Kinetics of high- and low-frequency waves in nonlinear media

V. E. Zakharov and E. A. Kuznetsov

L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences, Moscow
(Submitted 6 April 1978)
Zh. Eksp. Teor. Fiz. 75, 904–912 (September 1978)

An analysis is made of the kinetics of weak turbulence in a nonlinear medium in which there are two types—high- and low-frequency—interacting waves. Steady-state Kolmogorov spectra, corresponding to constant fluxes (over the spectrum of the integrals of motion) of the energy and number of high-frequency quasiparticles are derived. Two cases are considered: the case of identical power exponents of the dispersion laws of high- and low-frequency waves and the case of "differential approximation," when each scattering event changes the energy of high-frequency waves by a small amount. The conditions of locality of the Kolmogorov spectra are found. The results are applied to the interaction of the Langmuir and ion-acoustic waves in a plasma.

PACS numbers: 03.40.Kf, 52.35.Mw

INTRODUCTION

Scale-invariant turbulence spectra, which can be regarded as analogs of the Kolmogorov spectrum[1] of a developed hydrodynamic turbulence, occupy an important place in the theory of weak turbulence. In many problems of weak turbulence the Kolmogorov spectra[2-5] are the exact solutions of kinetic equations. These solutions play a fundamental role as the spectra with constant fluxes characterizing, for example, the effectiveness of a particular plasma heating method.

We shall consider the power spectra in the specific case of a weak turbulence which appears in a medium capable of supporting both high- and low-frequency waves. This situation occurs, for example, in a plasma in the course of interaction of high-frequency electromagnetic ($\omega \gg \omega_p$) and Langmuir waves, in the stimulated Brillouin scattering, in the interaction of the Langmuir and ion-acoustic waves, etc. The last problem is particularly interesting because the interaction of the Langmuir (l) and ion-acoustic (s) waves is the main mechanism of energy transfer in a nonisothermal plasma ($T_e \gg T_i$) in the range of moderately short wavelengths $[kr_d > \frac{1}{3}(m/M)^{1/2}]$.

The kinetics of the interaction of high- and low-frequency waves is largely determined by the structure of the kernel of the kinetic equations. We can distinguish two situations. The first is characterized by the fact that the kernel is a homogeneous function of the wave vectors k_i and, consequently, the Kolmogorov spectra are the exact solutions of the kinetic equations (§2). In the other case (§3) the kinetic equations have scale-invariant solutions only in the diffusion approximation, when each step of energy transfer along the spectrum is small. This approximation, used widely in studies of the stimulated scattering of waves in plasma,[6,7] describes well the average characteristics of the spectrum; however, it does not describe its structure due to the fine properties of the nuclei.

As in hydrodynamics, the Kolmogorov turbulence spectra are meaningful only in the inactive interval when, on the one hand, one can ignore the decay and pumping and, on the other, waves of the same scale interact effectively. The proof of this property—the locality of the spectrum—is given in §4. In §5 we shall consider the problem of the spectra of the Langmuir turbulence of a nonisothermal plasma.

§1. BASIC EQUATIONS

Let us assume that a medium supports high-frequency waves with the dispersion law ω_k and low-frequency waves with the dispersion law Ω_k, and that a_k and b_k are the respective amplitudes of these waves. Then, the Hamiltonian of the medium is of the form

$$H=\int \omega_k|a_k|^2dk+\int \Omega_k|b_k|^2dk+H_{int}.$$

We can find the interaction Hamiltonian by turning to the equations of motion

$$\frac{\partial a_k}{\partial t}+i\omega_k a_k=-i\frac{\delta H_{int}}{\delta a_k^*},\quad \frac{\partial b_k}{\partial t}+i\Omega_k b_k=-i\frac{\delta H_{int}}{\delta b_k^*},\tag{1}$$

from which it follows that the high-frequency amplitude a_k should occur in the Hamiltonian only in the combination a^*a. Assuming that H_{int} is linear in respect to b, we find—in the first order in respect of a^*a—that:[1)]

$$H_{int}=\int V_{kk_1k_2}\delta_{k-k_1+k_2}(b_k a_{k_1} a_{k_2}^* + b_k^* a_{k_1}^* a_{k_2})\,d\mathbf{k}\,d\mathbf{k}_1\,d\mathbf{k}_2.$$

