Investigation of singularities in superfluid He^3 in liquid crystals by the homotopic topology methods

G. E. Volovik and V. P. Mineev

L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences; Institute of Solid State Physics, USSR Academy of Sciences
(Submitted December 27, 1976)
Zh. Eksp. Teor. Fiz. 72, 2256–2274 (June 1977)

Various singularities in the superfluid He^3 are considered: vortices, disgyrations, pointlike singularities, vortices with ends, singular surfaces, and particle-like states, as well as disclinations in cholesteric liquid crystals. A classification is presented of the topologically stable singularities. The methods of homotopic topology are used and are described with examples of well known systems such as superfluid He II, an isotropic ferromagnet, and a nematic liquid crystal. The possibility of applying these methods to ordinary crytals and to liquid crytals of the smectic type is discussed.

PACS numbers: 67.50.Fi, 61.30.–v

1. INTRODUCTION

We shall consider singularities in superfluid He^3 and in liquid crystals. These substances are typical examples of systems with spontaneously broken symmetry. Such systems are characterized by the fact that their equilibrium states, at given homogeneous external conditions (temperature, pressure, external fields) are degenerate with respect to one or several parameters. In other words, there are non-equivalent equilibrium states in which these parameters are different but the thermodynamic potential is the same. Thus, for He II the degeneracy parameter is the condensate phase shift Φ, for an isotropic ferromagnet it is the direction of the spontaneous magnetization $\mathbf{m}$ (the degeneracy parameters for liquid crystals and for superfluid He^3 will be given in the text).

In the case of an inhomogeneous state of the substance the degeneracy parameter is a function of the coordinates and of the time. Inhomogeneous states are possible in which, at a certain point or on a line in space, the degeneracy parameter is not defined, and this singular point or line cannot be eliminated without destroying at the same time the ordered state in a large volume of matter. This, for example, is the situation with the vortex in He II. This vortex constitutes a singular line and the degeneracy parameter (the phase Φ) changes by 2π after circling this line. On the line itself, the phase Φ is indeterminate. This singular line can be eliminated only by destroying the superfluid state in a large volume of liquid. It is easily seen that the existence of a vortex in He II is connected with the fact that the region in which the phase Φ varies is a circle of unit radius.

It is natural to expect the existence of singular lines and points of other ordered substances also to depend on the global properties of the region where the degeneracy parameter varies, i.e., on its topological structure. The purpose of the present article is to describe a regular method of classifying topologically stable singularities of the degeneracy parameter of an ordered system, by starting from the topological structure of the region where the latter varies. This method is based on the use of the so called homotopic group. It makes it possible to find all the types of topologically stable singularities, i.e., those which can be eliminated only by destroying the ordered state in a large volume, and also to set in correspondence with each singularity a homotopic-group element, by the same token making it possible to classify the types of singularities. In addition, by using this method, it is possible to identify the type of singularity that results from coalescence of singularities.

The homotopic topology method has not been used in the literature known to us on the investigation of different singularities of ordered systems. (For the use of this method in field theory, see the paper of Monastyrskiĭ and Perelomov.)[1] In a number of cases, therefore, the classifications obtained in the literature were either incomplete or contained singularities that could be eliminated topologically.

In Secs. 2–4 of the present paper the method of homotopic groups is described with He II, a ferromagnet, and a nematic liquid crystal as examples. In the exposition of this method, we cite elementary information on homotopic topology, details of which can be found, for example, in the book of Huismoller and Spanier.[2] In Sec. 5, the homotopic groups are used to classify the singularities in superfluid He^3 (some of the results were already published earlier).[3] In Sec. 6, this method is applied to the classification of singularities in liquid crystals of the cholesteric type, and the question of its application to ordinary crystals and liquid crystals of smectic type is also considered. The Appendix contains a brief description of the method of calculating the homotopic groups used in the article (see also[1,2]).

2. THE FUNDAMENTAL GROUP AND LINEAR SINGULARITIES OF He II

It is known that He II is characterized by a complex order parameter $\Psi = |\Psi| e^{i\Phi}$. At equilibrium the modulus of the order parameter assumes a fixed value $|\Psi| = C(T, P)$, which minimizes the condensation energy $F_c(|\Psi|)$, while the phase Φ can assume any value and is therefore the degeneracy parameter. In the nonequilibrium state, when $|\Psi|$ and Φ vary in space, an additional gradient energy F_{grad}, which depends on $\nabla|\Psi|$ and $\nabla\Phi$, is added to the condensation energy. We consider weakly inhomogeneous states, when the characteristic distances over which $|\Psi|$ and Φ vary are much larger than the coherence length $\xi(T, P)$. In this case $F_{grad} \ll F_c$, so that we can assume $|\Psi|$ to be practically constant, and only the degeneracy parameter Φ varies in space, and the gradient energy can be regarded as dependent only on $\nabla\Phi$, i.e., $F_{grad} = \rho_s v_s^2/2$, where $\mathbf{v}_s = \hbar\nabla\Phi/m_4$ is the superfluid velocity and m_4 is the mass of the He^4 atom. This no longer holds near the core of the vortex. Indeed, since the phase Φ is indeterminate on the vortex axis and $|\Psi|$ should accordingly vanish, the gradient energy near the core becomes comparable with condensation energy. However, if we are not interested in a region of dimension near the core, and move over to a distance $r \gg \xi$, then we have here $F_{grad} \ll F_c$ and we can again assume that $|\Psi| = C(T, P)$ and only Φ changes.

From the mathematical point of view, the complex order parameter $\Psi(\mathbf{r})$ determines the continuous mapping of the set of points $\mathbf{r}$ of the vessel on the complex plane. If we consider only weakly inhomogeneous states and neglect regions with dimensions $\sim\xi$ near the lines on which the phase is not determined, then we obtain $|\Psi| = \text{const}$, and $\Phi(\mathbf{r})$ maps continuously the set of points $\mathbf{r}$ of a vessel with a notched line L, on which the phase Φ is not defined, into the circle that constitutes the region of variation of the phase Φ (this circle will henceforth be designated S^1).

We must ascertain which of these singular lines can be eliminated by continuous deformation of the field $\Phi(\mathbf{r})$, and which cannot be eliminated by any continuous change of the field $\Phi(\mathbf{r})$. To this end, we surround the investigated singular line L by a simple closed contour γ (which passes, of course, at a distance $r \gg \xi$ from the line), which starts at a fixed point and goes in a fixed direction. This contour is mapped by the function $\Phi(\mathbf{r})$ on the circle S^1 also into a closed contour Γ with a fixed circuiting direction; the point $\mathbf{r}_0$ is mapped thereby into the point $A = \Phi(\mathbf{r}_0)$ on S^1.

Let us imagine that we can contract the contour Γ to a point A, by continuously deforming it on S^1. It is then easily seen that the investigated line is topologically not singular, since it can be eliminated by continuously transforming the field $\Phi(\mathbf{r})$ into a constant field $\Phi(\mathbf{r}) = \text{const}$. In the case when the investigated line is the core of a vortex, the contour surrounds the circle S^1 once or several times and is closed at the point A. In this case we cannot contract the contour Γ into a point by any deformation. Consequently, no continuous change of the field $\Phi(\mathbf{r})$ is capable of eliminating the singularity on the core of the vortex.

Thus, the line L is topologically singular whenever the contours γ which surround it are mapped on contours Γ that cannot be contracted into points in the region of variation of the degeneracy parameters. This is valid also for other ordered systems. To investigate singular lines of an ordered system it is therefore necessary to investigate the possible continuous deformations of the contours Γ in the region of variation of the degeneracy parameters (this region will henceforth be denoted R)

In topology, continuous deformation is called homotopy. Two closed contours emerging from a point A are called homotopic relative to each other or homotopically equivalent, or else belonging to a single homotopic class, if they can be deformed in continuous fashion into each other while leaving the point A immobile. The contours that contract to a point are referred to as homotopic to zero. Obviously, contours with different numbers of circuits along a circle are homotopically not equivalent. The product of two contours Γ_1 and Γ_2 is defined as a contour $\Gamma_2\Gamma_1$ such that the mapping of a point that runs along the contour γ, after leaving the point A, first goes around contour Γ_1 and then Γ_2. The element reciprocal to Γ is defined as the contour Γ^{-1} with opposite circuiting direction. The set of contours Γ belonging to one homotopic class must be regarded as a single contour within the homotopy accuracy. Just as for the contours, we can introduce definitions of multiplication of classes of contours, of a class that is the inverse of a given class, and also a unit class of contours that are homotopic to zero. It can be verified that multiplication of classes is associative.

Thus, the aggregate classes of homotopic contours forms a group—the so called fundamental group of space R, which we designate $\pi_1(A, R)$. This group is generally speaking non-commutative. An example of such a non-commutative group is the fundamental of space R for a cholesteric liquid crystal. Deferring the discussion of non-commutative fundamental groups until we reach this case, we consider for the time being Abelian fundamental groups. The elements of the latter do not depend on the choice of the point A, so that the fundamental Abelian group will henceforth be designated $\pi_1(R)$.

