of the multiple-production processes will occur at these energies.

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Nonlinear interaction of a monochromatic wave with particles in a gravitating system

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A study is made of the motion of particles in a gravitational field corresponding to the proper characteristic monochromatic oscillatory mode of a gravitating collisionless cylinder. In the frame of reference rotating with the cylinder, the effect of the inertial forces on a gravitating particle is analogous to the effect of a longitudinal magnetic field on a test electric charge. In addition, the particles of the cylinder are magnetized, so that (approximately) they preserve their distance from the cylinder axis. For this reason, the equation of longitudinal motion of the particles reduces to an equation of the type of a mathematical pendulum, which can be solved in elliptic functions. An investigation is made of the nonlinear stage of the beam (two-stream) gravitational instability (see A. B. Mikhailovskiĭ and A. M. Fridman, Zh. Eksp. Teor. Fiz. **61**, 457 (1971); Sov. Phys. JETP **34**, 263 (1972)): the nonlinear evolution of the particle distribution function is studied, and the densities of the kinetic energy of the particles and the energy of the monochromatic wave, both averaged over a cylindrical layer, are found. The energy balance method is used to determine the time dependence of the nonlinear growth rate. The range of applicability of the theory is found and the amplitudes of steady oscillations estimated. In this way it is shown that in gravitating systems an important role can be played by a nonlinear mechanism of stabilization of a monochromatic density wave which is analogous to the mechanism investigated in a collisionless plasma by Mazitov (Zh. Prikl. Mekh. Tekh. Fiz. **1**, 27 (1965)) and O'Neil (Phys. Fluids **8**, 2255 (1965)).

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§1. INTRODUCTION

In real astrophysical objects, the velocity distribution functions of the particles (stars, gas) often have a beam nature. Of this kind are: all galaxies with heterogeneous structure in which flat subsystems rotate relative to elliptical and spherical subsystems; regions of active centers characterized by ejections of large gaseous masses; and so forth.

In[1], two of the present authors have shown that a beam (two-stream) instability can be excited in gravitating systems, this resulting in a growth in the amplitude of the density waves of the interacting subsystems. Initially,[1] this effect was studied on a gravitating cylinder. Later, in[2,3], the role of beam effects was investigated in more complicated systems consisting of two interacting disks and a sphere and ellipsoid.[4] It is very important to establish whether nonlinear stabilization of the amplitude takes place or whether the instability progresses and results in the collapse of the various density concentrations. We may mention that for the problem of the spiral structure of galaxies particular interest attaches to the interaction of a monochromatic density wave with the particles (stars).

In the present paper we draw attention to the fact that in gravitating systems an important role can be played by a nonlinear mechanism of stabilization of the monochromatic density wave analogous to the mechanism investigated in a collisionless plasma by Mazitov [5] and O'Neil. [6] The reasons for the analogy between the mechanisms of collective processes in gravitating and plasma media were investigated in[1,7,8]. In particular, it was noted that the kinetic equation of small oscillations of a simple model of a gravitating system—a rotating cylinder—can be cast into the same form as the kinetic equation for a collisionless magnetized plasma by a redefinition of the characteristic parameters. This is why a gravitating cylinder can sustain a beam instability described by the same relations as the one in a plasma with a magnetic field.

Replacing the doubled frequency of rotation of the cylinder by the cyclotron frequency, $2\Omega \to \omega_B$, and the square of the Jeans frequency by the negative square of the plasma frequency, $\omega_0^2 \to -\omega_p^2$, let us consider the upper hybrid branch of oscillations: $\omega^2 = \omega_p^2 + \omega_B^2$. In the case of a gravitating cylinder, this branch was called the rotational branch in[1]. It is characterized by the frequency $\omega^2 = \omega_0^2$, which by virtue of the equilibrium condition (Ref. 1) $\omega_0^2 = 2\Omega^2$ can also be represented in the form $\omega^2 = 2\Omega^2$.

In the presence of a beam moving along a generator of the cylinder, the rotational branch is excited with the linear growth rate[1]

$$\gamma_L \sim \alpha \left(\frac{v}{v_T}\right)^2 \left(\frac{k_z}{k}\right)^2 \omega_0, \tag{1}$$

where α is the ratio of the beam density to the density of the medium; v and v_T are the directed and the thermal velocity of the beam; k_z and k are the longitudinal and the total wave number (see Ref. 1). The expression (1) is valid if Cherenkov resonance, $\omega \approx k_z v$, predominates over the cyclotron resonance, $\omega \pm 2\Omega \approx k_z v$, i.e., under the condition $2\Omega/k_z \gg v_T$ (see Ref. 1). An estimate for the growth rate analogous to (1) also holds in the case of the two-stream instability in a plasma. It also remains in force in the absence of a magnetic field, i.e., in the limit $\omega_B \to 0$, when the upper hybrid branch goes over into the branch of electron plasma oscillations. In the case of a gravitating cylinder, this limit is prohibited by the equilibrium conditions mentioned above.

