band IV–VI compounds whose static dielectric constant is anomalously large at low temperatures. Thus, for example, soft TO modes were observed in PbTe,[5] SnTe,[6] $Pb_{1-x}Sn_xTe$[7] and others. It should be noted that the explanation offered by Kawamura *et al.*[8] for the existence of soft modes in the semiconductors, as consequences of the interaction of the electrons with the soft photons, seems unsatisfactory from our point of view. As follows from formulas (14) and (15) of the present paper, allowance for the polarization interaction in addition to the deformation interaction, makes the gap-renormalization effect illusory.

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Dynamics of Onsager-Feynman vortices in a rotating superfluid system of the pulsar type

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Equations are derived for the description of the dynamics of a vortex lattice in a rotating superfluid system of the pulsar type. The observed time dependence of the angular velocity of the normal part of the system is attributed to interaction of the system with Onsager-Feynman vortices.

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1. The nonstationary rotation of He II has been under intensive experimental investigation in recent years (see, e.g., the review)[1] in connection with a check on the premise that a pulsar is a superconducting system in which the interaction of the superfluid (neutron) component with the normal (proton) component is effected via Onsager-Feynman vortices.

It is known (detailed references to the original sources are contained, e.g., in the review)[1] that the time dependence of the angular velocity of pulsars has not yet found a satisfactory explanation. It was shown in experiments[2–3] that an analogous behavior of the angular velocity with time is observed also for rotating He II. These experiments have shown convincingly that the nature of the time dependence of the angular velocity should be the same for He II as for pulsars. We shall not present here the arguments (advanced already in[1]) that lead to this conclusion.

It is shown in the present paper that allowance for the motion of the vortex lattice and its interaction with the normal component in a rotating superfluid liquid, under conditions when there is no equilibrium between the actual angular velocity of the normal component and the number of vortices in the system, explains fully the experimentally observed time dependence of the angular velocity.

The application of our approach to the observed dependence of the angular velocity of a pulsar can explain the mechanisms of the radiative losses of the star, its structure (it permits measurements of the angular momenta of the superfluid and normal components and of the core) and yields quantitative information on the coefficients of viscous-friction of the vortices against the normal component.

We assume throughout that the normal component moves like a rigid body (i.e., it duplicates fully the rotation of the vessel or of the core). For pulsars it is legitimate to disregard the drag waves in the normal component, inasmuch as in pulsars the normal (charged) component is frozen into the core (if the latter exists) by the ultrastrong magnetic field of the stars. The analysis is carried out in the laboratory frame.

We note also the following important feature of the employed terminology. The symbol $\mathbf{v}_0$ denotes throughout the proper velocity of the superfluid-component velocity. By this we mean the velocity due to extraneous forces (usually connected with external sources of pres-

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sure or temperature gradients, with the gravitational field, etc.), which the superfluid component would have even if it contained no vortices. If there are no such forces and at a certain instant of time $\mathbf{v}_0$ is equal to zero, then it remains equal to zero subsequently, the onset of vortices notwithstanding. Thus, the proper velocity of the superfluid component is the difference between its total velocity and the combined (at this point) circulation velocity due to all the present Onsager-Feynman vortices. This subdivision of the total velocity of an element of the superfluid component seems to us extremely illustrative when it comes to separating the contribution of the vortices to the motion of the superfluid phase.

2. Each vortex is acted upon by the following three forces: (a) The Magnus force exerted by the superfluid component and determined by the vortex velocity relative to the superfluid component:

$$\mathbf{F}_M=\rho_s(\mathbf{v}_0-\mathbf{v}_L)\times\boldsymbol{\Gamma}_0\,, \tag{1}$$

where $\mathbf{v}_L$ is the instantaneous velocity of the vortex-axis element in the plane passing through this element perpendicular to the circulation vector $\boldsymbol{\Gamma}_0(\Gamma_0=h/m)$. (b) The force of interaction of this vortex with the remaining vortices, determined by the flow produced around the given vortex element by the circulation velocities of all the remaining vortices:

$$\mathbf{F}_c=\rho_s\mathbf{v}_c\times\boldsymbol{\Gamma}_0\,, \tag{2}$$

where $\mathbf{v}_c$ is the combined circulation velocity at the point of the axis of the given vortex element. (c) The force of friction of the vortex against the normal component, determined according to Hall and Vinen by the velocity of the vortex relative to the normal component:

$$\mathbf{F}_{fr}=-\rho_n\eta\Gamma_0(\mathbf{v}_L-\mathbf{v}_n)+\rho_n\beta(\mathbf{v}_L-\mathbf{v}_n)\times\boldsymbol{\Gamma}_0\,, \tag{3}$$

where $\mathbf{v}_n$ is the local velocity of the normal component at the point of the axis of the given vortex element.