For moderately high amplitudes of high-frequency waves we can go over to a statistical description of the equations of motion (1). Introducing the quantities

$$\langle a_k a_{k_1}^*\rangle=N_k\delta_{k-k_1},\quad \langle b_k b_{k_1}^*\rangle=n_k\delta_{k-k_1},$$

we obtain

$$\frac{\partial N_k}{\partial t}=\int (T_{k_2|kk_1}-T_{k_2|k_1k})\,d\mathbf{k}_1 d\mathbf{k}_2,\tag{2}$$

$$\frac{\partial n_k}{\partial t}=-\int T_{k|k_1k_2}d\mathbf{k}_1 d\mathbf{k}_2,\tag{3}$$

where

$$T_{k_2|kk_1}=2\pi|V_{k_2kk_1}|^2(N_{k_1}n_{k_2}-N_k n_{k_2}-N_k N_{k_1})\delta_{k-k_1-k_2}\delta(\omega_k-\omega_{k_1}-\Omega_{k_2}).$$

We shall now make some assumptions about the quantities ω_k, Ω_k, and $V_{kk_1k_2}$. First of all, we shall assume that the medium is isotropic. Then, all these functions depend on the moduli of their arguments. Finally, we shall assume that ω_k and Ω_k can be described by the power laws

$$\omega_k=\omega_0+c_1k^\alpha,\quad \Omega_k=c_2k^\beta$$

 0038-5646/78/090458-05$02.40

and that the quantity $V_{kk_1k_2}$ is a homogeneous function of the power exponent s:

$$V_{\varepsilon k \varepsilon k_1 \varepsilon k_2}=\varepsilon^s V_{kk_1k_2}.$$

Clearly, for this selection of ω_k the frequency ω_0 does not occur in Eqs. (2) and (3) and, therefore, without loss of generality we can assume that $\omega_0 = 0$. We shall next introduce a quantity $\epsilon_k = \omega_k N_k + \Omega_k n_k$, which represents the "energy" density of the waves in the k space. According to Eqs. (2) and (3), the evolution of this quantity is found from

$$\frac{\partial \varepsilon_k}{\partial t}=\int (\omega_k T_{k2|kk_1}-\omega_k T_{k_2|k_1k}-\Omega_k T_{k|k_1k_2})\,dk_1\,dk_2. \tag{4}$$

The conservation of energy and of the total number of waves follow directly from this equation and Eq. (2) so that these equations have steady-state thermodynamic-equilibrium solutions $N_k = T/(\omega_k + \mu)$, $n_k = T/\Omega_k$, where T and μ are the temperature and chemical potential. We shall be interested in other steady-state solutions for which $T = \mu = 0$. In contrast to thermodynamic-equilibrium solutions the latter should be characterized by fluxes (over the spectrum) of the number of high-frequency waves P_N and of the "energy" P_ϵ.

§2. KOLMOGOROV SPECTRA (EXACT SOLUTION)

We shall first consider the case $\alpha = \beta$. This situation occurs in the interaction of electromagnetic and acoustic waves—for example, in the stimulated Brillouin scattering and also in a strongly magnetized plasma ($8\pi nT/H^2 \ll 1$)-in the interaction of the Alfven and acoustic waves.[2)] In the case the kernel of the kinetic equations

$$U_{kk_1k_2}=2\pi\,|V_{kk_1k_2}|^2\delta_{k-k_1+k_2}\delta(\Omega_k-\omega_{k_1}+\omega_{k_2})$$

is a homogeneous function with the homogeneity index $y = 2s - \alpha - d$, where d is the dimensionality of space. Steady-state solutions of Eqs. (2)–(4) will be sought in the form

$$N_k=Ak^x, \qquad n_k=Bk^x. \tag{5}$$

We shall first consider the solutions of Eq. (2):

$$\int (T_{k_2|kk_1}-T_{k_2|k_1k})\,dk_1dk_2=0.$$

We shall map the integration domain of the second integral, governed by the decay conditions, into the integration domain of the first. For this purpose it is convenient to introduce in an arbitrary plane occupied by an external vector k a complex quantity $w = k_x + ik_y$, where k_x and k_y are the coordinates in that plane. The mapping is then performed as follows:

$$w=w'\frac{w}{w'}, \quad w_1=w\frac{w}{w'}, \quad w_2=w''\frac{w}{w'}. \tag{6}$$

In this way the expression under the integral is factorized:

$$\int T_{k_2|kk_1}[1-(k/k_1)^\gamma]\,dk_1dk_2=0,$$

where $\gamma = y + 3d + 2x$.

