The last expression with allowance for the relation, $E_x^{(1)} = -igE_y^{(1)}/\varepsilon_1$, between the components of the extraordinary wave, where $\varepsilon_1 = 1 - \omega_p^2/(\omega^2 - \Omega^2)$ and $g = \Omega\omega_p^2/\omega(\omega^2 - \Omega^2)$, reduces to the form

$$j_z^{(3)} = \frac{i\omega_p^2 E_z^{(2)*} E_y^{(1)} ek_1\Omega}{4\pi m\omega_1\omega_2(\omega_1^2 - \Omega^2 - \omega_p^2)}$$

(cf. (6)) irrespective of the degeneracy and the strength of the magnetic field. The remaining corrections are then $\sim 1/\eta^{1/2}\lambda \ll 1$.

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Nonlinear theory of the low-frequency oscillations in a weakly turbulent plasma

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A theory is developed for determining the frequency shift of electromagnetic waves in a weakly turbulent plasma as a function of the level of the turbulent pulsations. The case of magnetohydrodynamic waves is considered. It is shown that the dispersion laws for Alfvén and slow magnetosonic waves change markedly at low values of the longitudinal (parallel to the magnetic field) component of the wave vector. Modified dispersion laws are obtained for them and these are taken into account in a study of relaxation processes of excitations in the wave spectra.

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1. INTRODUCTION

As is well known, the interaction between particles or quasiparticles leads as a rule to a shift in their energies relative to the values of the energy corresponding to the free states of the particles. For example, the interaction of an atomic electron with the zero-point oscillations of an electromagnetic or electron-positron field leads to a shift in the energy levels of the atomic electron.[1]

A similar situation exists also for the energy spectra of electrons, photons and magnons in a solid (see, for example, Refs. 2 and 3) and for spectra of the natural oscillations in a plasma. In the last case, we need to take into account both the nonlinear interaction between

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the plasma waves (plasmons) and the interaction between the plasmons and the particles.

A large number of works have been devoted to the investigation of the nonlinear interaction of plasma waves. In particular, along with other problems of the nonlinear theory of oscillations in a turbulent plasma, the interaction of plasmons with oscillations at the eigenfrequencies has been studied in Refs. 4–7. In these works (see also Refs. 8 and 9), dispersion equations are obtained with account of the nonlinear interactions of the plasmons. However, the investigation of these equations, in view of their complexity, has been completed only for individual cases, pertaining generally to isotropic plasma. Here, in connection with problems of the excitation and damping of the waves, the effect of the interaction of the waves of the oscillations on the imaginary part of the eigenfrequency has been studied.

It was shown in Refs. 6 and 10 that account of the frequency shift can manifest itself in a number of cases in qualitatively new effects, and therefore, the nonlinear correction to the frequency of the Langmuir waves has been found. In a recently published paper,[11] the value of the nonlinear shift of the eigenfrequencies of the Langmuir waves, due to quadrupole resonance interaction, has been determined. In the case of low-frequency waves, of the type of magnetohydrodynamic waves (MHD), the determination of the shift in the frequencies is of special interest, since in the region of small values of the projection of the wave vector on the magnetic field this shift can turn out to be greater than the frequency of the free oscillations and the coefficient of absorption of the waves.

The purpose of the present work is the study of the shift of the magnetohydrodynamic spectra of the MHD waves—fast (f) and slow (s) magnetohydrodynamic waves and the Alfvén (a) wave in a weakly turbulent plasma. As we shall see, account of the nonlinear shift of the frequencies of the MHD plasma is necessary in the study of the relaxation of MHD plasmons.

In a weakly turbulent plasma, the mean intensity of the waves W is small in comparison with the energy of the particles nT, $w=W/nT \ll 1$ (n is the density of particles, T is their temperature) and a perturbation theory can be developed for finding the shift in the frequency, based on the smallness of the parameter w. Together with this, we shall assume that collective oscillations of rather high intensity are excited in the plasma with random phases, and at such intensities the lifetime of the wave in relation to processes of interaction between the waves can be significantly smaller than the wave lifetime due to interaction of the wave with the particles of the plasma. This leads to the inequalities[12,13]

$$\left(\frac{m_e}{m_i}\right)^{1/2}\frac{\nu_c}{\omega_e} \ll w^2 \ll 1,$$

where $m_{e,i}$ are the masses of the electron and ion, ν_c is the frequency of Coulomb collisions, ω_e is the electron cyclotron frequency.