The dynamics of a given vortex element is determined by the condition that the resultant force acting on the element be zero. Disregarding the possible pinning forces, we can obtain the equation of motion of a linear vortex by equating to zero the sum of the forces (1)–(3), i.e.,

$$\rho_s(\mathbf{v}_0-\mathbf{v}_L)\times\boldsymbol{\Gamma}_0+\rho_s\mathbf{v}_0\times\boldsymbol{\Gamma}_0 =\rho_n\eta\Gamma_0(\mathbf{v}_L-\mathbf{v}_n)-\rho_n\beta(\mathbf{v}_L-\mathbf{v}_n)\times\boldsymbol{\Gamma}_0\,. \tag{4}$$

When the system rotates uniformly, an equilibrium situation (without energy dissipation) corresponds to complete dragging of the vortices by the normal component, i.e., in this case

$$\mathbf{v}_L=\mathbf{v}_n=\boldsymbol{\omega}\times\mathbf{r}\,. \tag{5}$$

We emphasize that, according to Feynman, the equilibrium situation, when the vortices have a uniform density distribution

$$n=2\omega/\Gamma_0 \tag{6}$$

corresponds to formation of vortices in an immobile superfluid component, when $\mathbf{v}_0=0$. According to (4) we then obtain in this case

$$\mathbf{v}_c=\mathbf{v}_L=\boldsymbol{\omega}\times\mathbf{r}\,. \tag{7}$$

Indeed, the resultant circulation velocity at the axis point of the given element of the vortex is determined by the total number N of vortices inside the orbit of this vortex, i.e.,

$$\oint \mathbf{v}_c\,d\mathbf{l}=N\Gamma_0,$$

from which it follows that

$$v_c=\frac{N}{2\pi r^2}\boldsymbol{\Gamma}_0\times\mathbf{r}=\boldsymbol{\omega}\times\mathbf{r}$$

in full agreement with (7). Thus, the equilibrium situation corresponds to total dragging of the vortices by the normal component with cancellation of the forces (1) and (2) with the vortices produced in the immobile superfluid component having the Feynman density (6).

We note that we did not include in (4) any other forces except (1)–(3) (e.g., the forces of interaction of the vortex with the system boundary). It is easily seen that the Feynman equilibrium situation will be realized only when these forces are proportional to either $\mathbf{v}_L-\mathbf{v}_c$ or to $\mathbf{v}_L-\mathbf{v}_n$. But allowance for these forces is equivalent to a redefinition of the coefficients η and β. Therefore (4) should be regarded as the exact equation of motion of the vortex.

3. Assume now that, starting with the instant $t_0=0$, the angular velocity of the vessel begins to change. Then the vortices begin to move relative to the normal component, since now the force (1) cannot cancel the force (2), because the angular velocity of the normal component has changed, while the number of vortices inside the orbit of a given vortex is unchanged.

We denote the velocity of the given vortex element by

$$\mathbf{v}_L=\boldsymbol{\omega}\times\mathbf{r}+v_r\frac{\mathbf{r}}{r}+v_\varphi\frac{\boldsymbol{\Gamma}_0\times\mathbf{r}}{\Gamma_0 r}, \tag{8}$$

where $v_r=dr/dt$ and $v_\varphi=rd\varphi/dt$. Then Eq. (4) reduces to the system

$$\begin{gathered}\rho_s[r^2(0)\omega_0-r^2(t)\omega(t)]=[\rho_n\eta v_r+(\rho_s-\rho_n\beta)v_\varphi]r(t),\\ \rho_n\eta v_\varphi=(\rho_s-\rho_n\beta)v_r,\end{gathered} \tag{9}$$

where ω_0 is connected with the vortex density at the instant $t_0=0$ by the relation (6). (The second equation of the system (9) shows, in particular, that the number of vortices in the system remains unchanged at $\rho_n=0$ independently of the rotation of the vessel walls or of the core.)

Solving the system (9) with respect to $r(t)$, we obtain for the i-th vortex the equation

$$\frac{d}{dt}\frac{r_i^2(t)}{r_i^2(0)}=a\left[1-\frac{\omega(t)}{\omega(0)}\frac{r_i^2(t)}{r_i^2(0)}\right], \tag{10}$$

where $a = 2\eta\rho_n\rho_s\omega_0/[(\eta\rho_n)^2 + (\rho_s - \rho_n\beta)^2]$. It is seen therefore that the ratio

$$r_i^2(t)/r_i^2(0) = \theta(t) \tag{11}$$

is a universal function of the time, independent of the number of the vortex (a consequence of the hypothesis that the entire normal component rotates like a rigid body). The function $\theta(t)$ is described by the equation

$$\frac{d\theta}{dt} = a\left[1 - \frac{\omega(t)}{\omega_0}\theta(t)\right] \tag{12}$$

with the condition $\theta(0) = 1$.

