)), and as a result we have

$$V_l=-\frac{\pi^2p_0\tau}{\hbar}\int_{\varkappa/2}^{\infty}[J_{l+1/2}(x)]^2x^{-3}dx,\quad(20)$$

$J_{l+1/2}(x)$ is the Bessel function. It follows from (20) that all the V_l turn out to be negative.

It is convenient to represent V_l in the form

$$V_l=-\frac{\pi^2p_0\tau}{\hbar}\left\{\frac{2}{(l-1/2)(l+1/2)(l+3/2)}-\int_0^{\varkappa/2}[J_{l+1/2}(x)]^2x^{-3}dx\right\},\quad l\geq1,\quad(21)$$

where we have again used the value of

$$\int_0^{\infty}[J_{l+1/2}(x)]^2x^{-3}dx$$

from Ref. 8 (p. 244, of Russ. translation formula (24)). We shall show that the second term in (21), containing the integral, falls off at large l no more slowly than l^{-2l}, i.e., we can neglect it in comparison with the first term. For this, it suffices to calculate the integral by using the expansion $J_{l+1/2}(x)$ at small x (the actual value of the integral will be even smaller). We get

$$\int_0^{\varkappa/2} [J_{l+1/2}(x)]^2 \frac{dx}{x^3} < \int_0^{\varkappa/2} x^{2l-2} \cdot 2^{-2l-1}\left[\Gamma\left(l+\frac{3}{2}\right)\right]^{-2} dx \approx \frac{e^3}{4\pi\varkappa l^3}\left(\frac{e\varkappa}{4l}\right)^{2l}, \tag{22}$$

where e is the base of natural logarithms and use is made of Stirling's formula for the Γ function in finding the answer. It is seen that formula (22) begins to be applicable at $l \gtrsim l_0 = e\varkappa/4$. The corresponding estimates for the positive and negative ions with account of the values $R_+ \approx 5$ Å, $R_- \approx 15$ Å give $l_{0+} \approx 7$, and $l_{0-} \approx 20$. Thus, at $l > l_0$, the dependence of v_l on R practically disappears, as was to be expected.

At $q = 0$, the roots of the determinant equation (17), with account of (21), and (22), are given by the expressions

$$E_l(0) = \Delta + \frac{p_0^2}{2M} - \frac{\pi^2 p_0^6 \mu_0 \tau^2}{2\hbar^6[(l-1/2)(l+1/2)(l+3/2)]^2}. \tag{23}$$

Still one more limitation on l follows from this. Formula (23) has real meaning for those l which correspond to a depth less than $\tau/2R^4$ of the level of the bound state (the last term in (23)). Taking it into account that $\tau/2R_+^4 \approx 3.5$ K, we find that this is achieved at $l > 17$ (correspondingly, $\tau/2R_-^4 \approx 0.035$ K and $l > 36$).

At $q \neq 0$, the roots with different l are no longer separated, the degeneracy in m is removed, but m remains a good quantum number (compare with $\mathfrak{M}$, formula (6)). Expanding $B_{l'm}^{lm}$ with accuracy to q^2, we find, at small q,

$$B_{l'm}^{lm} = \frac{(2\mu_0)^{1/2} p_0^2}{4\pi\hbar^3[\Delta - E + p_0^2/2M]^{1/2}}\left\{\delta_{ll'} - \frac{q^2\delta_{ll'}}{4M\delta_l(0)}\right.$$
$$\times\left[1 - \frac{p_0^2}{2M\delta_l(0)}\right] + \frac{p_0 q N_l^m N_{l'}^m}{2M\delta_l(0)}\int_{-1}^{1} P_l^m(x)P_{l'}^m(x)P_1^0(x)\,dx$$
$$\left. + \frac{p_0^2 q^2}{4M^2\delta_l^2(0)} N_l^m N_{l'}^m \int_{-1}^{1} P_l^m(x)P_{l'}^m(x)P_2^0(x)\,dx + \ldots\right\}, \tag{24}$$

where $\delta_l(0) = \Delta + p_0^2/2M - E_l(0)$. The integrals in (24) are expressed in terms of the $3j$ symbol (see, for example, Ref. 9). Unfortunately, the presence of nondiagonal terms in the matrix $B_{l'm}^{lm}$ makes it difficult to find the explicit form of the dispersion law of the bound state. However, it can be seen that, since terms that are linear in q are contained only in the nondiagonal elements $B_{l'm}^{lm}$, terms $\propto q$ will be absent in the expansion of the determinant, and the first nonvanishing term in the dispersion law of the bound state will be $\propto q^2$.