For the description of the low-frequency waves, we use the model of magnetohydrodynamics which allow us to reduce the results of the theory to simple specific formulas.

Since the averaging is carried out over the phases of the waves, it is most natural to use the method of description of the oscillatory state of the plasma in terms of the number of plasmons $N_\mu(\mathbf{k})$, i. e., quasiparticles with different modes of oscillation ($\mu=a,f,s$) with wave vector $\mathbf{k}$. Of course, the plasma concept can be used only in that case in which the frequency of the oscillations $\omega_\mu(\mathbf{k})$ is large in comparison with the reciprocal of the lifetime of the plasmon, $\omega_\mu(\mathbf{k})\tau_\mu(\mathbf{k}) \gg 1$. Such a description is similar to the well-known description of the state of a solid by specifying the number of electrons and phonons with different quasimomenta and polarizations and, just as in the case of the solid, we can formulate a kinetic equation for $N_\mu(\mathbf{k})$.

Using the concept of plasmons, it is convenient to make use of the method of second quantization, although our problem is essentially purely classical. But the quantum perturbation theory is much simpler than the classical theory; therefore, we shall first use the quantum description and only at the end of the calculations will we go over into the classical limit.

Starting out from the structure of the Hamiltonian of a system that executes small oscillations, we first formulate the general integral equation for finding the shift in the frequencies of oscillation (Sec. 2). In order to apply this further to the plasma, we, as has already been said, make use of the magnetohydrodynamical model and formulate the equations in Lagrangian coordinates, in which the oscillations of the medium are described most simply (Sec. 3). Based on this formulation, we find the modified laws of dispersion of MHD waves in the case of a weakly turbulent plasma (Sec. 4). The frequencies of the s- and a-waves with account of the nonlinear shift are determined by Eqs. (25) and (32).

2. INTEGRAL EQUATION FOR THE DETERMINATION OF THE SHIFT IN OSCILLATION FREQUENCIES

In order to explain the method of finding the frequency shift of the oscillations, we first consider some general mechanical system with many degrees of freedom, in which oscillations are possible. The Hamiltonian of such a system in the case of small oscillations can be represented in the form of the sum of the Hamiltonians of the oscillators with natural frequencies $\omega_\mu(\mathbf{k})$ (μ is the mode of oscillation, $\mathbf{k}$ is the wave vector):

$$H_0=\frac{1}{2}\rho\sum_{\mu,\mathbf{k}}[\dot{\xi}_\mu(\mathbf{k})\dot{\xi}_\mu(-\mathbf{k})+\omega_\mu^2(\mathbf{k})\xi_\mu(\mathbf{k})\xi_\mu(-\mathbf{k})]; \quad (1)$$

here $\xi_\mu(\mathbf{k})$ is the spatial component of the Fourier displacement vector $\boldsymbol{\xi}=\boldsymbol{\xi}(\mathbf{r}_0, t)$ relative to the equilibrium position $\mathbf{r}_0$, and the index μ serves to identify the part of ξ_k corresponding to the natural μ-oscillation (ρ is the density).

If we now take into account the anharmonicity of the oscillations, then the Hamiltonian of the system will have the form $H=H_0+H_{\text{int}}$, where H_{int} is some nonquadratic function of the displacements $\xi(\mathbf{k})$. We can regard it as the

Hamiltonian of interaction of the oscillators or the quasiparticles corresponding to them. Precisely these interactions will lead to a change in the frequencies of the noninteracting waves. The charged (modified) frequencies will be denoted by $\tilde{\omega}_\mu(\mathbf{k})$, and we shall write their connection with $\omega_\mu(k)$ in the form $\tilde{\omega}_\mu^2(k)=\omega_\mu^2(k)+\delta_\mu^2(k)$, where the real quantities $\delta_\mu(\mathbf{k})$ denote the frequency shifts (neither the mode μ nor the wave vector $\mathbf{k}$ are changed!) We note that such a renormalization of the frequency in the Hamiltonian H_0 is necessary in those cases in which the frequency $\omega_\mu(\mathbf{k})$ can vanish, since in this case singularities appear in the integrals of the perturbation-theory series (see (9)).