It is easily seen here that the density of the vortices at the instant of time t is

$$n(t) = \frac{n(0)}{\theta(t)} = \frac{2\omega_0}{\Gamma_0\theta(t)}, \tag{13}$$

i.e., dilatation or contraction of the vortex lattice takes place with conservation of the homogeneity of the vortex density. If $\omega(t)$ is known, then Eq. (12) determines completely the dynamics of the vortex lattice in the system.

We, however, are interested in the following problem: Assume that at the initial instant of time the number of vortices in the system corresponded, according to formula (6), to a frequency ω_0, and the actual angular velocity of the normal component was $\omega(0) \neq \omega_0$. It is required to find the functions $\omega(t)$ and $\theta(t)$ in a free system with friction (against an external medium, or with radiation losses, as is the case in pulsars). To this end it is necessary to derive an equation for $\omega(t)$, which we now proceed to do.

4. The equation of motion of the normal component is, naturally, of the form

$$I_n\frac{d\omega}{dt} = \mathbf{K}_{fr} + \mathbf{K}_{ext}, \tag{14}$$

where I_n is the moment of inertia of the normal part of the system, $\mathbf{K}_{fr}$ is the moment of the friction forces against the normal component, and $\mathbf{K}_{ext}$ is the moment of the external friction forces, which we take in the form

$$\mathbf{K}_{ext} = -\gamma I_n\omega(t),$$

which corresponds, e.g., to the case of viscous friction against an external medium at low relative velocities. The moment of the friction forces $\mathbf{K}_{fr}$ can be easily calculated. The moment of the friction force of the i-th vortex is equal to $\mathbf{K}_i = \mathbf{r}_i \times \mathbf{F}_{fri}$ (per unit length of a vortex whose axis is located at a distance $\mathbf{r}_i$ from the system rotation axis). Using expression (3) for $\mathbf{F}_{fr}$ and the system (9), we readily see that

$$\mathbf{K}_i = \rho_s r_i v_{r_i}\Gamma_0.$$

Multiplying this by the length l of the vortex and summing over all the vortices, we obtain

$$\mathbf{K}_{fr} = \rho_s\Gamma_0 l\sum_{i=1}^{N(t)} r_i v_{r_i} = \frac{a\rho_s\Gamma_0 l}{2}\left[\frac{1}{\theta(t)} - \frac{\omega(t)}{\omega_0}\right]\sum_{i=1}^{N(t)} r_i^2(t).$$

Replacing here the summation by integration in accordance with the scheme

$$\sum_{i=1}^{N(t)} r_i^2(t) = 2\pi\int_0^R n(t)r^3\,dr, \tag{15}$$

we obtain

$$\mathbf{K}_{fr} = \frac{I_s\omega_0}{\theta^2}\frac{d\theta}{dt}, \tag{16}$$

where $I_s = \pi\rho_s R^4 l/2$ is the moment of inertia of the superfluid component.

Thus, the equation of motion of the normal component is given by

$$\frac{d}{dt}\left[I_n\omega(t) + \frac{I_s\omega_0}{\theta(t)}\right] = \mathbf{K}_{ext}. \tag{17}$$

We arrive thus at the important conclusion that the angular momentum of the superfluid component is equal to

$$\mathbf{M}_s(t) = I_s\omega_0/\theta(t) = \mathbf{M}_s(0)/\theta(t). \tag{18}$$

This can be verified also by indirect calculation. Indeed, it is well known (see, e.g.,)[8] the angular momentum of one vortex about the system rotation axis is equal to

$$\mathbf{m} = \mu_s\frac{\Gamma_0}{2\pi}\left(1 - \frac{r^2}{R^2}\right),$$

where r is the distance between the vortex and the rotation axis and μ_s is the mass of the entire superfluid component in a vessel of radius R.

Summing this expression over all the vortices, we get

$$\mathbf{M}_s = \mu_s\frac{\Gamma_0}{2\pi}\left[N(t) - \frac{1}{R^2}\sum_{i=1}^{N(t)} r_i^2(t)\right],$$

where $N(t)$ is the total number of the vortices at the instant t.

Taking into account (15) and the fact that $N(t) = N(0)/\theta(t)$ and $N(0)\mu_s\Gamma_0/2\pi = 2I_s\omega_0$, we obtain formula (18). Going over next to the dimensionless quantities

$$\Omega(t) = \frac{\omega(t)}{\omega(0)}, \quad p = \frac{I_s}{I_n}, \quad q = \frac{\omega(0)}{\omega_0}, \tag{19}$$

we can rewrite (17) in the form

$$\frac{d\Omega}{dt} = \frac{p}{q\theta^2}\frac{d\theta}{dt} - \gamma\Omega, \tag{20}$$

and (12) in the form

$$\frac{d\theta}{dt} = a[1 - q\Omega\theta] \tag{21}$$

with initial conditions $\Omega(0) = \theta(0) = 1$. The system (20)–(21) solves our problem completely. The parameter q represents the degree of disequilibrium between the number of the vortices and the rotation of the system at the initial instant of time.