In the case of He II, each vortex corresponds to a class of contours Γ—the transforms of the contours γ that surround the vortex. Therefore each vortex can be set in correspondence to an element of the fundamental group $\pi_1(S^1)$, which is called the fundamental group of the circle. The latter is isomorphic to the group of integers (which will henceforth be designated Z), since each class of contours Γ can be set in correspondence with an integer N—the number of circuits of S^1 in the positive direction. Consequently, each vortex can be characterized by a whole-number index, N, which is equal in this case to the number of circulation quanta of the superfluid velocity $\mathbf{v}_s$ around the core of the vortex

$$N = \frac{m_4}{2\pi\hbar}\oint \mathbf{v}_s\, d\mathbf{l}.$$

In our case, too, each linear singularity of the degeneracy parameters can be set in correspondence with an element of the fundamental group $\pi_1(R)$. A nonsingular configuration of the degeneracy-parameter field corresponds to a unit element of this group, and coalescence of the singularities corresponds to multiplication of elements of these groups. Since we are interested only in Abelian groups with a finite number of generators, each singularity can be characterized by a set of whole-number indices $\{N_i\}$. When singularities coalesce, the indices N_i add up in modulo p_i, where p_i is the order of the i-th generator (for example, if $p = 2$, then $1 + 1 = 0$).

Degeneracy-parameter field configurations characterized by identical indices N_i can be continuously transformed into one another. Nevertheless, they differ in their energy and therefore, a potential barrier is possible when they are transformed into one another, i.e., different locally stable singularities with identical indices are possible. In the investigation of the stability it is important to know the heights of the barriers that hinder transitions into configuration with lower energy. Thus, for example, in He II a barrier is possible when a vortex with two circulation quanta ($N=2$) decays (a process with decreasing energy) into vortices with one circulation quantum each ($N=1+1=2$). An estimate shows that if such a barrier exists then its value is $\sim F_c\xi^3$. However, if we want to transform a vortex with $N=2$ into a vortex with $N=1$ or to annihilate a vortex (processes with decrease of energy), then no continuous change of the phase Φ can accomplish this. To reach this goal it is necessary to go over in the intermediate states from the circle S^1 into a complex plane on which the entire order parameter Ψ changes. It is easy to see that the path with minimal barrier on which one vortex is transformed into another, is the one in which $|\Psi|$ vanishes in the intermediate state on a certain surface S that borders on the vortex line. The phase Φ is not defined on this surface. The height of the barrier turns out to be $\sim F_c\xi S \gg F_c\xi^3$. In view of the tremendous sizes of the potential barriers, the probability of processes with change of indices is vanishingly small. We can therefore assume that the dynamics of an ordered system is such that the summary indices of the singularities situated in a given volume of the system are conserved, i.e., the motion has the following invariants N_i:

$$\sum_l N_i^{(l)} \, (\mathrm{mod}\, p_i) = N_i, \qquad (2.1)$$

where the sum is taken over all the singularities in the given volume. The invariants N_i can change only when a singularity enters or leaves the volume. Homotopic topology makes it therefore possible to single out classes of singularities having identical indices with macroscopically large barriers to transitions from one class to another.

Thus, the procedure for classifying the linear singularities must be the following: A homotopic classification is first carried out to subdivide the singularities into classes with large topology-governed barriers to transitions from one class to another. For this purpose it is necessary to find the fundamental group of the space R. If this group is commutative, then each linear singularity can be set in correspondence with an element of this group (or with a set of whole-number indices), by identifying the class of contours of the space R into which the contour surrounding the singular line is mapped. By the same token, the singularities are subdivided into classes characterized by identical indices with large barriers to transitions from class to class. The subsequent analysis, which is no longer homotopic, should consist of finding the locally stable singularities within each class by minimizing the functional of the energy.

3. THE HOMOTOPIC GROUP $\pi_2(R)$ AND POINT SINGULARITIES OF A FERROMAGNET

We proceed to the investigation of the singular points. By way of example we consider an isotropic ferromagnet. The order parameter of an isotropic ferromagnet is the magnetization vector $\mathbf{M}$. The states of the ferromagnet are degenerate with respect to the directions of this vector, and therefore the degeneracy parameter is the unit vector $\mathbf{m}=\mathbf{M}/|\mathbf{M}|$, while the space R coincides with the two-dimensional sphere S^2. We note that an isotropic ferromagnet has no topologically stable linear singularities, since the fundamental group of the space S^2 is trivial ($\pi_1(S^2)=0$). Indeed, any contour on a sphere can be contracted into a point. This, of course, does not exclude the possible existence of locally stable singular lines of the vector $\mathbf{m}$ with a small barrier $\sim F_c\xi^3$ of nontopological character on going into a nonsingular configuration, where F_c is the ferromagnetic-ordering energy.

To investigate the singular points, we surround in the ferromagnet a point in which the vector $\mathbf{m}$ is not defined by a sphere of radius much larger than ξ. The function $\mathbf{m}(\mathbf{r})$ specifies the mapping of this sphere on a certain closed surface on the sphere S^2. If the sphere σ goes over in this case to a surface that is homotopic to zero, i.e., that can be contracted into a point on S^2, then the investigated singular point is topologically unstable, since the field $\mathbf{m}(\mathbf{r})$ can be made homogeneous by means of continuous deformation. On the other hand, if we consider the so called "hedgehog," i.e., a field of the type $\mathbf{m}(\mathbf{r})=\hat{\mathbf{r}}$ with a singular point at the origin (where $\hat{\mathbf{r}}$, $\hat{\theta}$, and $\hat{\varphi}$ are the unit vectors of a spherical coordinate system in ordinary space), then the sphere σ surrounding the singular point is mapped by the function $\mathbf{m}(\mathbf{r})$ on the entire sphere S^2. This surface cannot be contracted into a point and remain on the sphere S^2, since the singular point cannot be eliminated by any continuous transformation of the field $\mathbf{m}(\mathbf{r})$.

Thus, to classify the topologically stable singular points it is necessary to find all the classes of the surfaces on the sphere S^2 (in our case, in the space R) that are not homotopic to zero, into which the sphere σ can be mapped. These classes, together with the class of surfaces homotopic to zero, are the elements of a homotopic group of dimensionality 2, designated $\pi_2(R)$. In the case of He II the group $\pi_2(S^1)=0$ and there are no point singularities in He II.

In the case of an isotropic ferromagnet, the group $\pi_2(S^2)$ is isomorphic to the group of integers Z. Indeed, to each class of mappings of σ on S^2 one can set in correspondence an integer N that shows how many times the vector $\mathbf{m}$ runs over the sphere S^2, with allowance for the orientation, whenever $\mathbf{r}$ runs over σ. This number is called the degree of mapping and it can be expressed in terms of a surface integral of the field $\mathbf{m}(\mathbf{r})$, which coincides with the integral of the Gaussian curvature of the surface to which the vector $\mathbf{m}$ is the normal vector:

$$N=\frac{1}{4\pi}\int_\sigma d\theta\, d\varphi\, \mathbf{m}\left[\frac{\partial \mathbf{m}}{\partial\theta}\times\frac{\partial \mathbf{m}}{\partial\varphi}\right]. \qquad (3.1)$$

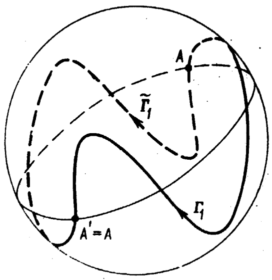

FIG. 1. Closed paths that are not homotopic to zero in the space S^2/Z_2.

Thus, each point singularity in the field of the unit vector $\mathbf{m}(\mathbf{r})$ is characterized by a whole-number index N that runs through the values from $-\infty$ to ∞. In the case of a "hedgehog" in the form $\mathbf{m}(\mathbf{r})=\hat{\mathbf{r}}$, we have $N=1$, and for the "hedgehog" of the type $\mathbf{m}(\mathbf{r})=-\hat{\mathbf{r}}$ the number is $N=-1$. When the singularities coalesce, the indices N add up. Of course, within a singularity class characterized by a single index N, there can be different locally stable singularities. For one such singularity to go over into another with a lower energy it is necessary to overcome barriers on the order of $F_c\xi^3$. But to transform a singularity of one class into a singularity of the other class it is necessary, in the intermediate state, to make $\mathbf{m}$ indeterminate on the entire line L that emerges from the singular point. On this line, the vector $\mathbf{M}$ vanishes, and therefore the barrier is macroscopically large: $\sim F_c\xi^2L \gg F_c\xi^3$.