Equation (4) is transformed similarly:

$$\int T_{k_2|kk_1}[\omega_k-\omega_{k_1}(k/k_1)^{\gamma+\alpha}-\Omega_{k_2}(k/k_2)^{\gamma+\alpha}]\,dk_1dk_2=0.$$

Transformations of the (6) type have been found by one of the present authors (Zakharov)[2] for the one-dimensional case and then generalized to the two- and three-dimensional cases by Kats and Kontorovich.[9] It follows directly from these equations that, apart from thermodynamic-equilibrium solutions ($T_{k|k_1k_2} \equiv 0$), there are also nonequilibrium solutions for which $\gamma = 0$ and $\gamma + \alpha = 0$ or $x_1 = -s - d + \alpha/2$ and $x_2 = -s - d$. One of the equations is then identical with Eq. (3):

$$\int T_{k|k_1k_2}\,dk_1dk_2=0, \tag{7}$$

which determines the relationship between the constants A and B in the distributions (5). Assuming that the diffusivity condition $\omega_k \gg \Omega_k$ is satisfied and that the integral in Eq. (7) is convergent, we obtain the estimate $A \sim Bc_1/c_2$. Thus, the energy of the low-frequency waves is of the same order as the energy $\omega_k N_k$.

We shall show that the first solution corresponds to a constant flux of the number P_N of the high-frequency waves and the second to a constant flux of the "energy" P_ϵ.

Integrating Eqs. (2) and (4), we find the following expressions for the fluxes in the case of the power-law distributions:

$$\left.\begin{aligned} P_N &= -\frac{fk^{\gamma+d}}{\gamma}\int T_{k_2|kk_1}(k^{-\gamma}-k_1^{-\gamma})\,dk_1dk_2,\\ P_\epsilon &= -\frac{fk^{\gamma+\alpha+d}}{\gamma+\alpha}\int T_{k_2|kk_1}(\omega_k k^{-\gamma-\alpha}-\omega_{k_1}k_1^{-\gamma-\alpha}-\Omega_{k_2}k_2^{-\gamma-\alpha})\,dk_1dk_2, \end{aligned}\right\} \tag{8}$$

where

$$f=2\pi \text{ for } d=2 \text{ and } f=4\pi \text{ for } d=3.$$

Going over to the limits $\gamma \to 0$ and $\gamma + \alpha \to 0$ in the above equations and applying Eq. (7), we find that the first solution corresponds to the Kolmogorov spectra with a constant flux of the number of the high-frequency waves for which $P_\epsilon = 0$ and the second corresponds to a constant flux of the "energy" P_ϵ and to $P_N = 0$ (compare with the treatment of Karas'[10] and Kats[11]). Hence, it follows in particular that A, $B \propto P^{1/2}$, which is in full agreement with the dimensional considerations.

As the final task, we shall determine the directions of the fluxes. We note that n_k can be found explicitly from Eq. (7). Substituting this distribution into Eq. (8) and symmetrizing the integrand, we find

$$P_N=\frac{fk^d}{8}\int R_{kk_1|k_2k_3}NN_1N_2N_3(N^{-1}+N_1^{-1}-N_2^{-1}-N_3^{-1}) \times \ln\frac{kk_1}{k_2k_3}\,dk_1dk_2dk_3,$$

$$P_\epsilon=\frac{fk^d}{8}\int R_{kk_1|k_2k_3}NN_1N_2N_3(N^{-1}+N_1^{-1}-N_2^{-1}-N_3^{-1}) \times(\omega\ln k+\omega_1\ln k_1-\omega_2\ln k_2-\omega_3\ln k_3)\,dk_1dk_2dk_3,$$