5. The system (20)–(21) is essentially nonlinear, but it is easy to obtain its solution in the cases of greatest interest.

First, we consider the case when there are no extraneous friction forces, i.e., $\gamma=0$. Then the total angular momentum of the system is conserved and the solution of the system (20)–(21) can be easily obtained:

$$\theta(t)=\frac{p+1}{q+p}+\frac{q-1}{q+p}e^{-a(p+1)t},$$

$$\Omega(t)=\frac{q+p}{q}\frac{1+(q-1)\exp[-a(p+1)t]}{1+p+(q-1)\exp[-a(p+1)t]}. \tag{22}$$

We see that immediately before the initial jump of the angular velocity of the normal component there is initiated a rapid (exponential) process of smoothing-out of the initial disequilibrium on account of the motion of the vortices (with a change, generally speaking, of the number of vortices in the system). The relaxation time of such a process is determined by the interaction between the vortices and the normal component, and is equal to

$$\tau_0=\frac{1}{a(p+1)}=\frac{I_n[(\eta\rho_n)^2+(\rho_s-\rho_n\beta)^2]}{2\eta\rho_n\rho_s\omega_0(I_n+I_s)}. \tag{23}$$

Equilibrium sets in after a time much longer than τ_0, when the number of the vortices in the system corresponds exactly, in accordance with the Feynman formula (6), to the angular velocity of the equilibrium rotation

$$\omega(\infty)=\frac{I_n\omega(0)+I_s\omega_0}{I_n+I_s}.$$

We consider now the case when $\gamma\neq 0$ but $\gamma/a\ll 1$ and $q=1+\Delta$, where $|\Delta|\ll 1$. Then the solution of the system (20) and (21) in first order in γ/a and Δ is of the form

$$\theta(t)=1+\frac{\gamma t}{p+1}-\left[\frac{\gamma}{a(p+1)^2}+\frac{\Delta}{p+1}\right](1-e^{-a(p+1)t}), \tag{24}$$

$$\Omega(t)=1-\frac{\gamma t}{p+1}-\left[\frac{\gamma p}{a(p+1)^2}+\frac{p\Delta}{p+1}\right](1-e^{-a(p+1)t}). \tag{25}$$

We see that in this case the initial jump of the angular velocity of the normal part of the system is immediately followed by an exponential equalization of the initial disequilibrium, with a relaxation time (23); this process then goes over into a slow (linear in first-order approximation) relaxation of the total angular momentum of the system on account of the extraneous friction forces; now characterized by a relaxation time $1/\gamma$. This is precisely the behavior of the angular velocity of pulsars which is observed by radioastronomores.[1]

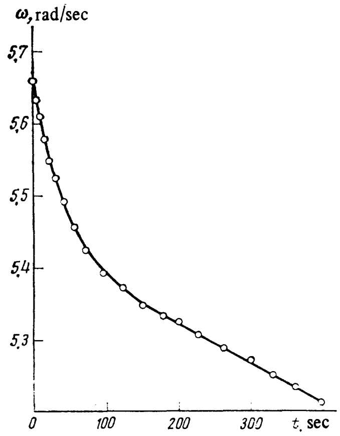

FIG. 1. Time dependence of the angular velocity of free rotation of a spherical vessel with He II at $T=1.46$ K after a jump $\Delta\omega=0.66$ rad/sec in the angular velocity at the instant $t=0$.

It is interesting that formula (25) for $\Omega(t)$ describes, within the limits of the experimental error, the behavior of the angular velocity of He II in experiments[2,3] for both cylindrical and spherical vessel. With the kind permission of Dzh. and S. Tsakadze, we show in the figure their experimental data on the rotation of a spherical vessel with He II. The angular velocity of the vessel prior to the instant $t=0$ was 5.0 rad/sec, and jumped to 5.66 rad/sec at the instant $t=0$. The solid curve corresponds to formula (25) and is given by

$$\omega(t)=5.66-5.47\cdot 10^{-4}t$$
$$-0.226[1-\exp(-2.516\cdot 10^{-6}\cdot t)].$$

The agreement is all the more surprising because the experimental accuracy was better than 0.2%, and formula (25) was derived for systems in which the normal component rotates as a unit.

The reduction of the function $\Omega(t)$ in accordance with (20) and (21) for pulsars would yield considerable information both on the mechanism of the radiative damping and on the internal structure of these stars.

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