When generalizing the foregoing arguments to the case of an arbitrary space R, it should be noted that we must stipulate in the definition of the group $\pi_2(R)$ that a certain point $\mathbf{r}_0 \in \sigma$ always goes over as a result of the mapping into the same point $A \in R$. If the fundamental group $\pi_1(R)$ is nontrivial, then it may turn out that when the point A is moved along a close contour that is not homotopic to zero the element of the group $\pi_2(R)$ goes over into another element of this group. This influence of the fundamental group $\pi_1(R)$ on the group $\pi_2(R)$ will be considered in the next section, using nematic liquid crystals as an example. On the other hand, if this phenomenon does not take place (for example, when $\pi_1(R)=0$), then the point singularities of ordered systems are classified in the same way as linear singularities in substances with an Abelian fundamental group of the sapce R. Namely, one finds the group $\pi_2(R)$, and then each point singularity is set in correspondence with an element of this group. When the singularities coalesce, multiplication of the elements of the group takes place. It is known that all the groups $\pi_2(R)$ are Abelian, and therefore the pointlike singularities can be classified by means of whole-number integers. We then obtain singularity classes having identical indices, with large barriers for the transition between classes. The succeeding, no longer homotopic, investigation of the energy functional should reveal the locally stable singularities within each class.

4. SINGULARITIES IN NEMATIC LIQUID CRYSTALS

A nematic liquid crystal (NLC) is characterized by the unit vector $\mathbf{d}$ of the director, the states with $\mathbf{d}$ and $-\mathbf{d}$ being indistinguishable. Therefore the region of variation of the degeneracy parameter R is the sphere S^2, in which two diametrically opposed points are equivalent. This space is written in the form $R=S^2/Z_2$ (i.e., the sphere S^2 is factorized with respect to a group of two elements Z_2).

In contrast to a ferromagnet, a nematic liquid crystal has a nontrivial fundamental group $\pi_1(R)=Z_2$ (Z_2 is also called the group of residues in modulo 2). Indeed, consider a contour Γ_1 that joins two diametrically opposite A and A' on the sphere S^2 (see Fig. 1). This contour is closed because $A'=A$ and, in addition, cannot be contracted into a point. Therefore Γ_1 belongs to one of the classes of contours that are non-homotopic to zero. There is only one such class. Indeed, consider the contour Γ_1^{-1} that goes in the opposite direction. This path is equivalent to the path $\tilde{\Gamma}_1$ consisting of the diametrically opposite points. But the path $\tilde{\Gamma}_1$ can be deformed into Γ_1 over the surface of the sphere, therefore $\Gamma_1^{-1}=\Gamma_1$ and consequently $\Gamma_1\cdot\Gamma_1=\Gamma_0$, where Γ_0 is the class of paths that are homotopic to zero. Thus, all the contours can be homotopic either to Γ_1 or to zero. Consequently the group $\pi_1(S^2/Z_2)$ consists of two elements (see also the Appendix). The singular lines of a nematic liquid crystal are characterized by a whole-number index N that assumes only two values, 0 and 1, and the addition of the indices occurs on modulo 2 (i.e., $1+1=0$).

The linear singularities in NLC (disclinations) are customarily characterized by the Frank index m. The contour γ that surrounds the disclinations with Frank index m is mapped into the contour Γ_1^m which is homotopic to zero for even m and homotopic to Γ_1 for odd m. Therefore disclinations with an even Frank index $m=2k$ belong to the class of singularities with index $N=0$, and consequently any possible barrier to the transformation of these disclinations into a nonsingular configuration is small, $\sim F_c\xi^3$ (here ξ is of the order of the interatomic distances, and $F_c \sim K/\xi^2$, where K is the elastic modulus of the liquid crystal). Let us examine for example the elimination of the disclination $\mathbf{d}=\hat{\boldsymbol{\rho}}$ with $m=2$ ($\hat{\boldsymbol{\rho}}$, $\hat{\mathbf{z}}$, and $\hat{\boldsymbol{\varphi}}$ are the unit vectors of a cylindrical coordinate system). The field $\mathbf{d}=\hat{\boldsymbol{\rho}}$ can be continuously deformed into a homogeneous field $\mathbf{d}=\hat{\mathbf{z}}$ by means of the continuous transition

$$\mathbf{d}=\hat{\boldsymbol{\rho}}\cos(\pi t/2)+\hat{\mathbf{z}}\sin(\pi t/2),$$

by varying t from zero to unity (see Figs. 2a–2c). On the other hand, if we go through a potential barrier when t is varied (see, e.g., the paper of Anisimov and Dzyaloshinskiĭ),[4] then it is more convenient to eliminate the singularity by breaking the filament (see Fig. 2d). We then change the field in a volume of the order of ξ^3, and possibly surmount a barrier

$$\sim\frac{K}{\xi^2}\xi^3\sim F_c\xi^3.$$

Disclinations with odd Frank indices $m=2k+1$ belong to the homotopic class $N=1$. To eliminate them it is therefore necessary to surmount a barrier $\sim F_c\xi S$, where S is the area of the surface that bears on the disclination line. An investigation of the locally stable disclinations inside each homotopic class can be found in[4].

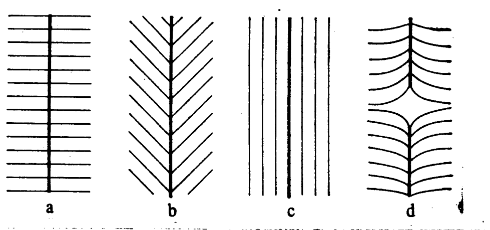

FIG. 2. Conversion of a disclination with $m=2$ into a nonsingular configuration: a)–c) by continuous deformation, d) by breaking the filament. Thick line—disclination filament. Thin lines—lines of the director field in a plane passing through the filament.

We proceed now to point singularities in NLC. The group is $\pi_2(S^2/Z_2)=Z$ (see the Appendix), i.e., there is a group of homotopic mappings of the sphere σ on the space $R=S^2/Z_2$; this group is homotopic to the group of integers. These mappings can be easily obtained from the mappings of on the sphere S^2 followed by mapping S^2 on S^2/Z_2. Therefore the elements of $\pi_2(R)$ are characterized by a whole-number invariant N of the type (3.1), where $\mathbf{m}$ must be replaced by $\mathbf{d}$. It is easy to see, however, that each singularity is characterized by two numbers, N and $-N$. Indeed, replacement of $\mathbf{d}$ by $-\mathbf{d}$ does not change the states, whereas in (3.1) N is replaced by $-N$. This is the consequence of the influence of the fundamental group $\pi_1(R)$ on the group $\pi_2(A, R)$. Let us examine this in greater detail.

Assume that we have a mapping of degree N of the sphere σ on R, and the point $\mathbf{r}_0$ goes over into a certain point $\mathbf{d}(\mathbf{r}_0)=A$. If we move the point $\mathbf{r}_0$ over a closed contour γ, then the point A will move over a closed contour Γ in the space R. If the contour γ does not enclose a singular line, then the vector $\mathbf{d}(\mathbf{r}_0)$ does not reverse sign after the circuit and N remains likewise unchanged. In the opposite case, when γ encloses a singular line, then $\mathbf{d}(\mathbf{r}_0)$ goes over into $-\mathbf{d}(\mathbf{r}_0)$ after the circuit, and consequently N reverses sign. The point A then runs over the contour Γ_1, which is not homotopic to zero. Thus, the influence of the fundamental group $\pi_1(R)$ on $\pi_2(R)$ consists in the fact that when the point A moves along a certain contour that is not homotopic to zero, the elements of the group $\pi_2(R)$ can go over into another element of this group. In this case the element N goes over into $-N$. In the general case of an arbitrary space R, we can state that each point singularity corresponds to an entire class of the group $\pi_2(R)$, whose elements are obtained from one another by motion of the point A over the contours of the group $\pi_1(R)$. Thus in this case each singularity corresponds to a class of a group of whole numbers with identical moduli, i.e., the singularities are characterized by the index $|N|$. When the singularities coalesce, multiplication of the classes takes place. In a nematic liquid crystal, the coalescence of singularities with $|N_1|$ and $|N_2|$ can result in a singularity having either the index $|N_1|+|N_2|$ or the index $||N_1|-|N_2|$ The particular singularity obtained depends on the manner of the coalescence. In order to clarify this, it is necessary to introduce continuously in the vicinity of the coalescence path, in place of the director field $\mathbf{d}$, the field of the true director $\mathbf{d}$ and add the indices of the singular points of the field $\mathbf{d}$ (see Fig. 3).