It is known from plasma theory[9] that an expression of the type (1) for the linear growth rate for excitation by a beam of plasma ocsillations can be used only when the wave amplitude is not too large, namely, when

$$\tau\gamma_L \gg 1, \tag{2}$$

where $\tau^2 \equiv m/ek_z^2\Psi_0$ (m and e are the electron mass and charge). This condition means that the back reaction of the wave field on the resonant particles is negligibly small. Otherwise, i.e., when $\tau\gamma_L \ll 1$, the wave field leads to trapping of the resonant particles, so that the expression of the type (1) for the growth rate is replaced by

$$\gamma(t) \approx \gamma_L F(t/\tau), \tag{3}$$

where $F(t/\tau)$ is a function whose explicit form is given in[6]. In this case

$$\int_0^\infty \gamma(t)\,dt \sim \gamma_L \tau. \tag{4}$$

The excitation of plasma waves by the beam stops when the amplitude of the field reaches values corresponding to τ such that

$$\tau\gamma_L \sim 1. \tag{5}$$

These results apply to a plasma without magnetic field, $\omega_B \to 0$, and to perturbations that propagate along the beam, $k \approx k_z$. However, one can show that for either a plasma with magnetic field or a gravitating medium with $\omega_B \sim \omega_p$ and $k_z \ll k$ (and it is this case that is interesting for our problem of a gravitating cylinder) the order-of-magnitude relations (1)–(5) remain in force. This enables us to extend the analogy between plasma and gravitating media to the region of nonlinear phenomena.

In the present paper, we investigate the nonlinear stage of the beam instability in the gravitating cylinder. In §2 we give the necessary results of the linear theory. In §3, we study the motion of particles in the gravitational field corresponding to the characteristic (monochromatic) oscillatory mode of the cylinder. In the frame of reference rotating with the cylinder, the effect of the inertial forces on a gravitating particle is analogous to the effect of a longitudinal magnetic field on a test charge. Moreover, the particles of the cylinder are "magnetized," so that (approximately) they keep their distance from the cylinder axis. For this reason, as is shown in §3, the equation of the longitudinal motion of the particles reduces to an equation of the type of a mathematical pendulum, which has been solved by Mazitov[5] and O'Neil[6] in elliptic functions.

In §4, we consider the nonlinear evolution of the distribution function of the particles and in Sect. 5 we find the densities of the kinetic energy of the particles and the energy of the monochromatic wave averaged over a cylindrical layer. The energy balance method (after radial averaging) is used to determine the time dependence of the nonlinear growth rate. In §6 we find the region of applicability of the theory. In §7 we estimate the amplitude of steady oscillations for different values of the parameters of the configuration.

§ 2. RESULTS OF THE LINEAR THEORY OF THE BEAM INSTABILITY

We consider a stationary collisionless system of gravitating particles in the form of a radially homogeneous cylinder of infinite length and radius R (see[1]). In the frame of reference rotating with the cylinder (with angular velocity $\Omega = (2\pi G\rho)^{1/2}$, where G is the gravitational constant and ρ the density) the particles move only along the axis, which we take as the axis of a cylindrical coordinate system (r and φ are, respectively, the radial and the angular coordinate).

Thus, the stationary radial and azimuthal velocities for all particles are equal to zero, $v_r = v_\varphi = 0$. We take the distribution over the longitudinal velocities, $\mathscr{F}(v_z)$, which is no way restricted by the equilibrium conditions, in the beam form[1]

$$\mathscr{F}(v_z)=f^M(v_z)+f(v_z)$$

with Maxwellian distribution of the main component $f^M(v_z)$:

$$f^M(v_z)=\frac{1}{\pi^{1/2}V_T}\exp(-v_z^2/V_T^2), \tag{6}$$

and beam distribution function $f(v_z)$:

$$f(v_z)=\frac{\alpha}{\pi^{1/2}v_T}\exp\left[-\left(\frac{v_z-v}{v_T}\right)^2\right], \tag{7}$$

and we assume the conditions

$$\alpha\ll 1,\quad v\gg v_T,\quad v\gg V_T. \tag{8}$$

In addition, we assume that the main component has a fairly large thermal spread V_T:

$$V_T\gg V,\quad V\equiv\Omega R. \tag{9}$$

Under these conditions, axisymmetric oscillations of the structure can propagate[1]:

$$\Phi(t,z,r)=\Phi_0 J_0(k_\perp r)e^{-i(\omega_0 t-k_z z)}e^{\gamma_F(t)},\quad r\leqslant R \tag{10}$$

within the cylinder and

$$\Phi(t,z,r)=\tilde{\Phi}_0 K_0(k_z r)e^{-i(\omega_0 t-k_z z)}e^{\gamma_F(t)},\quad r>R \tag{11}$$

outside it. Here, $\Phi(t, z, r)$ is the perturbation of the gravitational potential, t is the time variable, Φ_0 and $\tilde{\Phi}_0$ are constants that satisfy the conditions of matching[1] on the cylinder boundary, in particular the condition of continuity of the potential:

$$\Phi_0 J_0(k_\perp R)=\tilde{\Phi}_0 K_0(k_z R), \tag{12}$$

where J_0 and K_0 are the standard notation (see[10,11]) for the Bessel functions; ω_0 is the Jeans frequency, which is related to the angular frequency Ω by the equilibrium condition

$$\omega_0^2=2\Omega^2; \tag{13}$$

the longitudinal wave parameter k_z must satisfy the conditions[1]

$$k_z R\ll 1, \tag{14}$$

$$k_z V_T\ll\omega_0, \tag{15}$$

and the transverse parameter $k_\perp$ the conditions

$$J_0(k_\perp R)\approx 0,\quad k_\perp R\gg 1 \tag{16}$$

(note that instead of (16) we previously [1] erroneously assumed $J_1(k_\perp R)\approx 0$). In (10) and (11), γ_F is equal to the sum $\gamma_F=\gamma_M+\gamma$ of the damping rate γ_M of the wave interacting with the main component of the medium (see[1]) and the growth rate γ of the instability associated with the excitation of waves by the beam. In its turn, γ consists of two terms:

$$\gamma=\gamma_b+\gamma_c. \tag{17}$$

Here, γ_b is due to the Cerenkov resonance:

$$\gamma_b=\frac{\pi}{2}\frac{\omega_0^3}{k^2}f'\left(\frac{\omega_0}{k_z}\right)\operatorname{sign}k_z, \tag{18}$$

where $k^2\equiv k_z^2+k_\perp^2$, and the prime denotes the derivative with respect to the argument; γ_c is due to the cyclotron resonance,

$$\gamma_c=-\frac{\pi}{4}\frac{\omega_0^3}{|k_z|\Omega}\left[f\left(\frac{\omega_0-2\Omega}{k_z}\right)-f\left(\frac{\omega_0+2\Omega}{k_z}\right)\right]. \tag{19}$$

Under the conditions we have described, there is besides the beam instability (17) only the Jeans instability with exponentially small (if $V_T^2\gg V^2$) growth rate γ_J:

$$\gamma_J<\frac{V_T}{V}\exp\left(-\frac{V_T^2}{V^2}\right). \tag{20}$$

As will be shown below, the parameters of the configuration can be chosen in such a way that the beam growth rate (17) has its largest value and, in addition, is due basically to Cerenkov resonance:

$$\begin{gathered}\gamma_F\approx\gamma\approx\gamma_b,\quad \gamma_c\ll\gamma_b,\\ \gamma_M\ll\gamma,\quad \gamma_J\ll\gamma.\end{gathered} \tag{21}$$

We shall restrict ourselves to this case.

§3. MOTION OF PARTICLES IN THE FIELD OF THE WAVE

Suppose that at some initial time $t=0$ in the system described above a gravitational potential of the following form is switched on (see (10) and (11)):

$$\begin{gathered}\Phi(t,z,r)=-\Phi_0(t)J_0(k_\perp r)\cos(-\omega t+k_z z),\quad r<R,\\ \Phi(t,z,r)=-\tilde{\Phi}_0(t)K_0(k_z r)\cos(-\omega t+k_z z),\quad r>R.\end{gathered} \tag{22}$$

Following O'Neil, [6] to determine the motion of the particles we restrict ourselves to the zeroth order of perturbation theory in the small parameter $\Delta\Phi_0/\Phi_0$, i.e., we set

$$\Phi_0(t)\approx\text{const}\equiv\Phi_0,\quad \tilde{\Phi}_0(t)\approx\text{const}\equiv\tilde{\Phi}_0.$$

We shall assume that the potential (22) satisfies the conditions (12), (14)–(16).

