Interaction between first and second sound in He^3-He^4 solutions

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(Submitted December 20, 1976)
Zh. Eksp. Teor. Fiz. 72, 2224–2229 (June 1977)

Formulas are derived for the vertices describing the interaction between first and second sound in He^3-He^4 solutions in the hydrodynamic approximation by the Hamiltonian technique. Parametric excitation of second sound by first is analyzed.

PACS numbers: 67.60.–g

The transformation of first sound into second in HeII was first considered by Pushkina, Rudenko and Khokhlov.[1,2] However, Pokrovskiĭ and Khalatnikov have shown[3] that consideration of thermal broadening, which was neglected by the authors of Refs. 1 and 2, is essential and, on the basis of the Hamiltonian technique developed by them for the description of hydrodynamic phenomena in a quantum liquid,[4] Pokrovskiĭ and Khalatnikov investigated the interaction of first and second sound in He^4, taking into account both decay and Cerenkov processes. The purposes of the present work is the consideration of the interaction of first and second sound in superfluid He^3-He^4 solutions on the basis of the Hamiltonian technique.

In comparison with pure He^4, a new characteristic appears in the He^3-He^4 solutions—N, the number of particles of He^3 per unit volume, on which the thermodynamic energy density ε depends, along with the entropy density s and the mass density ρ. The differential is of the following form:

$$d\varepsilon=Td s+\zeta dN+\mu d\rho+(\mathbf{v}_n-\mathbf{v}_s)\,d\mathbf{j}, \tag{1}$$

where T is the temperature, ξ is the chemical potential of the He^3 particles, μ is the "ordinary" chemical potential, $\mathbf{v}_n$ is the velocity of normal flow, $\mathbf{v}_s$ is the velocity of superfluid flow, $\mathbf{j}$ is the normal momentum density.

The Hamiltonian is specified in the form (obtained by a Galilean transformation to the laboratory system of coordinates)[3,4]

$$H=\int d^3x\,({}^1/_2\rho v_s^2+\mathbf{j}\mathbf{v}_s+\varepsilon). \tag{2}$$

This Hamiltonian takes on the usual meaning upon the substitution[1)]

$$\mathbf{v}_s=\nabla\alpha,\quad \mathbf{j}=s\nabla\beta+N\nabla\xi,$$

where (ρ,α), (s,β) and (N,ξ) are pairs of canonically conjugate variables.

We write down the equations which are given by the Hamiltonian (2):

$$\frac{\partial\rho}{\partial t}=-\nabla\mathbf{I},\quad \frac{\partial s}{\partial t}=-\nabla s\mathbf{v}_n,\quad \frac{\partial N}{\partial t}=-\nabla N\mathbf{v}_n,$$
$$\frac{\partial\alpha}{\partial t}=-\mu-{}^1/_2v_s^2,\quad \frac{\partial\beta}{\partial t}=-T-\mathbf{v}_n\nabla\beta,\quad \frac{\partial\xi}{\partial t}=-\zeta-\mathbf{v}_n\nabla\xi. \tag{3}$$

The first three equations play the role of the laws of conservation of mass entropy, and number of He^3 particles; the pulse density $\mathbf{I}=\mathbf{j}+\rho\mathbf{v}_s$ is, as usual, composed of normal and superfluid parts. The system (3) is equivalent to a system of hydrodynamic equations for the superfluid He^3-He^4 solutions, which can be found in the literature.[5,6]

We note that in the expressions that have been written down and in all subsequent equations, the quantities s and N, T and ζ, β and ξ enter symmetrically; therefore it makes sense to introduce unified vector symbols for these pairs: $\mathbf{s}(s,N)$, $\boldsymbol{\beta}=(\beta,\xi)$, $\mathbf{T}=(T,\zeta)$, which we shall use in what follows. This method of writing down the quantities is convenient also in that all the formulas turn out to be suitable for the case of many-component solutions if we mean by $\mathbf{s}$ the quantities s, N_1, N_2 . . . .