In order to find the quantity $\delta_\mu(\mathbf{k})$, it is convenient first to renormalize the Hamiltonian H_0 by introducing the frequency shifts into it. In other words, we rewrite the Hamiltonian H in the form of the sum $H=\tilde{H}_0+\tilde{H}_{\text{int}}$, in which $\tilde{H}_0$ is described by the expression (1), but with modified frequency $\tilde{\omega}_\mu(k)$. In correspondence with this, $H_{\text{int}}=H_{\text{int}}-\Delta H_0$, where

$$\Delta H_0=\frac{1}{2}\rho\sum_{\mu,\mathbf{k}}\delta_\mu^2(\mathbf{k})\xi_\mu(\mathbf{k})\xi_\mu(-\mathbf{k}). \quad (2)$$

After such renormalization, the Hamiltonian of the interaction of the quasiparticles H_{int} no longer produces shifts in their frequencies. We shall use this condition in the derivation of the integral equation determining the shift in the frequencies of the oscillations. For this purpose, we first transform to the quantum mechanical description, i.e., we shall assume the quantities $\xi_1 \equiv \xi_{\mu_1}(\mathbf{k})$ $(1\equiv(\mu_1,\mathbf{k}_1);\ 2\equiv(\mu_2,\mathbf{k}_2);\ldots)$ to be operators satisfying the commutation relations

$$[\hat{\xi}_1,\dot{\hat{\xi}}_2]=i\rho^{-1}\Delta(1-2)$$

(Planck's constant is set equal to unity, $\Delta(1-2)$ is the Kronecker symbol). These operators can be expressed in terms of quasiparticle creation (c_1^+) and annihilation (c_1) operators satisfying the conditions $[c_1,c_2^+]=\Delta(1-2)$:

$$\hat{\xi}_1=\frac{1}{(2\rho\tilde{\omega}_1)^{1/2}}C_1=\frac{1}{(2\rho\tilde{\omega}_1)^{1/2}}(c_1+c_{-1}^+); \quad (3)$$

finally, we express the Hamiltonians $\tilde{H}_0$ and $\tilde{H}_{\text{int}}$ in terms of the operators c_1 and c_1^+.

The Hamiltonian $\tilde{H}_0$ is obviously represented in the form

$$\tilde{H}_0=\sum_1\tilde{\omega}_1(N_1+{}^1/_2), \quad (4)$$

where $N_1=c_1^+c_1$ is the operator of the number of quasiparticles $\mu_1,\mathbf{k}_1$ whose eigenvalues are $N_1=0,1,2,\ldots$. So far as the Hamiltonian $\tilde{H}_{\text{int}}$ is concerned, it is first necessary to expand H_{int} in a series in powers of the variables $\xi_\mu(k)$. As a result, we obtain

$$\tilde{H}_{\text{int}}=H_3+H_4+\ldots-\Delta H_0, \quad (5)$$

where

$$H_3=\sum_{1,2,3}V(1,2,3)C_1C_2C_3\Delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3),$$

$$H_4=\sum_{1,2,3,4}V(1,2,3,4)C_1C_2C_3C_4\Delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4), \quad (6)$$

$$\Delta H_0=\sum_1\frac{\delta_1^2}{\tilde{\omega}_1}C_1C_{-1},$$

and where $V(1,2,\ldots,n)$ are certain symmetrized functions of the wave vectors and modes of the oscillations $(\mathbf{k}_1,\mu_1;\ \mathbf{k}_2,\mu_2;\ldots;\mathbf{k}_n,\mu_n)$ having the meaning of the amplitudes of the interaction of the quasiparticles.

We shall now show how to find the shift in the frequencies of the interacting quasiparticles. For this purpose, we recall that the energy of the system with the Hamiltonian $H=H_0+H_{\text{int}}$ is not equal to the sum of the energies of the individual quasiparticles and will be a function of their distribution function. It is natural to determine the frequency of the quasiparticle ω_n as the variational derivative of the total energy E_i in the state $|i\rangle$ (the state $|i\rangle$ is determined by the set of the occupation numbers $|i\rangle=|\{N_n\}\rangle$):

$$\omega_n=\delta E_i/\delta N_n. \quad (7)$$

We can show that we achieve this same result if we use the Green's function formalism in the calculation of the real part of the self-energy of the quasiparticles.[14]