As an illustration, we consider the case in which only two constants, V_l and V_{l+1}, are different from zero and are given by the asymptote of formula (21) at large l. In this case, the equation for finding the spectrum reduces to the vanishing of a second order determinant, and we have for one of the roots,

$$E_l^m(q) \approx E_l(0) + \frac{q^2}{2M}\left\{1 - \hbar^8(l-1/2)(l+1/2)(l+3/2)\right.$$
$$\left.\times\frac{9(l+1/2)[l-1/2+(l^2-m^2)] + (l-1/2)^2[(l+1)^2-m^2]}{6\pi^2 p_0^6 \mu_0 M\tau^2}\right\}. \tag{25}$$

The coefficient at $q^2/2$ represents the reciprocal effective mass $(M^*)^{-1}$ of the bound quasiparticle. An interesting feature of formula (25) is the fact that the effective mass, being positive at not too large l, can change sign at definite l and m. Numerical estimates show that, beginning with $l \times 20$, the branches of the spectrum (25) with small m will have negative effective mass, and $M^* < 0$ will be the case for all branches at $l \geq 29$. Evidently, such a situation is preserved qualitatively in a more rigorous solution of Eq. (17).

We now investigate the problem of the end points of the spectrum of bound states. We rewrite formula (16) in the form

$$B_{l'm}^{lm} = \frac{(2\mu_0)^{1/2} p_0^2}{4\pi\hbar^3} N_l^m N_{l'}^m \int_{-1}^{1} P_l^m(x)P_{l'}^m(x)[\delta(q) + p_0 q(1-x)/M]^{-1/2} dx, \tag{26}$$

where $\delta(q) \approx \Delta - E + (p_0 - q)^2/2M$ represents the binding energy, as is easy to see—the energy separation of the level of the bound state from the boundary of the continuous spectrum. The end point of a certain branch corresponds to that value $q = q_{max}$ at which $\delta(q_{max}) = 0$. The expansions of $B_{l'm}^{lm}$ at small $\delta(q) > 0$ will be different at $m = 0$ and $m \neq 0$. In the first case,

$$B_{l'0}^{l0} = \frac{(2\mu_0 M)^{1/2} p_0}{4\pi\hbar^3}\left(\frac{p_0}{q}\right)^{1/2} N_l^0 N_{l'}^0 \left\{\int_{-1}^{1} \frac{P_l(x)P_{l'}(x)}{(1-x)^{1/2}}dx - 2\eta + O(\eta^3)\right\}, \tag{27}$$

in the second,

$$B_{l'm}^{lm} = \frac{(2\mu_0 M)^{1/2} p_0}{4\pi\hbar^3}\left(\frac{p_0}{q}\right)^{1/2} N_l^m N_{l'}^m \left\{\int_{-1}^{1} \frac{P_l^m(x)P_{l'}^m(x)}{(1-x)^{1/2}}dx\right.$$
$$\left. - \frac{1}{2}\eta^2 \int_{-1}^{1} \frac{P_l^m(x)P_{l'}^m(x)}{(1-x)^{3/2}}dx + O(\eta^3)\right\}, \quad m \neq 0, \tag{28}$$

where the notation $\eta^2 \equiv M\delta(q)/(p_0 q)$ has been introduced.

Now, expanding the determinant (17) in powers of η, it is easy to see that the values of q_{max} are given by the roots of the determinant (m is arbitrary)

$$\mathrm{Det}\left\{\left(\frac{q}{p}\right)^{1/2}\delta_{ll'} + V_{l'}\frac{(2\mu_0 M)^{1/2} p_0}{4\pi\hbar^3} N_l^m N_{l'}^m \int_{-1}^{1} \frac{P_l^m(x)P_{l'}^m(x)}{(1-x)^{1/2}}dx\right\} = 0. \tag{29}$$

We now estimate the region of momenta in which bound states can exist, i.e., we find q_{max} from (29), assuming that only a single coefficient V_l differs from zero and it is given by the asymptote of Eq. (21):

$$\left(\frac{q_{max}}{p_0}\right)^{1/2} > \frac{\pi(\mu_0 M)^{1/2} p_0^2 \tau}{2^{1/2}\hbar^4 l^3} \approx \frac{10^4}{l^3}. \tag{30}$$

In this estimate, the inequality

$$\int_{-1}^{1} [P_l^m(x)]^2(1-x)^{-1/2}dx > \int_{-1}^{1} [P_l^m(x)]^2 dx$$

has been taken into account. Thus, for $l \lesssim 20$, the region of existence of the bound state is $q_{max} \gtrsim p_0$.