In the region of wave vectors for which $\gamma\sim\gamma_{max}$, the condition of Cerenkov resonance can be represented in accordance with (7) and (18) in the form

$$\omega-k_z v\sim k_z v_T. \tag{23}$$

We study the motion of particles resulting from the switching on of the potential, using a frame of reference that moves with the wave along the z axis (and, as before, rotates with frequency Ω). In such a system, the potential (22) has the form

$$\Phi(z, r) = -\Phi_0 J_0(k_\perp r)\cos(k_z z), \quad r<R \tag{24}$$

with a similar expression for $r>R$.

The motion of the particles satisfies the equation[12]

$$d\mathbf{v}/dt = [\mathbf{v}\times 2\mathbf{\Omega}] - \nabla\Phi,$$

where $\mathbf{\Omega} = \Omega\hat{\mathbf{z}}$, and $\hat{\mathbf{z}}$ is the unit vector along the cylinder axis. This is the motion that an electric particle with unit charge and mass would execute in a constant homogeneous magnetic field $\mathbf{H} = 2\mathbf{\Omega}$ and electric potential Φ (by the choice of the system of units, the velocity of light is equal to unity).

All the particles of the cylinder can be divided into three groups in accordance with the value of their stationary velocity v_z:

1) "slowly" moving (in the system of the wave) particles, which are displaced along the z axis during the "cyclotron" period $T \equiv 2\pi/2\Omega = \pi/\Omega$ through a distance that is much less than the longitudinal wavelength $\lambda \equiv 2\pi/|k_z|$:

$$|v_z| \ll 2\Omega/|k_z|;$$

2) particles that are displaced through a distance of the order of the wavelength,

$$|v_z| \sim 2\Omega/|k_z|; \tag{25}$$

3) "fast" particles satisfying

$$|v_z| \gg 2\Omega/|k_z|.$$

The transverse motion of the slow particles is analogous to the motion of a charged particle in a longitudinal constant magnetic field and a slowly varying radial electric field, i.e., it consists of azimuthal drift; the radius of the orbit is of the order of the "cyclotron" radius:

$$r_{rot} \sim k_\perp \Phi_0/4\Omega^2. \tag{26}$$

If the potential Φ_0 is comparatively weak, the particle is not displaced much radially; the corresponding condition $r_{rot} \ll R$ can, with allowance for (9), (16), and (26), be written in the form

$$\Phi_0/V^2 \ll 1 \tag{27}$$

and means that the perturbation is small, which, of course, is also necessarily assumed by the linear theory.[1] For the nonlinear theory we are developing, we also assume that the inequality (27) is satisfied; we shall see below that the condition of applicability of the nonlinear theory gives a lower bound on the potential, but this bound need not contradict (27).

Remembering that the initial transverse velocity is zero, we find that the paths of slow particles have the form of epicycloids, and the longitudinal motion proceeds as if the particle remained during the whole time at the same distance r from the cylinder axis (r is the coordinate of the particle before the additional gravitational field is switched on).

The transverse motion of the fast particles consists of motion in the rapidly oscillating (with frequency much higher than the rotational frequency, $|k_z v_z| \gg \Omega$) gravitational field. The oscillations along r with amplitude

$$\Delta r \sim k_\perp \Phi_0/(k_z v_z)^2 \ll k_\perp \Phi_0/\Omega^2 \sim r_{rot}$$

are even less capable of changing the radial position of the particle significantly. Thus, the change in the radial position of the fast particles is even less than that of the slow ones, and in our study of their longitudinal motion we are even more justified in assuming that the radial component is constant.

It is only for the rotationally resonant particles (25) that the change in the radial coordinate cannot be ignored. But the fraction Δf of such particles can be estimated at

$$\Delta f \sim f\left(\frac{2\Omega}{|k_z|}\right)\Delta v_z \sim f\left(\frac{2\Omega}{|k_z|}\right)\frac{\Omega}{|k_z|},$$

and in accordance with (6)–(8), (13), and (23) it is exponentially small if

$$2/|k_z|R \gg v_T/V. \tag{28}$$

This last condition can be readily satisfied, and we therefore exclude these particles from our treatment. Thus, in considering the longitudinal motion, we shall assume that the radial coordinate of each particle is fixed at the initial value.

§ 4. NONLINEAR EVOLUTION OF THE DISTRIBUTION FUNCTION

We now consider the evolution of the distribution function with respect to the longitudinal velocity. The longitudinal field of the potential (24) is

$$E_z(z, r) \equiv -\frac{\partial}{\partial z}\Phi(z, r) = -\mathscr{E}_z(r)\sin(k_z z), \tag{29}$$

where the amplitude $\mathscr{E}_z(r)$ is given by the expression

$$\mathscr{E}_z(r) \equiv k_z \Phi_0 J_0(k_\perp r). \tag{30}$$

It can be seen that the amplitude $\mathscr{E}_z(r)$ depends only on the radial variable (which remains constant, as we have shown above, during the longitudinal motion of the particle). The amplitude varies from the maximal value on the cylinder axis to zero at the edge of the cylinder (see (16)).