where

$$R_{12|34}=T_{13|24}+T_{31|42}+T_{14|23}+T_{41|32},$$

$$T_{12,34}=\int dk'U_{k'|12}U_{k'|34}\left[\int U_{k'|56}(N_6-N_5)\,dk_5dk_6\right]^{-1},$$

$$N_1=N_{k_1}, \qquad N_2=N_{k_2} \text{ etc.}$$

These expressions are formally identical with the fluxes in the four-wave interaction and the "matrix" element R has a definite sign: this sign is opposite to sign ($x\alpha$). Then, following the treatment of Kats,[11] we obtain

$$\operatorname{sign} P_N = \operatorname{sign}(x_1+\alpha), \quad \operatorname{sign} P_\epsilon = -\operatorname{sign}(x_2+\alpha). \tag{9}$$

In the case of the interaction between electromagnetic (t) and acoustic (s) waves ($\alpha = 1, s = -\frac{1}{2}, d = 3$), we have

$$x_1 = -2, \quad x_2 = -{}^5/_2, \quad P_N < 0, \quad P_\epsilon > 0.$$

Thus, in this case the Kolmogorov solutions with a constant flux of the number of the high-frequency waves may be realized in the long-wavelength range where P_N is directed and the other solutions in the short-wavelength range where the "energy" is pumped.

§3. DIFFUSION APPROXIMATION

The homogeneity of the kernels of the kinetic equations (3) and (4), considered in §2, and the consequent existence of the exact solution in the form of the Kolmogorov spectra are related primarily to the coincidence of the indices α and β. This condition is not obeyed in the majority of the examples of practical importance. This comment applies particularly to the interaction of the high-frequency Langmuir waves with low-frequency ion-acoustic waves. A typical situation in turbulence of this kind is the one when the frequency obeys $\omega_k \gg \Omega_k$, the step in the transfer along the frequency axis of the high-frequency waves is small. In this situation, the equations can be simplified.

Expanding, in Eqs. (2)–(4), the δ function of the frequencies as a series in Ω and assuming that the wave distributions N_k and n_k are isotropic, we obtain

$$\frac{\partial n_k}{\partial t} = \int T_{k|k_1k_2} dk_1 dk_2, \tag{10}$$

$$\frac{\partial N_k}{\partial t} + \operatorname{div}_k \mathbf{p}_N = 0, \tag{11}$$

$$\frac{\partial \varepsilon_k}{\partial t} = -\operatorname{div}_k \omega_k \mathbf{p}_N + \int (\Omega_k T_{k|k_1k_2} - \Omega_{k_1} T_{k_1|kk_2})\, dk_1\, dk_2, \tag{12}$$

where

$$T_{k|k_1k_2} = 2\pi |V_{kk_1k_2}|^2 \left(N_{k_1}{}^2 + n_k \Omega_k \frac{\partial N_{k_1}}{\partial \omega_{k_1}}\right) \delta_{k-k_1+k_2} \delta(\omega_{k_1} - \omega_{k_2});$$

$$\mathbf{p}_N = \frac{\mathbf{k}}{k} \int \frac{\Omega_{k_2}}{\omega_k{}'} T_{k_2|kk_1} dk_1 dk_2$$

is the density of the high-frequency quasiparticle flux. These equations are two independent systems (10), (11) and (11), (12) whose identity can be checked by direct calculation. Like the initial system (2)–(4), Eqs. (10)–(12) have solutions in the form of the Rayleigh–Jeans distributions which cause the fluxes $P_N = 0$ and $P_\epsilon = 0$ to vanish ($\tilde{T}_{kk_1k_2} = 0$).

We shall consider solutions with the constant fluxes P_N and P_ϵ. The constancy of these fluxes corresponds to the power-law solutions:

$$N_k = Ak^x, \quad n_k = Bk^x \omega_k / \Omega_k. \tag{13}$$

In the case of the spectra corresponding to $P_N = \text{const}$, the power exponent x is found directly by calculating the power exponents in Eq. (11):

$$x_1 = -s - d + \alpha - \beta/2. \tag{14}$$

In this case we have $A,\ B \propto P_N^{1/2}$.