In the problem considered by us, the Hamiltonian of the system has the form $H=\tilde{H}_0+\tilde{H}_{\text{int}}$. We shall choose the Hamiltonian $\tilde{H}_0$ so that the Hamiltonian $\tilde{H}_{\text{int}}$ does not produce a change in the real part $\tilde{\omega}_n$ of the frequency. Therefore the integral equation for finding the quantity $\delta_\mu(\mathbf{k})$

$$\operatorname{Re}\left(\frac{\delta}{\delta N_n}(E_i-E_{0i})\right)=0, \quad (8)$$

should hold, where E_{0i} is the eigenvalue of the energy of the Hamiltonian $\tilde{H}_0$;

$$\tilde{H}_0|i\rangle=E_{0i}|i\rangle,\quad E_{0i}=\sum_1\tilde{\omega}_1(N_1+{}^1/_2).$$

In the Green's function formalism, this means that the real part of the self-energy of the already renormalized quasiparticles vanishes. The imaginary part of the self-energy describes the damping of the quasiparticles and in the case of small damping is equal to the reciprocal of their lifetime.[15]

Omitting the calculations, we give the integral equation for the determination of δ_n, limiting ourselves to the second approximation of perturbation theory:

$$\delta_n^2=48\tilde{\omega}_n\sum_1V(1,n,1,n)N_1+72\tilde{\omega}_n\mathscr{P}\sum_{1,2}|V(n,1,2)|^2$$
$$\times\left[\frac{\Delta(\mathbf{k}_n-\mathbf{k}_1-\mathbf{k}_2)}{\tilde{\omega}_n-\tilde{\omega}_1-\tilde{\omega}_2}N_1-\frac{\Delta(\mathbf{k}_n+\mathbf{k}_1-\mathbf{k}_2)}{\tilde{\omega}_n+\tilde{\omega}_1-\tilde{\omega}_2}(N_1-N_2)\right]; \quad (9)$$

where the symbol $\mathscr{P}$ denotes the principal value of the sum.

3. THE MAGNETOHYDRODYNAMIC APPROXIMATION

We now apply the general integral equation (9) to find the frequency shifts of the low-frequency oscillations in a plasma. With this aim, we shall make use of the magnetohydrodynamic approximation, i.e., we shall use the equations of magnetohydrodynamics for the description of the low-frequency oscillations. This allows us to find the specific form of the amplitude of the interaction of the plasmons $V(1,2,3)$, $V(1,2,3,4)$, which enter into the Hamiltonians H_3, H_4 which are in turn defined by the formulas (6). In order to stay within the framework of the general scheme of Sec. 2, we must introduce the quantity $\xi_\mu(\mathbf{k})$. Such a quantity will be the displacement of an element of the medium $\boldsymbol{\xi}(\mathbf{r}_0, t) = \mathbf{r}(\mathbf{r}_0, t) - \mathbf{r}_0$, where $\mathbf{r}$ is the location of the element at the instant of time t, $\mathbf{r}_0$ is its location at the time $t=0$, i.e., $\mathbf{r}_0$ is the Lagrangian coordinate of the medium.

We formulate the equations of magnetohydrodynamics in Lagrangian coordinates. In Eulerian variables, the equations of magnetohydrodynamics are well known.[16] In order to transform to Lagrangian variables, we note that

$$\nabla_i = (\nabla_i r_{0j}) \nabla_{0j} = (A^{-1})_{ij} \nabla_{0j}, \quad \left(\nabla_i = \frac{\partial}{\partial r_i}, \quad \nabla_{0i} = \frac{\partial}{\partial r_{0i}}\right), \tag{10}$$

where $(A^{-1})_{ij}$ are the elements of the matrix inverse to the matrix $A = \|\nabla_{0i} r_j\|$. We can show (see the Appendix) that

$$A^{-1} = D^{-1}\hat{u}, \quad \hat{u} = A^2 - S_1 A - {}^1/_2(S_2 - S_1^2) I, \tag{11}$$

where $D = \operatorname{Det} A = (1/6)(S_1^3 - 3S_1S_2 + 2S_3)$; $S_k \equiv \operatorname{Sp} A^k$ and I is the unit matrix. Introducing the displacement $\boldsymbol{\xi}(\mathbf{r}_0, t)$, we obtain

$$A = \|(\nabla_{0i}\xi_j) + \delta_{ij}\|, \tag{12}$$

$$D = 1 + d_1 + {}^1/_2(d_1^2 - d_2) + {}^1/_6(d_1^3 - 3d_1d_2 + 2d_3), \tag{13}$$

$$u_{ij} = (1 + d_1 + {}^1/_2 d_1^2 - {}^1/_2 d_2)\delta_{ij} - (1 - d_1)(\nabla_{0j}\xi_i) + (\nabla_{0e}\xi_i)(\nabla_{0j}\xi_e), \tag{14}$$

where

$$d_1 = (\nabla_0\boldsymbol{\xi}), \quad d_2 = (\nabla_{0j}\xi_i)(\nabla_{0i}\xi_j), \quad d_3 = (\nabla_{0j}\xi_e)(\nabla_{0e}\xi_i)(\nabla_{0i}\xi_j).$$