The field (29) leads to the following equation of the longitudinal motion (cf[6]):

$$\dot{v}_z \equiv \ddot{z} = -\mathscr{E}_z(r)\sin(k_z z). \tag{31}$$

The conservation of the energy of the longitudinal motion can be written in the form

$$\frac{v_z^2}{2} - \frac{\mathscr{E}_z(r)}{k}\cos(k_z z) = \text{const} \equiv w.$$

Following Galeev and Sagdeev,[9] we can find the distribution function because it is constant along the paths of the particles and is known at the initial time.

The distribution function of the particles trapped by the wave in their longitudinal motion has the form (in the system of the wave)

$$f_r(z,v_z,t)=f(0)+\left(\frac{\partial}{\partial v_z}f(0)\right)\sigma\left\{2\left[w(r,z,v_z)+\frac{\mathscr{E}_z(r)}{k_z}\right]\right\}^{1/2}$$
$$\times\mathrm{cn}\left\{F\left[\zeta(z),\frac{1}{\varkappa(z,v_z)}\right]-\frac{t}{\tau_r},\frac{1}{\varkappa(z,v_z)}\right\},\quad \varkappa>1. \tag{32}$$

For the untrapped particles, the distribution function has the form

$$f_r(z,v_z,t)=f(0)+\left(\frac{\partial}{\partial v_z}f(0)\right)\sigma\left\{2\left[w(r,z,v_z)+\frac{\mathscr{E}_z(r)}{k_z}\right]\right\}^{1/2}$$
$$\times\mathrm{dn}\left\{F\left[\frac{kz}{2},\varkappa(z,v_z)\right]-\frac{t}{\varkappa\tau_r},\varkappa(z,v_z)\right\},\quad \varkappa<1. \tag{33}$$

The notation here is as follows:

$$\sigma=\operatorname{sign}v_z,\quad w(r,z,v_z)=\frac{v_z^2}{2}+\frac{\mathscr{E}_z(r)}{k_z}\cos(k_z z),$$
$$F(\varphi,k)=\int_0^\varphi\frac{d\alpha}{(1-k^2\sin^2\alpha)^{1/2}}$$

is the elliptic integral of the first kind, $\mathrm{cn}(u, k)$ is the elliptic cosine (Jacobian elliptic function), $\mathrm{dn}(u, k)$ is the function defined by the relation

$$\mathrm{dn}\,[F(\varphi,k),k]=(1-k^2\sin^2\varphi)^{1/2},$$
$$\varkappa^2(z,v_z)=\frac{2\mathscr{E}_z(r)}{k_z w(z,v_z)+\mathscr{E}_z(r)},$$
$$\zeta(z,v_z)=\arcsin\left[\varkappa\sin(1/2 k_z z)\right],$$
$$\tau_r^2=\frac{1}{k_z\mathscr{E}_z(r)}=\frac{1}{k_z^2\Phi_0 J_0(k_\perp r)},$$

i.e.,

$$\tau_r^2=\tau_0^2/J_0(k_\perp r), \tag{34}$$

where

$$\tau_0^2=1/k_z^2\Phi_0. \tag{35}$$

Like Galeev and Sagdeev, [9] we conclude that in the region of the phase space (z, v_z) corresponding to trapped particles a plateau is formed and one can write down a time-averaged distribution function of the untrapped particles. The evolution differs from that in the problems of Mazitov and O'Neil[5,6] and Galeev and Sagdeev[9] in that our configuration does not evolve at the same rate at different distances from the axis. The period of the oscillations of the particles trapped by the wave increases in accordance with (34) from τ_0 on the axis to a (formally) infinite value at the edge of the cylinder.

§ 5. NONLINEAR EVOLUTION OF THE MONOCHROMATIC WAVE

To find the growth rate of the field by the method employed here, we use the energy balance equation

$$-dQ/dt=2W\gamma, \tag{36}$$

where Q is the mean (over the volume of the cylinder) density of the kinetic energy of the particles and W is the mean density of the energy of the wave (i.e., the sum of the field energy and the energy of the nonresonant particles; see, for example,[13]).