The laws according to which the binding energy vanishes at $q \to q_{max}$ will be different at $m = 0$ and $m \neq 0$. In order to be convinced of this, it is sufficient to write down Eq. (17) in the case of small η in the form

$$\mathrm{Det}\left\{\left(\frac{q}{p_0}\right)^{1/2}\delta_{ll'} + V_{l'}\frac{(2\mu_0 M)^{1/2} p_0}{4\pi\hbar^3} N_l^m N_{l'}^m \int_{-1}^{1} \frac{P_l^m(x)P_{l'}^m(x)}{(1-x)^{1/2}}dx\right\}$$
$$+ D_0\eta\delta_{m0} + D_m\eta^2(1-\delta_{m0}) = 0 \tag{31}$$

(the explicit form of D_0 or D_m is unimportant). Now, expanding the first term of Eq. (31) in terms of $q - q_{max}$ near one of the end points of the spectrum, we find

$$\delta(q)\sim(q_{max}-q)^2,\quad m=0,$$
$$\delta(q)\sim(q_{max}-q),\quad m\neq 0. \tag{32}$$

The analysis that has been given enables us to obtain the energy spectrum of the bound states as undamped quasiparticles. It can be seen, however, that they will be unstable relative to radiation of an energetic phonon, which is allowed by the laws of conservation of energy and momentum:

$$\Delta+p_0^2/2M-\delta_l(0)=Q^2/2M+c|Q|. \tag{33}$$

Here Q is the momentum of the emitted phonon, c is the velocity of first sound; the case is considered of the decay of the bound state with zero total momentum and the deviation of the phonon spectrum from linear has been neglected ($|Q|\lesssim 0.5$ Å^{-1}). Account of the virtual processes corresponding to Eq. (33), leads to the result that the matrix elements of the interaction will now contain an imaginary part $V_l\to V_l-iW_l$ and, correspondingly, the energies $\delta_l(0)$ turn out to be complex:

$$\delta_l(0)\approx\frac{\mu_0 p_0^4}{8\pi^2\hbar^6}(V_l^2-2iV_lW_l). \tag{34}$$

The calculation of W_l does not differ basically from that given in Ref. 4 and reduces to finding the imaginary part of the integral

$$\frac{f_l^2}{2\pi^2\hbar^3}\int\frac{Q^2d|Q|}{\Delta+p_0^2/2M-\delta_l(0)-Q^2/2M-c|Q|+i\varepsilon},\quad \varepsilon\to+0, \tag{35}$$

where f_l is the matrix element corresponding to the conversion of the pair ion-roton with angular momentum l into an ion and a phonon. Taking[4] $f_l\simeq V_l$ i.e., assuming that the interaction of the ion with a short-wavelength phonon is approximately the same as that with a roton, we obtain

$$\frac{W_l}{V_l}\approx\frac{|V_l|}{2\pi\hbar^3c}\left(\frac{\Delta}{c}\right)^2\approx\frac{\pi p_0\tau}{c\hbar^4}\left(\frac{\Delta}{c}\right)^2\frac{1}{l^3}\approx\frac{200}{l^3}, \tag{36}$$

which, for example, at $l=20$, gives $W_{20}/V_{20}\approx 0.025$. It is easy to obtain the result that the width of the level of the bound state at sufficiently large l turns out also to be small in comparison with the separation between levels $\xi_l(0)\equiv\delta_l(0)-\delta_{l+1}(0)$:

$$\xi_l(0)\approx 3\pi^2p_0^8\mu_0\tau^2/\hbar^8l^7,$$
$$\frac{2V_lW_l\mu_0p_0^4}{8\pi^2\hbar^6\xi_l(0)}\approx\frac{\pi\mu_0p_0^2\tau}{24\hbar^4l^2}\approx\frac{40}{l^2}. \tag{37}$$

Finally, we consider briefly the hydrodynamic interaction, not taken into account here, that arises because the ion, being in a bound state with the roton, for example at $q=0$, will move with a velocity $v_i\approx p_0/M$, creating a hydrodynamic current. This latter leads to roton-ion interaction of the form

$$U'(\rho)=\mathbf{p}[\mathbf{v}_s(\rho)-\mathbf{v}_n(\rho)], \tag{38}$$

where $\mathbf{v}_s(\rho)$ and $\mathbf{v}_n(\rho)$ are respectively the superfluid and normal velocity fields, created by the moving ion. The problem of finding the bound states in this case appears to be much more complicated than that considered above, since it includes the self-consistent determination of the velocity of the ion. We shall show that in the calculations in which we are interested, the interaction U' can be neglected in comparison with the polarization (18). Direct comparison of the potentials U and U' is difficult since the latter is of alternating sign. It is more convenient to compare, for example, the contributions n_r and n_r', which each makes to the total number of localized rotons. In Ref. 10, the problem was solved of the localization of rotons near an ion, due to the appearance of an interaction of the form (38) at ion drift velocities different from zero. Extrapolating the expression given in Ref. 10 for the total number of localized rotons (formulas (5), (6)), we get, in the region of small v_i,

$$n_r'\approx\frac{A}{6}N_\infty R^3\frac{p_0^2}{(kT)^2}v_i^2, \tag{39}$$

where k is Boltzmann's constant, T is the temperature N_∞ is the equilibrium number of rotons per unit volume of helium, and $A\approx 5-8.5$.