It is clear that in the Lagrangian variables $\rho = D^{-1}\rho_0$, where $\rho_0 = \rho(\mathbf{r}_0, 0)$ and $p = p(D^{-1}\rho_0)$.

From the condition of freezing-in of the lines of force of the magnetic field, we can show[17] that

$$\mathbf{B}(\mathbf{r}_0, t) = D^{-1}B_0(\mathbf{b} + \mathbf{d}),$$

where

$$\mathbf{B}_0 = \mathbf{B}_0(\mathbf{r}_0, 0), \quad \mathbf{b} = \mathbf{B}_0/B_0, \quad \mathbf{d} = (\mathbf{b}\nabla_0)\boldsymbol{\xi}.$$

Noting that the particle velocity of the medium is given by $\mathbf{v} = \dot{\boldsymbol{\xi}}$, we can rewrite the equation of motion of the particles in the form

$$\rho_0\ddot{\boldsymbol{\xi}} = -(\hat{u}\nabla_0)(p + {}^1/_2 B_0^2 D^{-2}(1 + 2(\mathbf{bd}) + d^2)) + D^{-1}B_0(\mathbf{b} + \mathbf{d})(\hat{u}\nabla_0)(D^{-1}B_0(\mathbf{b} + \mathbf{d})). \tag{15}$$

The energy of the magnetohydrodynamic medium in Lagrangian coordinates is determined by the formula

$$E = \frac{1}{2}\int \rho_0[\dot{\boldsymbol{\xi}}^2 + D^{-1}V_A^2(1 + 2(\mathbf{bd}) + d^2) + \varepsilon]\, dV_0, \tag{16}$$

where $V_A = B_0/\sqrt{\rho_0}$ is the Alfvén velocity and ε is the internal energy density.

The expression for the energy E (16) includes the energy E_0 of the medium at rest; therefore the Hamiltonian of the medium must be defined as the difference $E - E_0$. This Hamiltonian can be expanded in powers of the displacement $\boldsymbol{\xi}$: $H = H_0 + H_3 + H_4 + \cdots$, where H_0 is a quadratic form in $\boldsymbol{\xi}$ and H_n is a form which contains $\boldsymbol{\xi}$ in the power n $(n = 3, 4, 5\ldots)$:

$$H_0 = \frac{1}{2}\int \rho_0\, dV_0[\dot{\boldsymbol{\xi}}^2 + (V_S^2 + V_A^2)d_1^2 + V_A^2(d^2 - 2d_1(\mathbf{bd}))], \tag{17}$$

$$H_3 = \frac{1}{2}\int \rho_0\, dV_0\left[\left(V_S^2 - \rho_0\frac{\partial V_S^2}{\partial\rho_0}\right)d_1^3 - (V_S^2 + V_A^2)d_1d_2 + V_A^2(d_1^2(\mathbf{bd}) + d_2(\mathbf{bd}) - d_1d^2)\right], \tag{18}$$

$$H_4 = \frac{1}{24}\int \rho_0\, dV_0\left[\left(V_A^2 + V_S^2 + \rho_0^2\frac{\partial^2 V_S^2}{\partial\rho_0^2}\right)d_1^4 + (V_A^2 + V_S^2)d_1(3d_1d_2 + 8d_3) - 12\left(V_S^2 - \rho_0\frac{\partial V_S^2}{\partial\rho_0}\right)d_1^2d_2 + 12V_A^2(d_1^2 + d_2)d^2 - 8V_A^2(\mathbf{bd})(d_1^3 + 3d_1d_2 + 2d_3)\right] \tag{19}$$

($V_S = (\partial p/\partial\rho_0)^{1/2}s$ is the sound velocity). All these expressions are valid both for homogeneous and for inhomogeneous plasma. (We note that H_0 is usually employed for the study of the stability of an inhomogeneous plasma.[18])

In what follows, we shall consider a weakly turbulent plasma and limit ourselves in the expansion of H to only the first three terms. The Hamiltonians H_3 and H_4 describe the interaction between the third and four plasmons. Account of H_4 is necessary since the change in the dispersion of the magnetohydrodynamic waves is determined both by H_3 and H_4. Moreover, account of H_4 is also necessary in the study of the relaxation of Alfvén waves. For the study of low-frequency waves in a cold, weakly turbulent plasma, the components H_n with $n > 4$ are unimportant.