5. CLASSIFICATION OF SINGULARITIES IN SUPERFLUID He^3

The order parameter in superfluid He^3 is a complex 3×3 matrix A_{ik}. In the equilibrium state, the order parameter that minimizes the condensation energy F_c (see Leggett's review), [5]

$$F_c=-\alpha A_{ik}A_{ik}^*+\beta_1|A_{ik}A_{ik}|^2+\beta_2(A_{ik}A_{ik}^*)^2 +\beta_3A_{ik}A_{il}A_{mk}^*A_{ml}^*+\beta_4A_{ik}A_{il}^*A_{mk}A_{ml}^* +\beta_5A_{ik}A_{il}^*A_{mk}^*A_{ml}, \quad (5.1)$$

takes the following form for the A phase and the B phase of He^3, respectively

$$A_{ik}=\mathrm{const}\cdot V_i(\Delta_k'+i\Delta_k''),\quad A_{ik}=\mathrm{const}\cdot e^{i\Phi}R_{ik}(\omega), \quad (5.2)$$

where $\mathbf{V}$, Δ', and Δ'' are arbitrary unit vectors connected by the relation $\Delta'\perp\Delta''$ (the vector product $\Delta'\times\Delta''$ specifies the direction $\mathbf{l}$ of the orbital angular momentum of the Cooper pair); Φ is the phase of the condensate, $R_{ik}(\omega)$ is the matrix of the rotation through the angle $|\omega|\leq\pi$ around the ω axis, and $\mathrm{const}\approx(\alpha/\beta)^{1/2}$. In the B phase of He^3 the equilibrium states are degenerate with respect to the phase Φ and the rotation matrix R_{ik}, therefore the space R for the B phase is the product of S^1 (the region of variation of the phase Φ) by the space of the three-dimensional rotations SO_3, i.e.,

$$R_B=S^1\times SO_3. \quad (5.3)$$

In the A phase the space R is the product of S^2 (the region of variation of the vector $\mathbf{V}$) by SO_3 (rotations that specify the orientation of the triplet of vectors Δ', Δ'', $\mathbf{l}$). It must be recognized here that the states with $\mathbf{V}$,

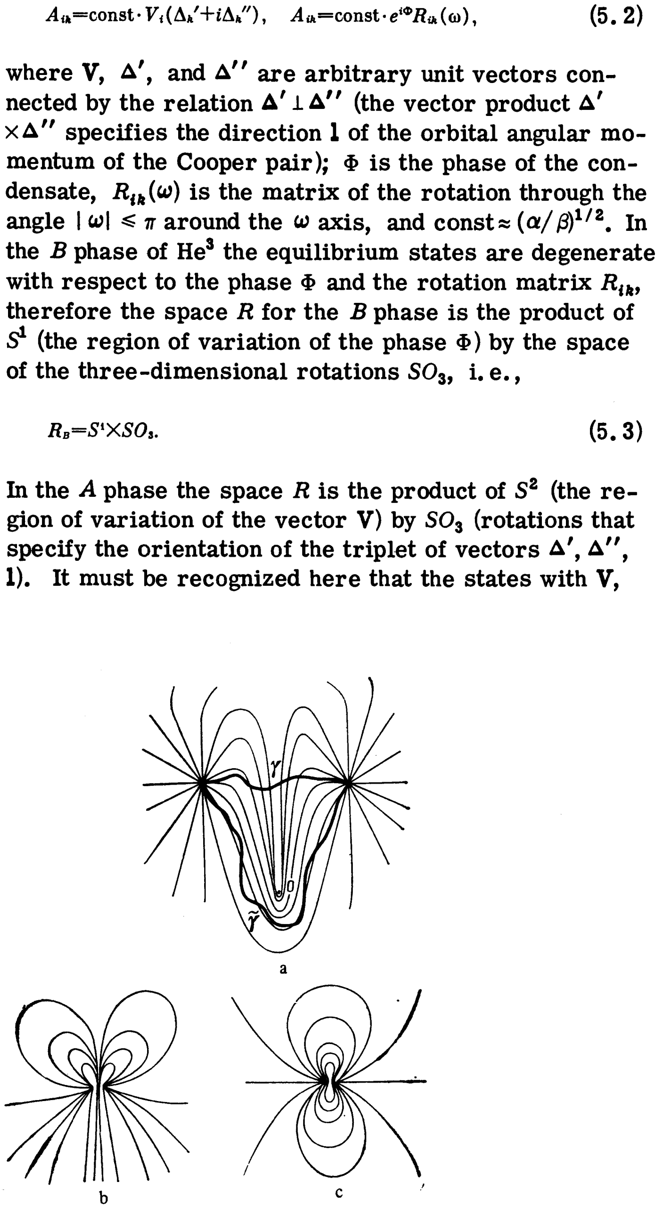

FIG. 3. Coalescence of point singularities with indices $|N_1|=|N_2|=1$ in a nematic liquid crystal: a) coalescence along the paths γ and $\hat{\gamma}$ passing on opposite sides of the disclination line (the point O) perpendicular to the plane of the figure yields respectively b) a point singularity with $|N|=2$ and c) a point singularity with $|N|=0$.

Δ', Δ'' and $-\mathbf{V}$, $-\Delta'$, $-\Delta''$ are indistinguishable, so that the space $S^2 \times SO_3$ must be factorized with respect to the group Z_2, i. e.,

$$R_A = (S^2 \times SO_3)/Z_2. \tag{5.4}$$

(Chechetkin[6] has incorrectly defined the spaces R_A and R_B, and this resulted in an incorrect classification of the singularities in the A and B phases of He^3).

The degeneracy in the A and B phases is partially lifted on account of the weak spin-orbit interaction F_{sl}:

$$F_{sl} = \lambda(|A_{ii}|^2 + A_{ik}A_{ki}^*), \quad \lambda \ll \alpha. \tag{5.5}$$

In the A phase, the vector $\mathbf{V}$ is fixed in either the direction $\mathbf{l}$ or $-\mathbf{l}$, while in the B phase the degeneracy remains with respect only to those matrixes R_{ik} which describe rotation through a fixed angle $|\omega| = \theta_0 = \arccos(-\frac{1}{4}) \approx 104°$ relative to an arbitrary axis ω. The regions R_A and R_B go over in this case into the following:

$$\tilde{R}_A = O_3, \quad \tilde{R}_B = S^1 \times S^2, \tag{5.6}$$

where the sphere S^2 is the region of variation of the unit vector ω/θ_0.

In the inhomogeneous state, the gradient energy

$$F_{grad} = \gamma\left(2\frac{\partial A_{ik}}{\partial x_k}\frac{\partial A_{ip}^*}{\partial x_p} + \frac{\partial A_{ik}}{\partial x_p}\frac{\partial A_{ik}^*}{\partial x_p}\right) \tag{5.7}$$

is added to F_c and F_{sl}, and the result is two length scales $\xi \sim (\gamma/\alpha)^{1/2}$ and $R_c \sim (\gamma/\lambda)^{1/2}$, with $R_c \sim (10^2 - 10^3)\xi \gg \xi$. If the characteristic distances over which A_{ik} varies are $r \gg R_c$, then the gradient energy $F_{grad} \ll F_{sl} F_c$, $F_{grad} \ll F_c$ does not change the structure of the order parameter and the region of variation of R takes the form (5.6). On the other hand, if the characteristic distances are $\xi \ll r \ll R_c$, then $F_{sl} \ll F_{grad} \ll F_c$, and therefore the spin-orbit interaction can be neglected, and does not influence the form of the order parameter, and consequently the region R is given by formulas (5.3) and (5.4). Therefore the classification of the singularities in superfluid He^3 depends essentially on the dimensions of the investigated region of the liquid. We shall consider only the A phase, since the classification of the linear and point singularities of the B phase is given quite completely in the preceding paper.[3] Exceptions are the singular surfaces (see below).

We consider the first when one of the characteristic dimensions of the region is $\xi \ll r \ll R_c$. In this case the space R is given by formula (5.4). The homotopic groups of this space are

$$\pi_1((S^2 \times SO_3)/Z_2) = Z_4, \quad \pi_2((S^2 \times SO_3)/Z_2) = Z$$

(see the Appendix). Here Z_4 is a group of residues in modulo 4, therefore the linear singularities are characterized by a whole-number index N that takes on values 0, 1, 2, and 3. Coalescence of the singularities leads to addition of the indices in modulo 4, for example, $3 + 3 = 2$. The barrier for the transition of a singularity from one class to another is $\sim F_c \xi S$.

To find locally stable singularities within each class it is necessary to minimize the gradient-energy functional $\int d^3r F_{grad}$ with respect to the degeneracy parameters. Upon variation of this functional, equations are obtained for $\mathbf{V}$, Δ', and Δ''. We write out several solutions for these equations for linear singularities of various classes ($\hat{\mathbf{z}}, \hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\varphi}}$ and $\hat{\mathbf{z}}, \hat{\mathbf{x}}, \hat{\mathbf{y}}$ are unit vectors of cylindrical and Cartesian coordinate systems, respectively, with the $\hat{\mathbf{z}}$ axis along the singular line):

$N=1$:

$$\Delta' + i\Delta'' = e^{i\varphi/2}(\hat{\mathbf{x}} + i\hat{\mathbf{y}}), \quad \mathbf{V} = \hat{\mathbf{x}}\cos\frac{\varphi}{2} - \hat{\mathbf{y}}\sin\frac{\varphi}{2}, \quad \mathbf{v}_s = \frac{\hat{\boldsymbol{\varphi}}}{4m_3\rho} \tag{5.8}$$

$N=2$:

$$\Delta' = \hat{\boldsymbol{\varphi}}, \quad \Delta'' = \hat{\mathbf{z}}, \quad \mathbf{l} = \hat{\boldsymbol{\rho}}, \quad \mathbf{V} = \text{const}, \quad \mathbf{v}_s = 0; \tag{5.9a}$$

$$\Delta' + i\Delta'' = e^{i\varphi}(\hat{\mathbf{x}} + i\hat{\mathbf{y}}), \quad \mathbf{V} = \text{const}, \quad \mathbf{v}_s = \hat{\boldsymbol{\varphi}}/2m_3\rho; \tag{5.9b}$$

$$\Delta' = \hat{\boldsymbol{\varphi}}, \quad \Delta'' = -\hat{\mathbf{z}}, \quad \mathbf{l} = -\hat{\boldsymbol{\rho}}, \quad \mathbf{V} = \text{const}, \quad \mathbf{v}_s = 0; \tag{5.9c}$$

$$\Delta' = \hat{\boldsymbol{\varphi}}, \quad \Delta'' = \hat{\mathbf{r}}, \quad \mathbf{l} = \hat{\boldsymbol{\theta}}, \quad \mathbf{V} = \text{const}, \quad \mathbf{v}_s = \hat{\boldsymbol{\varphi}} 2m_3 r; \tag{5.9d}$$