An estimate of the total number of rotons found in a bound roton-ion state, with account of the interaction U, gives (see Ref. 5)

$$n_r\approx 2\pi N_\infty\tau/RkT. \tag{40}$$

Writing down the ratio n_r'/n_r in the case of values of the velocity v_i that are characteristic for an ion in a bound state with a roton, $v_i\approx p_0/M$, we get

$$n_r'/n_r=AR^4p_0^4/12\pi\tau M^2kT, \tag{41}$$

when $(n_r'/n_r)_+\approx 0.0125$ for positive ($M_+\sim 45m_4$) and $(n_r'/n_r)_-\approx 0.05$ for negative ($M_-\sim 200m_4$) ions. Thus, the contribution of the interaction U' to the thermodynamic equilibrium number of bound rotons, at least at not too large values of the total momentum of the ion-roton system, turns out to be unimportant and we neglect it also in the quantum-mechanical calculation of the energy spectrum of the bound states.

3. The results of the foregoing sections refer to the case of the bound state of the ion with a single roton, and this actually assumed that the system is at absolute zero temperature. In fact, the presence of other elementary excitations can complicate the picture and bound states of the ion with more than one roton can turn out to be important. However, it can be shown (this follows from Ref. 5) that in the range of temperatures important for rotons ($T\gtrsim 0.8$ K) the mean number of rotons localized near the ion, $n_r\lesssim 1$. It was noted earlier[5] that the presence of localized rotons creates an additional dissipation channel in the case of ionic motion and this leads to a lower mobility in comparison with the bare ion. In recent experiments[11] on the acceleration of ions in He II in external fields,[1)] an increase in mobility was observed in the case of an increase in the drift velocity of the ion (the corresponding experiments were carried out in a temperature region in which the mobility is limited by the rotons). A qualitative explanation of this fact can be given on the basis of the results obtained above. As has been noted, at $T\sim 1$ K, the principal contribution to the retardation of the ions is made by the localized rotons. In weak fields, i.e., in the case of small drift velocities of the ion, the binding energy of the roton localized in its vicinity is large and therefore corresponding contribution to the dissipation is large also. With increase in the drift velocity (this corresponds to an increase in the translational momentum q of the ion-roton system, see above) the binding energy of the localized roton decreases, the ion becomes free and its mobility increases (we note that the

phenomena described take place at drift velocities $v_0 \lesssim 10$ m/sec and nonlinear effects associated with the dependence of the mobility of the "bare" ion on v_0 are evidently still insignificant: $v_0 \ll \bar{v}_r \sim 90$ m/sec—the mean thermal velocity of the roton at $T \sim 1$ K.

It must be noted that the described effect appears in experiments with negative ions at pressures of about 25 bar. Under such conditions, the radius of the electronic bubble decreases by almost one half in comparison with its value at a pressure of saturated vapor and the stability of the bound state of the roton with the ion relative to thermal decay increases at high pressures. It should be expected that with decrease in the pressure (this corresponds to an increase in the size of the negative ion) the bound roton-ion states become less effective and the phenomenon of growth of the mobility with increase in the external field should disappear—the mobility of the electron "bubble" will approach the mobility of the bare ion.

We note that the most direct experimental confirmation of the existence of bound roton-ion states would be the observation of resonance effects in light scattering by superfluid helium in the presence of ions. Here a peak should be observed in the spectrum of the scattered light, corresponding to a transfer of energy $\approx \Delta$ (and not 2Δ as in the case of two-roton bound state), and its intensity will be proportional to the total number of ions. In correspondence with the conservation law and with account of the smallness of the wave vector of light, the total momentum of the compound particle roton-ion will be practically equal to the original momentum of the bare ion.

In conclusion, we show that the results obtained above can be applied to the description of bound states of rotons with impurity excitations in He^3-He^4 solutions. However, the absence of any sort of reliable data on the character of roton-impurity interactions does not permit us to draw definite conclusions as to the existence of such states.

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1)The review of Shikin is devoted to various aspects of the behavior of ions in helium, including problems of mobility.[12]

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