Transforming to the Fourier components of the displacement vector we get (after diagonalization) the Hamiltonian H_0 in the form (1), where μ denotes the type of plasmon $\mu = a, s, f$ and $\omega_\mu(\mathbf{k})$ are the frequencies of the noninteracting magnetohydrodynamic waves:

$$\omega_a = |k_\parallel| V_A,$$
$$\omega_{f,s} = k[(V_A^2 + V_S^2 + 2V_AV_S\varkappa_\parallel)^{1/2} \pm (V_A^2 + V_S^2 - 2V_AV_S\varkappa_\parallel)^{1/2}]^{1/2}; \tag{20}$$

where $k_\parallel = (\mathbf{k}\cdot\mathbf{b})$, $\varkappa = \mathbf{k}/k$, $\varkappa_\parallel = (\boldsymbol{\varkappa}\cdot\mathbf{b})$.

Choosing $\tilde{H}_0$ as the basic Hamiltonian and introducing the operators c_1 and c_1^+, we obtain $\tilde{H}_0$ in the form (4) and $\tilde{H}_{\text{int}}$ in the form (6). The interaction amplitudes in these expressions have the following form:

$$V(1,2,3) = i\frac{V_A^2}{12(2\rho)^{1/2}}\frac{k_1k_2k_3}{(\tilde{\omega}_1\tilde{\omega}_2\tilde{\omega}_3)^{1/2}}F(1,2,3),$$
$$V(1,2,3,4) = \frac{V_A^2}{96\rho}\frac{k_1k_2k_3k_4}{(\tilde{\omega}_1\tilde{\omega}_2\tilde{\omega}_3\tilde{\omega}_4)^{1/2}}F(1,2,3,4), \tag{21}$$

where $F(1,2,3)$ and $F(1,2,3,4)$ are certain dimensionless functions of the angles between the wave vectors and the magnetic field (their specific form is found in Ref. 15; the normalized volume in Eq. (21) should be equal to unity).

We now make the assumption that the gas of plasmons is found in the equilibrium state with temperature T^*. Then the number of plasmons will be determined by the Rayleigh-Jeans distribution

$$N_1 = T^*/\tilde{\omega}_1 \quad (22)$$

(Boltzmann's constant, as also Planck's constant, is taken to be unity) and Eq. (9) for the determination of $\delta_\mu(\mathbf{k})$ takes the form

$$\delta_n^2 = \frac{T^* V_A^2}{4\rho} k_n^2 \sum_1 \frac{k_1^2}{\tilde{\omega}_1^2}\Big[2F(1,n,1,n) + \mathscr{P}\sum_2 \frac{k_2^2 V_A^2 |F(1,2,n)|^2}{\tilde{\omega}_2^2} \times\Big(\frac{\tilde{\omega}_2\Delta(\mathbf{k}_n-\mathbf{k}_1-\mathbf{k}_2)}{\tilde{\omega}_n-\tilde{\omega}_1-\tilde{\omega}_2} + \frac{(\tilde{\omega}_1-\tilde{\omega}_2)\Delta(\mathbf{k}_n+\mathbf{k}_1-\mathbf{k}_2)}{\tilde{\omega}_n+\tilde{\omega}_1-\tilde{\omega}_2}\Big)\Big]. \quad (23)$$

4. MODIFICATION OF THE DISPERSION LAWS OF MAGNETOHYDRODYNAMIC WAVES

The change in the dispersion laws of magnetohydrodynamic waves is important for a- and s-waves in the region of small values of $k_\parallel = 0$. Therefore, in the calculation of the frequency shifts δ_a and δ_s, we need to take into account in (23) only the terms that differ from zero at $k_\parallel = 0$.