$N=3$:

$$\Delta' + i\Delta'' = e^{-i\varphi/2}(\hat{\mathbf{x}} + i\hat{\mathbf{y}}), \quad \mathbf{V} = \hat{\mathbf{x}}\cos\frac{\varphi}{2} + \hat{\mathbf{y}}\sin\frac{\varphi}{2}, \quad \mathbf{v}_s = -\frac{\hat{\boldsymbol{\varphi}}}{4m_3\rho}; \tag{5.10}$$

$N=0$:

$$\Delta' = \hat{\mathbf{x}}, \quad \Delta'' = \hat{\mathbf{y}}, \quad \mathbf{l} = \hat{\mathbf{z}}, \quad \mathbf{V} = \text{const}, \quad \mathbf{v}_s = 0; \tag{5.11}$$

$$\Delta' + i\Delta'' = e^{2i\varphi}(\hat{\mathbf{x}} + i\hat{\mathbf{y}}), \quad \mathbf{V} = \text{const}, \quad \mathbf{v}_s = \boldsymbol{\varphi}/m_3\rho; \tag{5.11b}$$

$$\Delta' = \hat{\mathbf{x}}, \quad \Delta'' = \hat{\mathbf{y}}, \quad \mathbf{l} = \hat{\mathbf{z}}, \quad \mathbf{V} = \hat{\boldsymbol{\rho}}, \quad \mathbf{v}_s = 0. \tag{5.11c}$$

Here m_3 is the mass of the He^3 atom and $v_s^i = \Delta' \nabla^i \Delta''/2m_3$.

Singularities with $N=1$ and $N=3$ are vortices in which the circulation quantum of the superfluid velocity $\mathbf{v}_s$ is equal to $\frac{1}{2}$, superimposed on these vortices are disclinations of the vector $\mathbf{V}$ with a Frank index $m=1$ (the vector $\mathbf{V}$ turns out to be analogous to the vector $\mathbf{d}$ in the NLC). The written-out singularities with $N=2$ are the vortex (5.9b) with one circulation quantum and the disgyrations (see de Gennes' paper[7]) (5.9a) and (5.9c). An analysis of the energy functional $\int d^3r F_{grad}$ shows that both disgyrations are locally stable (see[8]) and have identical and apparently the lowest energies from among all singularities of the class $N=2$. The vortex (5.9b) is locally unstable and should go over without a barrier into one of the stable disgyrations. The locally stable solution (5.9d) is a junction of two disgyrations (5.9a) and (5.9c) at the point $\mathbf{r}=0$. Indeed, at $z>0$, $\rho \to 0$ we have $\mathbf{l} \to \hat{\boldsymbol{\rho}}$, $\Delta'' \to \hat{\mathbf{z}}$, i. e., the solution takes the form (5.9a), and at $z<0$, $\rho \to 0$ we have $\mathbf{l} \to -\hat{\boldsymbol{\rho}}$; $\Delta'' \to \hat{\mathbf{z}}$ and the solution takes the form (5.9c).

Singularities with $N=0$—a vortex with two circulation quanta (5.11b) and a disclination in the field of the vector $\mathbf{V}$ (5.11c) with a Frank index $m=2$ can relax without a barrier into the homogeneous state (5.11a). A barrier $\sim F_c \xi^3$ can appear when account is taken of the structure of the cores of these singularities.

The point singularities of the A phase in the region $\xi \ll r \ll R_c$ are singularities in the vector field $\mathbf{V}$. They are analogous to the singularities of the vector $\mathbf{d}$ in a nematic liquid crystal and are characterized by the index $|N|$ in accordance with formula (3.1), in which $\mathbf{m}$ must be replaced by $\mathbf{V}$. The corresponding solutions of the equations for $\mathbf{V}$, Δ', and Δ'' are not simple in form even for $|N|=1$. The "hedgehog" of the form $\mathbf{V}=\mathbf{r}$, $\Delta' + i\Delta'' = \text{const}$ is not a solution of these equations and relaxes to a stable singularity with $|N|=1$ (we note that

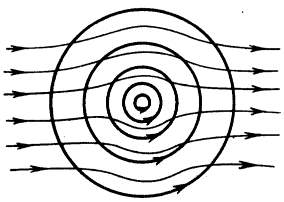

FIG. 4. Qualitative distribution of the vector fields in a plane perpendicular to the axis of a stable vortex with $N=2$ at distances $\rho > R_c$ from the axis. Thin lines—lines of vector field **l**. Thick lines—streamlines of superfluid velocity $\mathbf{v}_s$.

point singularities in the field of the vector **l**, in contrast to the statement made by Blaha,[9] are topologically removable.)

Let us see now what happens if we extend the region of the liquid to distances $r \gg R_c$, where the range of variation of the degeneracy parameters R_A narrows down to $\tilde{R}_A$ of (5.5). This gives rise to two groups of singularities of different origin. The first group includes the singularities characterized by elements of homotopic groups of space $\tilde{R}_A$. Let us examine first these singularities.

The homotopic groups are $\pi_1(\tilde{R}_A) = Z_2$ and $\pi_2(\tilde{R}_A) = 0$, i.e., there are no point singularities, and there are two classes of linear singularities. It is convenient to characterize them by the same index N as the linear singularities in the region $\xi \ll r \ll R_c$, but now N can assume only two values, 0 and 2, with addition in modulo 4.

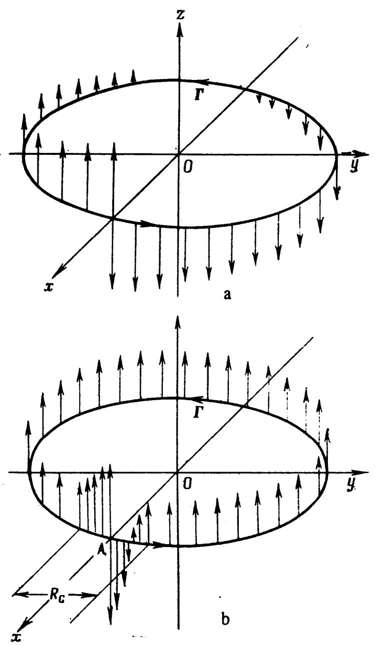

FIG. 5. Formation of singular surface in the propagation of a linear singularity from the region of distances $\rho < R_c$ into the region $\rho > R_c$ from the singular line. a) region $\rho < R_c$. The field of the vector ω on the contour Γ surrounding the disclination line (z axis) with $m=1$ in the B phase of He^3: $\omega = \hat{z}(\pi - \varphi)$ (see[3]). b) region $\rho > R_c$. Field of the vector ω on the contour Γ surrounding the disclination line (z axis) and crossing at the point A the singular surface bearing against the disclination line. In the region of distances larger than R_c from the singular surface we have $|\omega| \approx 104°$.

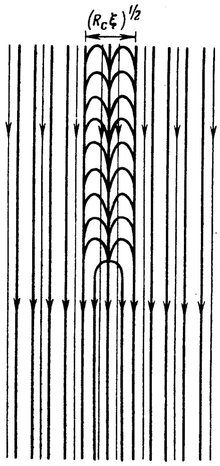

FIG. 6. Formation of singular lines in the propagation of the point singularity of the field of the vector **V** with $|N| = 1$ from the region of distances $r < (R_c\xi)^{1/2}$ into the region $r > (R_c\xi)^{1/2}$ from the singular point. The qualitative distribution of the lines of the vector field **l** is shown by thin lines, and those of the field **V** by thick lines.

Among the singularities of the class $N=2$, the lowest energy is possessed by a vortex with one circulation quantum and vectors **l** and **v** perpendicular to the vortex axis. The corresponding solution has no simple form. The lines of the fields **l** and $\mathbf{v}_s$ in a plane perpendicular to the vortex axis are shown in Fig. 4. At distances $r \lesssim R_c$, this solution merges with one of the locally stable solutions of the class $N=2$ (see (5.9)). We note that in addition to linear singularities, the state $\tilde{R}_A$ admits of the existence of singular surfaces.[1)] The reason is that the space O_3 is doubly connected (in other words $\pi_0(O_3) = Z_2$, where the homotopic group $\pi_0(R)$ determines the connectivity of the space R). There is therefore one class of singular surfaces joining the region with $\mathbf{V} = \mathbf{l}$ and the region with $\mathbf{V} = -\mathbf{l}$. The width of the surface is $\sim R_c$. These surfaces recall the domain walls in magnets.