The equation of the longitudinal motion (31) coincides with the corresponding equation of O'Neil[6]; essentially the same is true of the distribution functions (32) and (33) at fixed r. Therefore, for the rate of change of the density of the kinetic energy of the particles in the annular cylinder (r, $r+dr$), averaged over the volume of the annular cylinder, we obtain (cf Eq. (30) of[6])

$$\frac{dQ_r}{dt}=-\frac{\pi}{2}\frac{\omega_0^3}{k_z^2}f'\frac{\mathscr{E}_z^2(r)}{4\pi G}\sum_{n=0}^{\infty}\frac{64}{\pi}\int_0^1 d\varkappa$$
$$\times\left\{\frac{2n\pi^2\sin(\pi nt/\varkappa K\tau_r)}{\varkappa^5K^2(1+q^{2n})(1+q^{-2n})}+\frac{(2n+1)\pi^2\varkappa\sin[(2n+1)\pi t/2K\tau_r]}{K^2(1+q^{2n+1})(1+q^{-2n-1})}\right\} \tag{37}$$
$$K=F\left(\frac{\pi}{2},\varkappa\right),\quad q=\exp\left\{\frac{\pi}{K}F\left[\frac{\pi}{2},(1-\varkappa^2)^{1/2}\right]\right\}.$$

This expression must be averaged over the radius of the annular cylinder:

$$\frac{dQ}{dt}=\int_0^R\frac{dQ_r}{dt}\frac{2\pi r\,dr}{\pi R^2}. \tag{38}$$

Note that the radial dependence enters (37) only through $E_z(r)$ and τ_r.

To calculate W in (36), we must take the integral (in accordance with (12) and (16), the external field is already very small at the edge of the cylinder and decreases rapidly with increasing r, so that its contribution to the energy can be ignored)

$$I_0=\frac{1}{\pi R^2}\int_0^R\frac{\mathscr{E}_z^2+\mathscr{E}_r^2}{8\pi G}2\pi r\,dr. \tag{39}$$

Here, as in (30),

$$\mathscr{E}_r(r)=k_\perp\Phi_0 J_1(k_\perp r), \tag{40}$$

so that the radial field strength $E_r(z, r)$ is

$$E_r(z,r)=-\frac{\partial\Phi(z,r)}{\partial r}=-\mathscr{E}_r(r)\cos(k_z r).$$

With allowance for (40) and (30), the calculation of (39) reduces to calculation of the integrals

$$I_1=\frac{2}{R^2}\int_0^R J_0^2(k_\perp r)r\,dr,\quad I_2=\frac{2}{R^2}\int_0^R J_1^2(k_\perp r)r\,dr. \tag{41}$$

Using the well-known expression[10]

$$\int xJ_\mu^2(\alpha x)\,dx=\frac{x^2}{2}\left[J_\mu^2(\alpha x)-J_{\mu-1}(\alpha x)J_{\mu+1}(\alpha x)\right]$$

and taking into account (16), we find that

$$I_1=I_2=J_1^2(k_\perp R). \tag{42}$$

Finally, the mean energy density of the wave is

$$W=\frac{1}{8\pi G}k^2\Phi_0^2J_1^2(k_\perp R), \tag{43}$$

where

$$k^2 = k_z^2 + k_\perp^2.$$

We shall not calculate explicitly the cumbersome general expression for the growth rate $\gamma(t)$. We shall merely show that in the limiting case we obtain exactly the linear growth rate. Indeed, in this case, following O'Neil,[6] we obtain from (37)

$$-\frac{dQ_r}{dt} = \frac{\pi}{2}\frac{\omega_0^3}{k_z^2} f' \frac{\mathscr{E}_z(r)}{4\pi G}\left\{\frac{64}{\pi}\int_0^1 d\varkappa \frac{2\pi^2 \sin(\pi t/\varkappa K \tau_r)}{\varkappa^5 K^2 (1+q^2)(1+q^{-2})} + O\left(\frac{t}{\tau_0}\right)\right\}$$
$$= \frac{\pi}{2}\frac{\omega_0^3}{k_z^2} f' \frac{\Phi_0^2 J_0^2(k_\perp r)}{4\pi G}\left\{1 + O\left(\frac{t}{\tau_0}\right)\right\}.$$

Hence, averaging in accordance with (38), we find

$$-\frac{dQ}{dt} \approx \frac{\pi}{2}\frac{\omega_0^3}{k_z^2} f' \frac{\Phi_0^2 k_z^2}{4\pi G}\left[\frac{2}{R^2}\int_0^R J_0^2(k_\perp r) r\,dr\right] = \frac{\pi}{2}\omega_0^3 f' \frac{\Phi_0^2}{4\pi G} J_1^2(k_\perp R). \quad (44)$$

In deriving the last equation, we have used the relations (41) and (42).