We shall also consider the case of a plasma of low pressure, when the relation $V_S^2/V_A^2 \ll 1$ holds. We first consider the s-wave; for it, it is necessary to take into account scattering processes of fourth order:

$$s+a \rightleftarrows s+a, \quad s+f \rightleftarrows s+f$$

and processes of third order

$$s \rightleftarrows a+f, \quad s \rightleftarrows f+f.$$

In Eq. (23), the indices 1, 2, and n for these processes denote $1 \equiv (\mathbf{k}_1, a)$, $1 \equiv (\mathbf{k}_1, f)$ and $2 \equiv (\mathbf{k}_2, f)$, $n \equiv (\mathbf{k}, s)$.

Making use of the expressions for $F(1, n, 1, n)$ and $F(1, 2, n)$ from Ref. 15, and making the transition from summation to integration in the space of $\mathbf{k}_1$ and $\mathbf{k}_2$, we get

$$\delta_s^2 = \alpha_s k_\perp^2 V_s^2 w_s, \quad w_s = W/\rho V_s^2, \quad (24)$$

where $W = T^*\Gamma$ is the energy density of the turbulent pulsations, Γ is the phase volume, occupied by the waves,[12,13,19] α_s is a numerical coefficient of the order of 0.1.

Thus, the modified dispersion law for slow magnetosonic waves has the form

$$\tilde{\omega}_s(\mathbf{k}) = V_s(k_\parallel^2 + \alpha_s k_\perp^2 w_s)^{1/2}. \quad (25)$$

We see that the correction to the ordinary dispersion law for the s-wave is different from zero at $k_\parallel = 0$ and is proportional to the square root of the level of the turbulent pulsations W.

We consider the a-waves further. In the calculation of δ_a, singularities develop in the integrals in the region of small $k_\parallel$ for processes with participation of the s-waves $a+s \rightleftarrows a+s$, $a \rightleftarrows s+f$ $(n \equiv (\mathbf{k}, a)$, $1 \equiv (\mathbf{k}_1, s)$, $2 \equiv (\mathbf{k}_2, f))$ and also for the processes $a+a \rightleftarrows a+a$ and $a \rightleftarrows a+f$ $(1 \equiv (\mathbf{k}_1, a)$, $2 \equiv (\mathbf{k}_2, f))$. The singularities for the first group of processes are removed if we take into account the modified dispersion law for the s-waves (25). As a result, it turns out that the contribution from these processes to δ_a^2 is equal to $k_\parallel^2 V_A^2 \sqrt{w_s}$ with accuracy to within a numerical factor. Since this contribution is proportional to $k_\parallel^2$, it is unimportant. The principal contribution to δ_a is made by the second group of processes.

Taking into account the nonvanishing terms in the expressions for $F(a, a, a, a)$ and $F(a, a, f)$ in the case $k_\parallel = 0$,[16] we obtain

$$\delta_a^2(\mathbf{k}) = \frac{9T^* V_A^2}{8\rho} \sum \frac{(\mathbf{k}[\mathbf{k}_1\mathbf{b}])^4}{\tilde{\omega}_1^2 k_\perp^2 k_{1\perp}^2}. \quad (26)$$

The solution of this integral equation for δ_a must be sought in the form

$$\delta_a^2(\mathbf{k}) = c k_\perp^2 V_A^2, \quad (27)$$

where c is a certain constant. Actually, making the transition in (26) from the sum to the integral, we obtain the following equation for the determination of c:

$$c = \frac{27}{2^8\pi^2}\frac{T^*}{\rho V_A^2}\int dk_\parallel \int \frac{k_\perp^3\, dk_\perp}{ck_\perp^2 + k_\parallel^2}. \quad (28)$$

After integration, this equation takes the form

$$c^2 = \frac{27}{2^5} w_a \Big(1 - \frac{1}{c}\ln(1+c) + 2c^{1/2}\,\mathrm{arctg}\,\frac{1}{c}\Big). \quad (29)$$

Since the level of the turbulent pulsations is assumed to be small, then the constant c will be small and therefore the expression in the parentheses can be expanded in a series in powers of c, keeping only the principal term in it, equal to $\pi\sqrt{c}$. As a result, we undergo transition to the following expression for c:

$$c = \alpha_a w_a^{2/3}, \quad w_a = W/\rho_0 V_A^2, \quad d_a \sim 1. \quad (30)$$

Thus

$$\delta_a^2(\mathbf{k}) = \alpha_a k_\perp^2 V_A^2 w_a^{2/3} \quad (31)$$

and the frequency of the Alfvén plasmons will be equal to

$$\tilde{\omega}_a(\mathbf{k}) = V_A(k_\parallel^2 + \alpha_a k_\perp^2 w_a^{2/3})^{1/2}. \quad (32)$$

We see that the frequency of the a-waves at $k_\parallel = 0$ differs from zero and is equal to

$$\tilde{\omega}_a(\mathbf{k}) \approx k_\perp V_A w_a^{1/3},$$

i.e., the dispersion law for the a-waves changes significantly in the region of small $k_\parallel$.