The second group of singularities occurs when linear singularities with $N=1$ and 3 and point singularities propagated from the region $\xi \ll r \ll R_c$ into the region $r \gg R_c$. In this case, inasmuch as we cannot satisfy the condition $\mathbf{V} \parallel \mathbf{l}$ in the region $r \gg R_c$, the energy of these singularities becomes proportional to $F_{sl} V$, where V is the volume of space in which $\mathbf{V} \neq \pm\mathbf{l}$. In the case of linear singularities, this volume is minimal if $\mathbf{V} \neq \pm\mathbf{l}$ on a certain surface of thickness R_c, bearing on the singular line. Thus, the singular lines with $N=1$ and 3 go over into singular surfaces that border on these lines. A similar surface exists also in the B phase (see Fig. 5).

Analogously, the point singularity of the field **V** with index $|N|$ goes over into a singular line of thickness $(R_c\xi)^{1/2}$, which starts out from a singular point (see Fig. 6). In the same homotopic class there exists another singular line, albeit less convenient because of the larger gradient energy, but apparently locally stable. This singular line is a vortex with an end (vorton), the possible existence of which was noted by Blaha and by us[9,10]. The exact solution for this singularity does not have a simple form, and we therefore present one of the possible configurations of the fields **V** and **l**, which are characterized by the index $|N| = 1$ (see Fig. 7):

$$\mathbf{V} = \hat{\mathbf{r}}, \quad \mathbf{v}_s = \frac{1 - \hat{\mathbf{z}}\mathbf{l}}{2m_3\rho}\hat{\boldsymbol{\varphi}}, \quad \mathbf{l} = \begin{cases} \hat{\mathbf{z}}, & z < 0, \quad \rho \leqslant R_c \\ \hat{\mathbf{r}} & \text{in the rest of space} \end{cases}$$

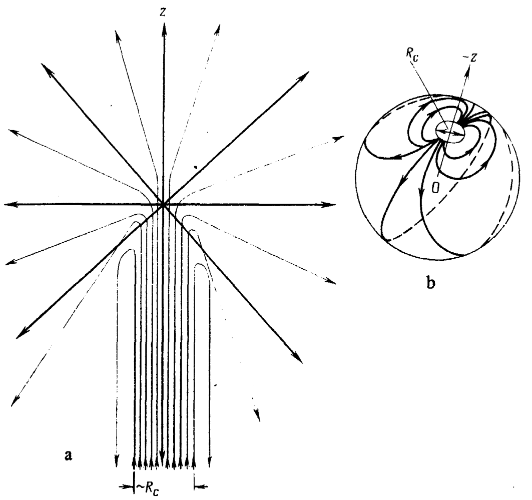

FIG. 7. Vortex with free and with $|N|=1$. a) The qualitative distribution of the lines of the vector field **1** is shown by thin lines, and those of the field **V** by thick lines. b) Distribution of the lines of the vector field Δ' on a sphere at distances $\rho > R_c$ from the vortex line coinciding with the semiaxis $z<0$.

At a distance $\rho > R_c$ from the lower semiaxis the field of the superfluid velocity $\mathbf{v}_s$ and the field $\mathbf{l}$ take the form

$$\mathbf{v}_s = \frac{\hat{\varphi}}{2m_3\rho}(1-\cos\theta), \quad \mathbf{l}=\hat{\mathbf{r}},$$

which corresponds to a vortex with two circulation quanta that terminate at the center of the "hedgehog" (see[10]). The width of the core of the vortex is $\sim R_c$. In the region $r< R_c$ there are no singularities in the fields $\mathbf{v}_s$ and $\mathbf{l}$. Such a vorton cannot vanish because of the topologically stable point singularity in the field **V** with index $|N| = 1$ at the origin. However, it is separated by an energy barrier $\sim F_{sl}R_c^3$ from the singular line with the lowest energy, shown in Fig. 6. On the other hand, if there is no singularity in the field of the vector **V**, then such a vortex relaxes into a nonsingular configuration.

We have considered all types of singular lines, points, and surfaces in the A phase in the absence of external fields. The presence of a magnetic field complicates the situation, since a third length scale appears, namely the magnetic lengths $R_H \sim (\gamma\alpha/\beta\chi H^2)^{1/2}$, where χ is the magnetic susceptibility of He^3. The classification depends on the relations between ξ, R_c, and R_H and on the length interval in which the characteristic dimensions of the investigated regions of the liquid are situated. For lack of space, we shall not describe this classification. It is obtained by the same method as in the case of two lengths.

We note in conclusion that the topology admits of the existence in the A phase of He^3 of particle-like solutions that have no singularities. By particle-like solutions we mean solutions characterized by a topological invariant that does not perturb the field of the degeneracy parameters at large distances from the particle, so that at infinity the field of the degeneracy parameters is homogeneous, i.e., all of infinity is mapped on a single point of the space R. The usual three-dimensional space R^3, all the points of which are equivalent at infinity, has the same topology as a three-dimensional sphere S^3 in four-dimensional space R^4 (exactly just as the plane R^2, all of the infinitely remote points of which are mapped on a single point via, e.g., a stereographic projection, is equivalent to a two-dimensional sphere S^2). Thus, the field of the degeneracy parameter specifies the mapping of S^3 in R. The homotopic classes of these mappings form the group $\pi_3(R)$. For the A phase, for example, in the region $r > R_c$ we have $\pi_3(\tilde{R}_A)=Z$ (see the Appendix). Therefore particle-like solutions are characterized by a whole-number invariant, which in this case can be written in the form

$$N=(m_3/2\pi\hbar)^2\int d^3r\, \mathbf{v}_s \operatorname{rot} \mathbf{v}_s. \tag{5.13}$$

From dimensionality considerations, the momentum and energy of such particles are of the order of

$$p\sim\rho_s\frac{\hbar r_0^2}{m_3}, \quad E\sim\rho_s\frac{\hbar^2}{m_3^2}r_0, \quad E\sim p^{1/2}, \tag{5.14}$$

where ρ_s is the density of the superfluid component, r_0 is the characteristic dimension of the region of space where the fields **V**, Δ', and Δ'' are inhomogeneous. The spectrum of such particles $E\sim p^{1/2}$ is reminiscent of the spectrum of vortex rings in He II, but the field of the vectors **V**, Δ', and Δ'' has no singularities anywhere; in addition, the spectrum can be anisotropic.

The particles can also have dimensions smaller than R_c, since $\pi_3(R_A)=Z+Z$ (see the Appendix). The smallest dimension of these formations is $\sim\xi$. If $r_0 \lesssim \xi$, then $F_{grad}\sim F_c$ in this region, and consequently the order parameter A_{ik} changes already not in the vicinity of R_A, but in the entire linear space R^{18}. Since π_3 is trivial for any linear space R, it follows that the topological invariant N ceases to exist. Thus, if the particle momentum decreases to $\rho_s\hbar\xi^2/m_3$, then the particle can go over continuously into a homogeneous state.

6. CHOLESTERIC LIQUID CRYSTALS

In cholesteric liquid crystals (CLC) the spatial distribution of the director **d** about an arbitrary point $\mathbf{r}_0$ takes the following form (see the review of Stephen and Straley[11]):

$$\mathbf{d}(\mathbf{r})=\mathbf{d}(\mathbf{r}_0)\cos\left\{\frac{2\pi}{L}\mathbf{t}(\mathbf{r}_0)(\mathbf{r}-\mathbf{r}_0)\right\}+[\mathbf{t}(\mathbf{r}_0)\mathbf{d}(\mathbf{r}_0)]\sin\left\{\frac{2\pi}{L}\mathbf{t}(\mathbf{r}_0)(\mathbf{r}-\mathbf{r}_0)\right\}, \tag{6.1}$$

i.e., at each point $\mathbf{r}_0$ of space there is specified an orthonormal basis **t**, **d**, $\mathbf{t}\times\mathbf{d}$ (analogous to **l**, Δ', Δ'' in the A phase of He^3); **t** is a unit vector of the helix axis, and L is the pitch of the helix. Let us find the space R for CLC. We note for this purpose that the region of variation of the triplet of vectors **t**, **d**, $\mathbf{t}\times\mathbf{d}$ is the three-dimensional group SO_3 of rotations of this triplet relative, e.g., the bases $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$. It is known that each three-dimensional rotation corresponds to two 2×2 complex unimodular matrices $\hat{U}$ and $-\hat{U}$. If we express the matrix $\hat{U}$ in the form

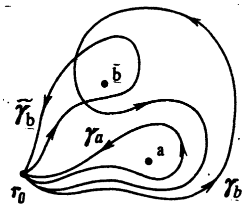

FIG. 8. The paths $\gamma_a, \gamma_b, \tilde{\gamma}_b = \gamma_a^{-1}\gamma_b\gamma_a$ surrounding the singular lines marked by the points a and b.

$$\hat{U}=\begin{pmatrix} x_1+ix_2 & x_3+ix_4 \\ -x_3+ix_4 & x_1-ix_2 \end{pmatrix}, \tag{6.2}$$

then, by virtue of unimodularity we have $x_1^2+x_2^2+x_3^2+x_4^2=1$, and consequently the region of variation of the matrix $\hat{U}$ is the three-dimensional sphere S^3 in four-dimensional space. Each three-dimensional rotation corresponds to two diametrically opposite points on the sphere S^3. Consequently $SO_3=S^3/Z_2$.