Finally, in accordance with (36) and using (44) and (43), we find

$$\gamma = -\frac{1}{2W}\frac{dQ}{dt} \approx \frac{\pi}{2}\frac{\omega_0^3}{k^2} f', \quad (45)$$

which really does coincide with the linear growth rate (see (21), (18)),

We now estimate the total gain K of the wave. In accordance with (36),

$$\mathscr{K} = \int_0^\infty \gamma(t)\,dt = -\frac{1}{2W}\int_0^\infty \frac{dQ}{dt}\,dt = -\frac{1}{2W}\Delta Q. \quad (46)$$

Following O'Neil,[6] we find

$$-\Delta Q = -\int_0^R \Delta Q_r \frac{2\pi r\,dr}{\pi R^2} = O\left[\int_0^R \gamma_L \tau_r k^2 \frac{\Phi_0^2}{4\pi G} J_0^2(k_\perp r)\frac{2r\,dr}{R^2}\right]$$
$$= O\left[\gamma_L k^2 \frac{\Phi_0^2}{4\pi G}\tau_0 \frac{2}{R^2}\int_0^R J_0^{4/3}(k_\perp r) r\,dr\right] = O\left(\gamma_L k^2 \frac{\Phi_0^2}{4\pi G}\tau_0\right). \quad (47)$$

In accordance with (43)

$$2W \sim k^2 \Phi_0^2 / 4\pi G,$$

and from (46) and (47) we therefore obtain

$$\mathscr{K} = O(\gamma_L \tau_0).$$

From the last equation we obtain the condition of validity of the approximation of constancy in time of the wave amplitude which we have adopted in our study of the motion of the particles:

$$\gamma_L \tau_0 \ll 1. \quad (48)$$

§6. RANGE OF APPLICABILITY OF THE THEORY

Our theory is applicable when all our adopted assumptions hold, i.e., (48), the condition (27) of a small cyclotron radius, and the condition (21) that Cerenkov resonance predominates. We must also include the condition[1] that the growth rate of the hydrodynamic two-stream instability is small:

$$v_T/v < \alpha^{1/3}.$$

We denote $v_T/V \equiv \tilde{v}_T$, $v/V \equiv \tilde{v}$, $V_T/V \equiv \tilde{V}_T$. We take

$$\tilde{v}_T \sim 1, \quad k_\perp R \sim 1,$$
$$1/\tilde{v} \sim \tilde{v}_T/\tilde{v} \sim \alpha^{1/3}, \quad (49)$$

and then, with allowance for (6)–(8), (14), (16), (17)–(21), (23), and (34), we arrive at the inequalities

$$\Phi_0/V^2 \ll 1, \quad r_{rot}/R \ll 1, \quad (50)$$

$$\tilde{v}\Phi_0^{1/2}/V \gg 1, \quad \gamma_L \tau_0 \ll 1; \quad (51)$$

$$1 \gg \tilde{v}\exp(-\tilde{v}^2), \quad \gamma_b \gg \gamma_c,$$
$$1/\tilde{v}^2 \gg \tilde{V}_T \exp(-\tilde{V}_T^2), \quad \gamma \gg \gamma_J, \quad (52)$$
$$\frac{1}{\tilde{v}^2} \gg \frac{\tilde{v}}{\tilde{V}_T}\exp\left[-\left(\frac{\tilde{v}}{\tilde{V}_T}\right)^2\right], \quad \gamma \gg \gamma_M.$$

If now, keeping $V \sim v_T$, we increase $\tilde{V}_T$ and $\tilde{v}$ in accordance with the law $\tilde{v} \sim \tilde{V}_T^2$, decreasing α simultaneously in accordance with the law (49) and k_z in accordance with the law $k_z R \sim 1/\tilde{v}$ (see (23), (9), and (13)), then the right-hand sides of (52) decrease exponentially, while the left-hand sides decrease not stronger than a power. Therefore, from certain sufficiently large V_T and v onward, all the inequalities (52) will be satisfied. The inequality (51) will also be satisfied for a sufficiently large beam velocity v no matter what is the value of the amplitude Φ_0 provided it satisfied (50). At the same time, the relations (8), (9), (14), (15), and (28) are not violated.