For the f-wave, the account of the nonlinear interactions of the waves leads only to an unimportant renormalization of its phase velocity:

$$\tilde{\omega}_f(\mathbf{k}) = kV_A(1 + \alpha_f w_s^{1/2}), \quad \alpha_f \approx 0.1. \quad (33)$$

In finding the frequency shifts, we took into account only the Hamiltonian H_3 and H_4. It can be shown that account of the Hamiltonians at $n>4$ leads only to a small additional correction just as in the higher terms of perturbation theory.

5. RELAXATION IN A GAS OF MAGNETOHYDRODYNAMIC PLASMONS

The relaxation times of plasmons was calculated in Ref. 15 with account of triple processes only, described by the Hamiltonian H_3. Account of four processes, described by the Hamiltonian H_4, leads, as can be shown, to diverging expressions for the probabilities of relaxation of the plasmons, if we do not take the modifications of the dispersion laws of magnetohydrodynamic into account. It is therefore of interest to consider the role of four processes, and to take into account in this case the shift of the frequencies of the magnetohydrodynamic waves. It can be shown that such an account leads to a removal of the divergences, and it turns out that the contribution of the four processes to the probability of relaxation of the a-waves will be of the same order as the contribution of the triple processes. For the relaxation process of magnetosonic waves, account of four processes leads only to small corrections.

Using the well known expression[15] for the relaxation times of plasmons $\tau_\mu(\mathbf{k})$, we can show that the relaxation time of an a-plasmon due the four scattering processes $a+a\rightleftarrows a+a$ is determined by the formula

$$1/\tau_a(\mathbf{k})\sim\omega_{ak_\perp}{}^4V_A{}^3/k_{max}\bar{\omega}_a{}^2, \tag{34}$$

while the relaxation time of an a-plasmon due to four processes of interaction with s-plasmon $a+a\rightleftarrows a+s$ and $a+s\rightleftarrows a+s$—by the formula

$$1/\tau_a(\mathbf{k})\sim w_sk_\perp{}^2V_A/k_{max}. \tag{35}$$

We now turn our attention to the fact that these expressions, just as in the corresponding expressions for the reciprocal relaxation times $1/\tau_a(\mathbf{k})$ of plasmons due to triple processes, are proportional to the level of turbulent pulsations W.

In the calculation of the relaxation times of excitations in the spectrum of s-waves due to the processes $s\rightleftarrows s+s$, divergent expressions also appear if we do not take into account the frequency shifts of the s-plasmons. In Ref. 20, these divergences are removed by the inclusion of dissipation processes. At small damping of the waves, the interaction between the s-waves led to a very rapid relaxation.

If we use the modified dispersion laws of magnetohydrodynamic waves, then the divergences are eliminated, the relaxation time does not depend on the damping of the waves and is of the same order as the relaxation time calculated with account of the other types of three-wave interactions:

$$1/\tau_s(\mathbf{k})\sim\bar{w}_sk_\parallel{}^2V_S/k_{max}. \tag{36}$$

APPENDIX

We now explain how to obtain Eq. (11) for the inverse matrix.

If $A=\|a_{ik}\|$ is some matrix of order n, then for the calculation of the inverse matrix A^{-1}, we must use the method of Leverieu, according to which the matrix A^{-1} is determined by the expression

$$A^{-1}=\frac{1}{p_n}(A^{n-1}-p_1A^{n-2}-\ldots-p_nI)$$

(see Ref. 21). The coefficients p_n are found from the recurrence relations according to the Newton formulas

$$kp_k=S_k-p_1S_{k-1}-\cdots-p_{k-1}S_1\quad(k=1,2,\ldots,n),$$
$$p_n=(-1)^{n-1}\operatorname{Det}A,\quad S_k=\operatorname{Sp}A^k.$$

At $n=3$, we obtain the formula (11).

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