In CLC, however, the factoring of S^3 is not confined to the inversion group in four-dimensional space. Indeed, the states obtained from the initial one by making in (6.1) the substitutions

$$\mathbf{d}(\mathbf{r}_0)\to-\mathbf{d}(\mathbf{r}_0), \quad \mathbf{t}(\mathbf{r}_0)\to-\mathbf{t}(\mathbf{r}_0), \tag{6.3}$$

are equivalent to the initial state. Therefore each point on the sphere S^3 has already 7 equivalent points that are obtained by inversion and rotations in three-dimensional space through an angle π about the axes $\mathbf{t}$, $\mathbf{d}$, $\mathbf{t}\times\mathbf{d}$. Thus, for example, the point corresponding to the unit matrix $\hat{U}=\hat{\sigma}_0$ on the sphere S^3 has equivalent points corresponding to the matrices

$$-\hat{\sigma}_0, \quad \pm i\hat{\sigma}_x, \quad \pm i\hat{\sigma}_y, \quad \pm i\hat{\sigma}_z, \tag{6.4}$$

where $\hat{\sigma}_x$, $\hat{\sigma}_y$, $\hat{\sigma}_z$ are Pauli matrices. The matrices (6.4) together with the unit matrix form a group that is isomorphic to the group Q of the quaternion units, which consists of 8 elements ($1, -1, i, -i, j, -j, k, -k$), such that $ij=-ji=k$, $jk=-kj=i$, $ki=-ik=j$, $ii=kk=jj=-1$. It can be shown that any 8 equivalent points on S^3 also form a group isomorphic to Q. Therefore the space R for CLC is $R=S^3/Q$.

The homotopic groups of this space are $\pi_2(R)=0$ and $\pi_1(A, R)=Q$ (see the Appendix). CLC have no pointlike topologically stable singularities. The fundamental group $\pi_1(A, R)=Q$ is noncommutative. As a result, each linear singularity in CLC is characterized not by an element of the fundamental group, but by a class of conjugate elements of this group. Let us consider the two singular lines marked in Fig. 8 by the points a and b. We take the point $\mathbf{r}_0$ and surround a and b by contours γ_a, γ_b, and $\tilde{\gamma}_b$ that start out from this point. It can be verified that the contour $\tilde{\gamma}_b$ can be represented in the form

$$\tilde{\gamma}_b=\gamma_a^{-1}\gamma_b\gamma_a.$$

If γ_a is mapped into element a of the group $\pi_1(R)$ and γ_b into element b, then $\tilde{\gamma}_b$ is mapped into the element $\tilde{b}=a^{-1}ba$. Let $a=i$ and $b=j$; then $\tilde{b}=-j$. This means that the singularity b is characterized by two elements, j and $-j$, i.e., by the class $\{j, -j\}$. There are five such classes:

$$e=\{1\}, \quad e_1=\{-1\}, \quad a=\{i, -i\}, \quad b=\{j, -j\}, \quad c=\{k, -k\}.$$

When the singularities coalesce the classes are multiplied. We present the results of the multiplication

	e	e_1	a	b	c
e	e	e_1	a	b	c
e_1	e_1	e	a	b	c
a	a	a	e, e_1	c	b
b	b	b	c	e, e_1	a
c	c	c	b	a	e, e_1

When a is multiplied by a we obtain either an element of the class e or an element of the class e_1. An analogous result is obtained for bb and cc. This means that in the foregoing three cases, and only in these cases, the type of the resultant singularity depends on the coalescence path, in analogy with coalescence of singular points in NLC (see Fig. 9).

Kleman and Friedel[12] (see also[11]) have presented a classification of the disclinations in CLC. According to this classification there are three types of disclinations: $\chi^{(m)}$, $\lambda^{(m)}$, $\tau^{(m)}$, where m is the Frank index of the disclination. We write out the distribution of these singularities among the classes of the elements of group Q:

e: $\chi^{(4k)}$, $\lambda^{(4k)}$, $\tau^{(4k)}$;
e_1: $\chi^{(4k+2)}$, $\lambda^{(4k+2)}$, $\tau^{(4k+2)}$;
a: $\chi^{(2k+1)}$,
b: $\lambda^{(2k+1)}$,
c: $\tau^{(2k+1)}$.

We recall that within each class the disclinations can continuously go over into one another. The question of the local stability of the disclinations within each class is no longer homotopic and calls for an examination of the functional of the energy.

A few words now concern smectic liquid crystals and ordinary crystals. The states of an ordinary crystal are degenerate with respect to translation through an arbitrary vector $\mathbf{u}$ that runs through the linear space R^3, and with respect to the three-dimensional rotations defined in the space SO_3. In the space $R^3\times SO_3$, each point corresponds to a set of equivalent points that transform the crystal into a state identical with the initial one. This set forms a subgroup G of the a crystal space group whose elements contain no inversion. Therefore for a crystal the space is $R=(R^3\times SO_3)/G$. The homotopic group is $\pi_2(R)=0$, so that there are no point singular-

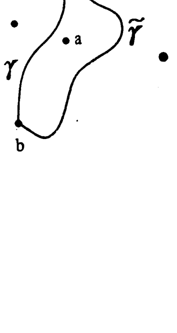

FIG. 9. Coalescence of singular lines in a cholesteric liquid crystal. The results of the coalescence is $b\cdot b=e_1$ along the path γ and $b\cdot b=e$ along the path $\tilde{\gamma}$.

ities of topological character in the crystals. The fundamental group $\pi_1(R)$ depends on the form of G and admits, besides dislocations, of the existence of disclinations. These disclinations, however, have an energy proportional to the volume, which is the result of the requirement that the distances between the crystal planes be equal; they can exist only in very soft crystals. The homotopic classification of the dislocations yields nothing new in comparison with the known classification in accordance with the Burgers vector. From the point of view of homotopic topology, edge and screw dislocations, characterized by an identical Burgers vectors belong to one homotopic class.

A smectic liquid crystal constitutes a system of equidistant surfaces separated from one another by atomic distances. These surfaces can bend, so that in smectic liquid crystals there are no dislocations. However, the requirement that the distances between layers be constant makes impossible many continuous deformations and therefore limits the applicability of the homotopic classification. In CLC the requirement that the pitch L of the helix be constant is not so stringent, since L is much larger than the dimension of the molecules and can vary slowly from point to point.

In conclusion, the authors thank S. P. Novikov, O. I. Bogoyavlenskiĭ, M. I. Monastyrskiĭ, and V. L. Golo for valuable consultations on topology, and also I. E. Dzyaloshinskiĭ for interesting discussions of the questions touched upon in the paper.

APPENDIX

The spaces R used in this paper have the general form P/G, where G is a discrete group, for example Z_2 or the group of quaternion units Q; $P=R_1\times R_2$, where R_1 and R_2 are spaces with known homotopic groups. In the calculation of the homotopic groups of the space R of this type it is necessary to use certain very simple rules which make it possible to express $\pi_n(R)$ in terms of the homotopic groups R_1 and R_2. First,

$$\pi_n(P)=\pi_n(R_1)+\pi_n(R_2). \tag{A.1}$$

Second, the sequence of homomorphisms of the groups

$$\rightarrow\pi_n(G)\rightarrow\pi_n(P)\rightarrow\pi_n(P/G)\rightarrow\pi_{n-1}(G)\rightarrow\ldots \tag{A.2}$$

is exact, i.e., for each triplet of successive groups from this sequence $G_1\rightarrow G_2\rightarrow G_3$ the transform of the homomorphism $G_1\rightarrow G_2$ is the kernel of the homomorphism $G_2\rightarrow G_3$. Third, π_0 from a connected space is trivial, and

$$\pi_0(G)=G. \tag{A.3}$$

1. Nematic liquid crystal

The spaces $R=S^2/Z_2$. It is known that $\pi_2(S^2)=Z$ and $\pi_1(S^2)=\pi_0(S^2)=0$. To find $\pi_1(R)$ and $\pi_2(R)$ we write down the exact sequence (A.2):

$$\pi_2(Z_2)\rightarrow\pi_2(S^2)\rightarrow\pi_2(S^2/Z_2)\rightarrow\pi_1(Z_2)\rightarrow$$
$$\rightarrow\pi_1(S^2)\rightarrow\pi_1(S^2/Z_2)\rightarrow\pi_0(Z_2)\rightarrow\pi_0(S^2),$$

or, substituting the known homotopic group, we obtain

$$0\rightarrow Z\rightarrow\pi_2(S^2/Z_2)\rightarrow 0\rightarrow 0\rightarrow\pi_1(S^2/Z_2)\rightarrow Z_2\rightarrow 0.$$