§7. ESTIMATES OF THE AMPLITUDE OF STEADY MONOCHROMATIC WAVES

It follows from the above relations that the beam instability of the gravitating cylinder is saturated when

$$\frac{\Phi}{\Psi} \sim \frac{\alpha^2}{(kR)^4}\left(\frac{Vv}{v_T^2}\right)^2,$$

where Ψ is the equilibrium gravitational potential. It can be seen that the ratio Φ/Ψ as a function of kR is maximal for $kR \sim 1$; at the same time

$$\Phi/\Psi \sim \alpha^2 (Vv/v_T^2)^2.$$

This ratio increases with decreasing thermal spread of the beam; at the limit of applicability of our ideas on the kinetic instability, i.e., for $v_T/v \sim \alpha^{1/3}$ (see[13]),

$$\Phi_{max}/\Psi \sim \alpha^{4/3}(V/v)^2.$$

It is interesting to note that when a rotating gravitating medium interacts with a beam of comparable density, $\alpha \sim 1$, and comparable velocity, $v \sim V$, the perturbed gravitational potential Φ has the same order of magnitude as the equilibrium potential Ψ.

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Polarization mechanism for nonconservation of parity and the effect of a weak neutral interaction in heavy μ-mesic atoms

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An estimate is made of the effect of nonconservation of parity of the $3d$-orbit meson states for mesic atoms with odd nuclei in the $Z = 56$–60 and $Z \sim 83$ ranges in which the hyperfine structure terms of the $3p\,^3/_2$–$3d\,^5/_2$ and $3p\,^1/_2$–$3d\,^5/_2$ orbits respectively intersect. The angular distribution coefficient α in $W(\theta) = 1 + \alpha\cos\theta$ for quanta emitted by polarized mesic atoms in the $3d \to 1s$ transition is determined. The effect of a weak neutral interaction between a muon and a nucleus and the Coulomb polarization mechanism for transferring the nonconservation of parity of nuclear states to mesic atom states in the chain of nonradiative transitions of a meson are considered. In the nuclear ranges indicated above the effect of the weak interaction for $3d$- and $3p$-orbits is smaller by two–three orders of magnitude than the effect of the polarization mechanism. Under optimal conditions for the intersection of $3d\,^5/_2$ and $3p\,^3/_2$ terms of a mesic atom which, as is shown, can be realized in the range $Z = 56$–60 this mechanism leads to a value of the coefficient α which is approximately equal to the amplitude for the nonconservation of parity for nuclear states $\beta(I_\nu E_\nu)$ lying at $E_\nu \approx \hbar\omega(3d \to 1s)$ and $E_\nu \approx \hbar\omega(3d \to 2p)$. If dynamic amplification of the nonconservation of parity of nuclear states occurs at $E_\nu \approx 5$–6 MeV, i.e., $\beta(I_\nu E_\nu) \sim 10^{-5}$, then for the quanta arising from a $3d \to 1s$ transition of a meson under real conditions for the transfer of polarization of the meson spin in the cascade of transitions populating the $3d\,^5/_2$-orbit the anisotropy coefficient α can attain a value of $\sim 10^{-5}$.

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INTRODUCTION

1. The weak neutral interaction between a muon and a nucleus leads to a mixing of mesic atom states of opposite parities. For the $2s\frac{1}{2}$ and $2p\frac{1}{2}$ meson orbits this effect has been investigated in a number of papers. [1–7] with the calculation in Ref. 2 being carried out for the range $3 \leqslant Z \leqslant 82$. In the case of light ($Z < 10$) mesic atoms the observation of the effect of the weak interaction can be significantly impeded by a number of accompanying processes: configuration mixing in the electron shell of the mesic atom, the Stark effect of the electric field of the medium, etc. (cf., Ref. 8), while for heavy mesic atoms the role played by these processes is insignificant. Since the amplitude of the admixture of the state of opposite parity is determined by the ratio of the matrix element of the interaction to the difference between the energies of the states being mixed, it is natural to seek in the spectrum of a mesic atom states of opposite parity close in energy. However, it is necessary that the meson should penetrate the volume of the nucleus sufficiently effectively if our aim is to determine the magnitude of the weak interaction between a muon and a nucleus. A preliminary calculation of the terms of a mesic atom in the Coulomb field of a uniformly charged sphere ($R_0 = 1.24 \times 10^{-13} A^{1/3}$ cm) has picked out three ranges of Z in which the $3dj_1$ and $3pj_3$ terms of a mesic atom intersect:

Range of Z:	$55 \leqslant Z \leqslant 60$	$65 \leqslant Z \leqslant 70$	$82 \leqslant Z \leqslant 85$
Mesic atom terms:	$3d\,^5/_2$–$3p\,^3/_2$	$3d\,^3/_2$–$3p\,^1/_2$	$3d\,^5/_2$–$3p\,^1/_2$
Nuclear spin:	$I \geqslant {}^1/_2$	$I \geqslant {}^1/_2$	$I \geqslant 1$

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