The second solution corresponding to $P_\epsilon = \text{const}$ is found by carrying out in Eq. (12) a transformation analogous to Eq. (6), which then gives

$$x_2 = -s - d - (\beta - \alpha)/2, \quad A, B \propto P_\epsilon^{1/2}. \tag{15}$$

For both solutions the relationship between the constants A and B is found, as before, from the steady-state equation (11) by showing also that the other flux vanishes: $P_\epsilon = 0$ for $P_N = \text{const}$, and vice versa. The signs of the fluxes are given by the previous expressions which now no longer have the power exponent β. This is due to the fact that the quantity $n_k \Omega_k$ rather than n_k is excluded from Eqs. (11) and (12) in the $\alpha = \beta$ case.

By way of example, we shall consider the interaction of electromagnetic and Langmuir waves in the $k_t c \gg \omega_p$ case. In this case we have $\alpha = 1,\ \beta = 0,\ s = 0$ (Ref. 6). Therefore, $x_1 = -2$ and $x_2 = -\frac{5}{2}$.

§4. LOCALITY OF TURBULENCE

Our solutions of the diffusion and exact equations are meaningful, firstly, only in the inactive interval representing an intermediate region of the k space where there is no decay or pumping and, secondly, when the interaction of waves in this region is local. Therefore, the local Kolmogorov spectra are independent of the growth and decay increments, but are governed only by the magnitudes of the fluxes.

To prove the locality of the spectra it is sufficient to demonstrate the convergence of the integrals in the diffusion equations (10)–(12). This can be done only if we know the asymptotes of the matrix elements $V_{kk_1k_2}$. We shall assume that

$$V_{kk_1k_2} \to Dk^{s-\xi} k_\perp{}^\xi \quad \text{for} \quad k_1 \gg k.$$

Then, the convergence of the integrals from above is ensured for

$$x + 2\xi + d - 2\alpha < -1,$$

from below for

$$x + \alpha + \beta + d + 2(s - \xi) > 1.$$

§5. SPECTRA OF THE LANGMUIR TURBULENCE IN A NONISOTHERMAL PLASMA

We shall find the isotropic spectra of the Langmuir turbulence due to the decay of the Langmuir (l) waves into the Langmuir and ion-acoustic (s) waves in a nonisothermal plasma ($T_e \gg T_i$). To be specific, we shall assume that the excitation of this turbulence occurs in a narrow range Δk near $k = k_0$ with a characteristic increment γ much greater than the linear decay of waves which we shall ingore. In this situation the steady-state spectra should be of the flux type. To determine them in the diffusion approximation it is sufficient to use the general expressions (13)–(15). True, we have to know also the matrix element of the interaction and the dispersion laws of waves. They are of the form

$$V_{kk_1k_2} = \omega_p \frac{(k_1 k_2)}{k_1 k_2} \left(\frac{\Omega_k}{8 n_0 t}\right)^{1/2},$$

$$\omega_k = \omega_p + {}^3/_2 \omega_p (k r_d)^2, \quad \Omega_k = k c_s.$$

In the case of the Langmuir oscillations the dispersion correction is small and, therefore, the energy spectrum of the Langmuir waves ϵ_k^l and the energy flux along the spectrum agree, to within a factor ω_p, with the number of waves and their flux, respectively.

Following the results of §3 we can say, firstly, that the turbulence spectra corresponding to a constant flux of the number of high-frequency waves are realized in the range $k < k_0$, whereas the spectra corresponding to P_ϵ = const are realized in the range $k > k_0$; secondly, these fluxes are directed in opposite ways: P_N to the long-wavelength oscillations, where the Langmuir waves are dissipated by collapse, and P_ϵ to the range of short waves, where the dissipation of the ion-acoustic due to the Landau damping by electrons is important. According to Eqs. (14) and (15), the spectra have the following form in these regions:

$$\varepsilon_k^l = \omega_p A_1/k^2, \quad \varepsilon_k^s = {}^3/_2 \omega_p r_d^2 B_1 \quad \text{for} \quad k < k_0,$$
$$\varepsilon_k^l = \omega_p A_2/k^3, \quad \varepsilon_k^s = {}^3/_2 \omega_p r_d^2 B_2/k \quad \text{for} \quad k > k_0,$$

where the constants A_1, B_1 and A_2, B_2 are deduced from the constancy of the fluxes P_N and P_ϵ and from the condition of their matching with the growth region. The condition for P_N can be written in the form

$$P_N \sim \gamma \Delta k k_0^2 N_{k_0},$$

and hence it follows from Eq. (11)

$$A_1 \sim B_1 \sim n_0 T \frac{\gamma}{\omega_p} \frac{\Delta k}{k_{dif}^2},$$

where $k_{dif} = r_d^{-1}(m/M)^{1/2}$ is the characteristic step in the transfer of the Langmuir waves along the spectrum.

The quantities A_2 and B_2 are determined from the continuity of the energy flux of the ion-acoustic waves P_s at $k = k_0$. We can easily see why such a flux appears for $k < k_0$. This is due to the fact that as a result of the transfer of the Langmuir waves to the region $k \approx 0$ and their dissipation in this region, sound is generated and accumulated. We recall that for $k < k_0$, where P_N = const,

$$P_\epsilon = {}^3/_2 \omega_p (k r_d)^2 P_N + P_s = 0,$$

i.e.,

$$P_s = -{}^3/_2 \omega_p (k r_d)^2 P_N,$$

and is directed toward higher values of k. For a similar reason in the region $k > k_0$, where P_ϵ = const and $P_N = 0$, the energy flux of the ion-acoustic waves is identical with P_ϵ. Hence, it follows from the condition of continuity of P_s at $k = k_0$ that

$$A_2 \sim B_2 \sim n_0 T \frac{\gamma}{\omega_p} \frac{\Delta k k_0}{k_{dif}^2}.$$

It is clear from the above solutions that the intensity of the ion-acoustic waves is fairly high:

$$\varepsilon_s/\varepsilon_l \sim (k r_d)^2.$$

The main proportion of the energy of the ion sound is concentrated in the short-wavelength range. This circumstance has a decisive influence on the dynamics of the collapse, resulting particularly in additional damping because of the conversion of the long-wavelength Langmuir oscillations interacting with the short-wavelength sound.[12] This aspect is outside the scope of the present paper; it will be discussed elsewhere.

We shall now estimate the influence of other weakly turbulent mechanisms on the kinetics of the Langmuir turbulence. The most important of these is the conversion of the Langmuir waves interacting with sound into long-wavelength electromagnetic oscillations. We shall turn to the kinetic equation for the occupation numbers of the transverse (t) waves N_k^t:[3)]

$$\frac{\partial N_k^t}{\partial t} = \pi \int |V_{k_2 k k_1}^{tl}|^2 (N_{k_1}^l n_{k_2} - N_k^t n_{k_2} - N_k^t N_{k_1}^l) \delta(\omega_k^t - \omega_{k_1}^l - \Omega_{k_2})$$
$$\times \delta_{k-k_1-k_2} dk_1 dk_2 - \pi \int |V_{k_2 k_1 t}^{tl}|^2 (N_k^t n_{k_2} - N_{k_1}^l n_{k_2} - N_k^t N_{k_1}^l) \delta(\omega_{k_1}^l - \omega_k^t - \Omega_{k_2})$$
$$\times \delta_{k_1-k-k_2} dk_1 dk_2, \qquad (16)$$

where

$$|V_{k_2|k k_1}^{tl}|^2 = |V_{k_2|k_1 k}^{lt}|^2 = \omega_p^2 \frac{\Omega_{k_2}}{8 n_0 T} \frac{[\mathbf{k}\mathbf{k}_1]^2}{k^2 k_1^2},$$
$$\omega_k^t = \omega_p + k^2 c^2 / 2\omega_p.$$

We note that in the $k_{rd}^l \gg (m/M)^{1/2}$ range the frequencies of the l and t waves participating in the conversion processes are similar. Therefore, the wave vector of an electromagnetic wave is small compared with the wave vector of a Langmuir wave $(k_t/k_l \sim v_{Te}/c)$. This, in its turn, means that electromagnetic waves have a smaller phase volume [by a factor $(v_{Te}/c)^3$] than the Langmuir waves. It is the smallness of this phase volume that allows us to ignore the conversion processes. This can be demonstrated as follows.