Using the definition of the exact sequence, we obtain

$$\pi_1(S^2/Z_2)=Z_2,\quad \pi_2(S^2/Z_2)=Z. \tag{A.4}$$

2. Cholesteric liquid crystals

The space is $R=S^3/Q$. It is known that $\pi_0(S^3)=\pi_1(S^3)=\pi_2(S^3)=0$ and $\pi_3(S^3)=Z$; using the exact sequence, we obtain

$$\pi_1(S^3/Q)=Q,\quad \pi_2=(S^3/Q)=0,\quad \pi_3(S^3/Q)=Z. \tag{A.5}$$

3. The B phase of He[3]

The space is $R_B=S^1\times SO_3$. We first obtain $\pi_n(SO_3)$. For this purpose we use the fact that $SO_3=S^3/Z_2$ (see Sec. 6) and then we have in analogy with (A.5) $\pi_1(SO_3)=Z_2$, $\pi_2(SO_3)=0$, $\pi_3(SO_3)=Z$. Using (A.1) we obtain

$$\begin{gathered}\pi_2(R_B)=\pi_2(S^1)+\pi_2(SO_3)=0,\\ \pi_1(R_B)=\pi_1(S^1)+\pi_1(SO_3)=Z+Z_2,\quad \pi_3(R_B)=Z.\end{gathered} \tag{A.6}$$

4. The A phase of He[3]

The space is $R_A=(S^2\times SO_3)/Z_2$. We have

$$\begin{gathered}\pi_2((S^2\times SO_3)/Z_2)=\pi_2(S^2\times SO_3)=\pi_2(S^2)+\pi_2(SO_3)=Z,\\ \pi_3(R_A)=\pi_3(S^2)+\pi_3(SO_3)=Z+Z,\quad \text{since}\quad \pi_3(S^2)=Z.\end{gathered} \tag{A.7}$$

To calculate $\pi_1(R_A)$ we make the exact sequence

$$\pi_1(Z_2)\rightarrow\pi_1(S^2\times SO_3)\rightarrow\pi_1(R_A)\rightarrow\pi_0(Z_2)\rightarrow\pi_0(S^2\times SO_3)$$

or, substituting in the known homotopic groups, we obtain

$$0\rightarrow Z_2\rightarrow\pi_1(R_A)\rightarrow Z_2\rightarrow 0.$$

It follows therefore that either $\pi_1(R_A)=Z_4$ or $\pi_1(R_A)=Z_2+Z_2$. It can be shown that for the A phase the first possibility is realized.

The space is $\tilde{R}_A=O_3=SO_3\times Z_2$. We have

$$\pi_0(\tilde{R}_A)=Z_2,\quad \pi_1(\tilde{R}_A)=Z_2,\quad \pi_2(\tilde{R}_A)=0,\quad \pi_3(\tilde{R}_A)=Z. \tag{A.8}$$

[1)] This circumstance was called to our attention by I. E. Dzyaloshinskiĭ.

[1] M. I. Monastyrskiĭ and A. M. Perelomov, Pis'ma Zh. Eksp. Teor. Fiz. 21, 94 (1975) [JETP Lett. 21, 43 (1975)]; M. I. Monastyrsky and A. M. Perelomov, Preprint, Institute for Theoretical and Experimental Physics, 56, Moscow, 1974.
[2] D. Huismoller, Stratified Spaces (Russ. Transl.) Mir, 1970. E. Spanier, Algebraic Topology, McGraw, 1966 (Russ. transl. Mir, 1971).
[3] G. E. Volovik and V. P. Mineev, Pis'ma Zh. Eksp. Teor. Fiz. 24, 605 (1976) [JETP Lett. 24, 595 (1976)].
[4] S. I. Anisimov and I. E. Dzyaloshinskiĭ, Zh. Eksp. Teor. Fiz. 63, 1460 (1972) [Sov. Phys. JETP 36, 774 (1973)].
[5] A. J. Leggett, Rev. Mod. Phys. 47, 331 (1975).
[6] V. R. Chechetkin, Zh. Eksp. Teor. Fiz. 71, 1463 (1976)

[Sov. Phys. JETP 44, 766 (1977)].
[7]P. G. de Gennes, Phys. Lett. 44A, 271 (1973).
[8]F. Fishman and I. A. Privorotskiĭ, J. Low Temp. Phys. 25, 225 (1976).
[9]S. Blaha, Phys. Rev. Lett. 36, 874 (1976).
[10]G. E. Volovik and V. P. Mineev, Pis'ma Zh. Eksp. Teor. Fiz. 23, 647 (1976) [JETP Lett. 23, 593 (1976)].
[11]M. J. Stephen and J. P. Straley, Rev. Mod. Phys. 46, 617 (1974).
[12]M. Kleman and J. Friedel, J. Phys. (Paris) 30, Suppl. C4, 43 (1969).

Translated by J. G. Adashko

On absorption of sound in ferrodielectrics

V. S. Lutovinov, O. A. Ol'khov, and S. P. Semin

Institute of Chemical Physics, USSR Academy of Sciences
(Submitted December 27, 1976)
Zh. Eksp. Teor. Fiz. 72, 2275–2285 (June 1977)

Expressions are found for the sound attenuation due to interaction between the sound and magnons at temperatures $T_0 < T < \Theta_0^2/\Theta_c (T_0 = \Theta_c (\mu M_0/\Theta_c)^{4/7})$ and frequencies $\tau_{m\text{-}ph}^{-1}$, $\tau_{ph\text{-}m}^{-1} < \omega < \tau_m^{-1}$($\tau_m$, $\tau_{m\text{-}ph}$, and $\tau_{ph\text{-}m}$ are the magnon-magnon, magnon-phonon and phonon-magnon collision times). In a broad frequency range, the attenuation exceeds that of sound due to anharmonism. It is shown that the method proposed by Akhiezer for calculation of sound attenuation is valid over a broader frequency range than was previously assumed.

PACS numbers: 43.35.Rw

1. INTRODUCTION

In an ideal ferrodielectric at temperatures $T \ll \Theta_c$ (Θ_c is the Curie temperature) sound absorption is due to its interaction with magnons and phonons and depends essentially on the relation between the frequency of the sound wave ω and the mean collision times: magnon-magon τ_m, phonon-phonon τ_{ph}, magnon-phonon τ_{m-ph} and phonon-magnon τ_{ph-m} (in a ferrodielectric, over a wide temperature ranges, $\tau_m \ll \tau_{ph-m}, \tau_{m-ph}$).[1] In the calculation of sound absorption, as a rule, two approaches are employed, the choice of which is also determined by the frequency interval under investigation.

At high frequencies, the sound absorption is usually represented as the result of the collisions of a sound quantum with phonons and magnons of the crystal. Such phonon-phonon damping at $\tau_{ph}^{-1} \ll \omega$, due to triple anharmonism, is determined for transverse sound in second order perturbation theory,[2] and for longitudinal sound, by the method of account of the anharmonism in all orders of perturbation theory[3] (the latter corresponds to account of the lifetime of the interacting phonons[4]). Sound absorption due to interaction with magnons at $\tau_m^{-1} \ll \omega$ is also considered in second order perturbation theory.[5,6] It was shown in Ref. 7 that at temperatures $T \ll \Theta_0^2/\Theta_c$ (Θ_0 is the Debye temperature), it is not sufficient to limit ourselves to second order for the calculation of phonon-magnon damping, rather it is necessary to take into account the contribution from fourth order perturbation theory. In this case, it turns out that the considered contribution can appreciably exceed the phonon-phonon damping.

At low frequencies, the sound absorption is usually calculated by the Akhiezer method.[8] In this case, the sound wave is considered as an external field, which produces a departure from equilibrium in the gas of magnons and phonons. Knowing the change in the particle distribution function under the action of the sound field, we can determine the change in the entropy of the gas and thus calculate the dissipation of energy of the sound wave. The phonon-phonon damping in dielectrics at $\omega \ll \tau_{ph}^{-1}$ was calculated by this method.[8] In what follows, the indicated method for phonon-phonon damping was developed in other researches.[9,10] Phonon-magnon damping of sound in ferrodielectrics was considered by the same method for the frequencies $\omega \ll \tau_m^{-1}$ at low temperatures $T \ll T_0 = \Theta_c(\mu M_0/\Theta_c)^{4/7}$ (μ is the Bohr magneton, M_0 the magnetization saturation), when the equilibrium in the magnon gas is established through dipole-dipole processes.[5,11]

In the present work, we consider the attenuation of sound in a ferrodielectric due to interaction with magnons at temperatures $T_0 \ll T$, when it is necessary to take exchange scattering into account, and at frequencies $\tau_{ph-m}^{-1}, \tau_{m-ph}^{-1} \ll \omega \ll \tau_m^{-1}$. The sound attenuation in the given frequency range was considered previously in the work of Kaganov and Chikvashvili,[5] who neglected the contribution to the damping from the anisotropic part of the phonon-magnon interaction and, in addition, in contrast to the case considered by us, assumed that $T_0 \sim \Theta_0^2/\Theta_c$.

As will be shown, account of the anisotropy has a significant effect on the results. It is also important here that the basic contribution to the damping is made by the interaction of the sound with subthermal intermediate magnons, while the phonon-magnon damping at $T \ll T_0$ and the phonon-phonon damping due to anharmo-

 0038-5646/77/4506-1196$02.40