We shall assume first that the occupation numbers obey $N_{kt}^t \ll N_{kl}^l$. Then,

$$\frac{\partial N_k^t}{\partial t} \approx 2\pi \int |V_{k_2 k k_1}^{tl}|^2 N_k^l n_{k_2} \delta(\omega_k^t - \omega_{k_1}^l) \delta_{k_1-k_2} dk_1 dk_2 > 0,$$

i.e., the electromagnetic waves grow. To find the level at which stabilization takes place, we shall consider the opposite limiting case: $N_{kt}^t \gg N_{kl}^l$. We can then ignore all the terms in Eq. (16), apart from the terms proportional to $N_k^t n_{k_2}$ $(n_k \gg N_k^l)$:

$$\frac{\partial N_k^t}{\partial t} \approx -2\pi \int |V_{k_2|k k_1}^{tl}|^2 N_k^t n_{k_2} \delta_{k_1-k_2} \delta(\omega_k^t - \omega_{k_1}^l) dk_1 dk_2 < 0.$$

Hence, we can see that N_k^t decays at the rate characterized by the increment

$$\gamma^c \sim \omega_p W_e / nT.$$

Therefore, in equilibrium we have $N^l \sim N^t$ and the total energy of the electromagnetic waves

$$W^t = \omega_p \int N_k^t dk \sim W^l (v_{Te}/c)^3$$

is small compared with the energy of the Langmuir waves. Hence, an estimate of the characteristic time of the process follows directly:

$$\frac{1}{\tau} \sim \omega_p \frac{W_e}{nT} \left(\frac{k_{dif}}{k}\right)^2 \left(\frac{v_{Te}}{c}\right)^3.$$

Thus, the influence of conversion can be ignored.

We shall conclude with some comments on the locality of the Langmuir turbulence spectra and the range of validity of the weak turbulence approximation. The locality can be determined directly from the results in §4. In the present case we have $\xi = 0$ and $s = \frac{1}{2}$. Therefore, the spectra are local. The following requirement is most important for the validity of the weak turbulence approximation. It is essential that the reciprocal of the randomization time of the wave phases, governed by the linear effects, is high compared with the reciprocal time of the nonlinear process. The former quantity should be the frequency of the low-frequency waves and the latter the decay increment of a monochromatic high-frequency wave. Hence, in particular, we obtain the following criterion for the Langmuir turbulence:

$$W_e/nT<kr_d(m/M)^{1/2}.$$

This criterion implies also the absence of the collapse of the whole Langmuir wave spectrum.

1) If two or more high-frequency oscillation branches with similar frequencies have to be allowed for, the following substitution is required:

$$V_{kk_1k_2}\to V_{kk_1k_2}^{\lambda\lambda_1\lambda_2},\quad a_k\to a_k^{\lambda},\quad \omega_k\to\omega_k^{\lambda},\quad dk_1\to\sum_{\lambda_1}dk_1,\ \text{etc.},$$

where λ is the index representing each branch.

2) This example corresponds to a medium with a strong anisotropy, when the kernels of the kinetic equations are bihomogeneous functions of k_z and $k_\perp$, i.e., they permit independent elongations along and across a magnetic field.[8]

3) Here, as before, we are considering isotropic distributions. In particular, the averaging over the polarizations of the t waves is already carried out in this equation.

[1] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 299 (1941).

[2] V. E. Zakharov, Zh. Prikl. Mekh. Tekh. Fiz. No. 4, 35 (1965); Zh. Eksp. Teor. Fiz. **51**, 688 (1966); **62**, 1745 (1972) [Sov. Phys. JETP **24**, 445 (1967); **35**, 908 (1972)].

[3] V. E. Zakharov and N. N. Filonenko, Dokl. Akad. Nauk SSSR **170**, 1292 (1966) [Sov. Phys. Dokl. **11**, 881 (1967)]; N. N. Filonenko, Zh. Prikl. Mekh. Tekh. Fiz. No. 5, 62 (1967).

[4] V. E. Zakharov and R. Z. Sagdeev, Dokl. Akad. Nauk SSSR **192**, 297 (1970) [Sov. Phys. Dokl. **15**, 439 (1970)].

[5] E. A. Kuznetsov, Zh. Eksp. Teor. Fiz. **62**, 584 (1972) [Sov. Phys. JETP **35**, 310 (1972)].

[6] V. N. Tsytovich, Nelineĭnye éffekty v plazme, Nauka, M., 1967 (Nonlinear Effects in Plasma, Plenum Press, New York, 1970).

[7] B. N. Breĭzman, V. E. Zakharov, and S. L. Musher, Zh. Eksp. Teor. Fiz. **64**, 1297 (1973) [Sov. Phys. JETP **37**, 658 (1973)].

[8] E. A. Kuznetsov, Preprint IYaF 81-73, Novosibirsk, 1973.

[9] A. V. Kats and V. M. Kontorovich, Zh. Eksp. Teor. Fiz. **64**, 153 (1973); **65**, 206 (1973) [Sov. Phys. JETP 37, 80 (1973); **38**, 102 (1974)].

[10] V. I. Karas', Pis'ma Zh. Tekh. Fiz. **1**, 1020 (1975) [Sov. Tech. Phys. Lett. **1**, 438 (1975)].

[11] A. V. Kats, Zh. Eksp. Teor. Fiz. **71**, 2104 (1976) [Sov. Phys. JETP **44**, 1106 (1976)].

[12] A. A. Galeev, R. Z. Sagdeev, V. D. Shapiro, and V. I. Shevchenko, Zh. Eksp. Teor. Fiz. **73**, 1352 (1977) [Sov. Phys. JETP **46**, 711 (1977)].

Translated by A. Tybulewicz

Spectrum of bound roton-ion states in helium II

V. N. Bondarev

Physics Research Institute, Odessa State University

(Submitted 29 Janaury 1978)
Zh. Eksp. Teor. Fiz. **75**, 913–923 (September, 1978)

It is shown that the formation of a bound roton-ion state in superfluid helium is possible for arbitrarily weak attraction. An equation for the energy spectrum of the bound states and its solutions for a zero total momentum of the compound quasiparticle are obtained on the basis of the roton-ion interaction potential found previously. The momentum dependence of the binding energy near the end points of various branches of the spectrum is found. The problem of experimental observation of the phenomena is discussed.

PACS numbers: 67.40.Db

The problem of the bound states of elementary excitations in various systems evokes appreciable interest. It was shown earlier[1,2] that an arbitrarily weak attraction is sufficient for the formation of bound states of two rotons in superfluid helium. The problem of the coupling of two elementary excitations in a crystal near the special points of the Brillouin zone was investigated in a recent work of Pitaevskiĭ.[3] At these points, the bound states develop at any nonzero interaction constant. Further investigation of the properties of two-roton states was undertaken in a paper by Pitaevskiĭ and Fomin,[4] in which such states were classified according to the value of the angular momentum of the system, and a dependence of the binding energy on the momentum was also found there.

In the present paper, we solve the problem of the binding of rotons with Newtonian particles—ions—in liquid helium. The problem of the dynamics of similar systems is nontrivial: the impossibility of complete separation of the motion of the center of mass and the relative motion makes the problem practically unsolvable in the general case. Here the smallness of the effective mass of the roton $\mu_0=0.16m_4$ in comparison with the mass of the ion $M\sim 50m_4$, where m_4 is the mass of the He^4 atom, does not mean that in the collision of the roton with the ion we can neglect the effect of the recoil of the latter (the characteristic momentum of the roton p_0 is close to the thermal momentum of the ion $\sim(MT)^{1/2}$ at $T\sim 1$ K).

1. It was shown in a previous paper of the author[5] that at distances that are large in comparison with the interatomic distances, forces of attraction of polarization origin operate between the ion and the roton. We write down the classical Hamiltonian of the ion-roton

0038-5646/78/090462-06$02.40