Similar to Pokrovskiĭ and Khalatnikov,[3,4] we can obtain expressions for the momentum flux density and the pressure (see Refs. 5, 6):

$$p=\mathbf{sT}+\rho\mu+(\mathbf{v}_n-\mathbf{v}_s)\,\mathbf{j}-\varepsilon. \tag{4}$$

All the quantities in the sound wave oscillate about equilibrium values, which we shall designate in the following by the subscript 0. The additions to the equilibrium values will be denoted by δ. Linearizing the system (3), we obtain

$$\frac{\partial\delta\rho}{\partial t}=-\nabla^2(\mathbf{s}_0\delta\boldsymbol{\beta}+\rho_0\delta\alpha),$$
$$\frac{\partial\delta\mathbf{s}}{\partial t}=-\mathbf{s}_0\nabla^2\left(\frac{1}{\rho_n}\mathbf{s}_0\delta\boldsymbol{\beta}+\delta\alpha\right), \tag{5}$$
$$\frac{\partial\delta\alpha}{\partial t}=-\delta\frac{\partial\varepsilon}{\partial\rho},\quad \frac{\partial\delta\boldsymbol{\beta}}{\partial t}=-\delta\frac{\partial\varepsilon}{\partial\mathbf{s}},$$

where the normal density ρ_n is introduced such that $\mathbf{j}=\rho_n(\mathbf{v}_n-\mathbf{v}_s)$. It is seen from these equations that $\delta\mathbf{s}\propto\mathbf{s}_0$; therefore, we can introduce ν so that $\delta\mathbf{s}=\mathbf{s}_0\nu$; we also introduce $\psi=\mathbf{s}_0\delta\boldsymbol{\beta}$, $\varphi=\rho_0\delta\alpha$, and $\delta\rho=\rho_0\eta$. We note that the pairs (ν,ψ) and (η,φ) are canonically conjugate variables. In these terms, the set of equations (5) for the plane wave take the form

$$\partial\eta_k/\partial t=k^2(\psi_k+\varphi_k),\quad \partial\nu_k/\partial t=k^2((1+\Gamma)\psi_k+\varphi_k),$$
$$\partial\varphi_k/\partial t=-\partial_\rho\partial_s\varepsilon\nu_k-\partial_\rho^2\varepsilon\eta_k,\quad \partial\psi_k/\partial t=-\partial_s^2\varepsilon\nu_k-\partial_\rho\partial_s\varepsilon\eta_k, \tag{6}$$

where $\Gamma=\rho_s/\rho_n$ and the notation $\partial_\rho=\rho_0(\partial/\partial\rho)$, $\partial_s=\mathbf{s}_0(\partial/\partial\mathbf{s})$ and $\partial_\rho^2=\rho_0^2(\partial/\partial\rho)^2$ is introduced.

 0038-5646/77/4506-1169$02.40

A dispersion equation is obtained from the set (6), having the form

$$(\rho c^2)^2-\rho c^2(\partial_\rho^2\varepsilon+2\partial_\rho\partial_s\varepsilon+(1+\Gamma)\partial_s^2\varepsilon)-\Gamma((\partial_\rho\partial_s\varepsilon)^2-\partial_\rho^2\varepsilon\partial_s^2\varepsilon)=0. \tag{7}$$

(This equation can be found, in other variables, in Refs. 5, 6.)

For the velocities of first (c_1) and second (c_2) sound, we get from (7)

$$\begin{gathered}\rho(c_1^2+c_2^2)=(\partial_\rho+\partial_s)p+\Gamma\,\partial_s^2\varepsilon,\\ \rho^2c_1^2c_2^2=\Gamma((\partial_\rho+\partial_s)p\,\partial_s^2\varepsilon-(\partial_s p)^2).\end{gathered} \tag{8}$$

We express the Fourier components of the quantities which describe the wave in terms of the classical analogs of the annihilation operators of phonons of first (a) and second (b) sound. We have the relations

$$\{a_k,\ a_{k'}^*\}=i\delta_{kk'},\quad \{b_k,\ b_{k'}^*\}=i\delta_{kk'}, \tag{9}$$

where $\{\ldots\}$ are Poisson brackets. We seek η in the form

$$\eta_k=\partial_s pA(a_k+a_{-k}^*)+\partial_s pB(b_k+b_{-k}^*). \tag{10}$$

From the system (6), we can obtain

$$\begin{gathered}\nu_k=(\rho c_1^2-\partial_\rho p)A(a_k+a_{-k}^*)+(\rho c_2^2-\partial_\rho p)B(b_k+b_{-k}^*),\\ ik\varphi_k=\rho c_1(\partial_\rho\partial_s\varepsilon+\rho c_2^2/\Gamma)A(a_k-a_{-k}^*)+\rho c_2(\partial_\rho\partial_s\varepsilon+\rho c_1^2/\Gamma)B(b_k-b_{-k}^*),\\ ik\psi_k=\rho c_1(\partial_s^2\varepsilon-\rho c_2^2/\Gamma)A(a_k-a_{-k}^*)+\rho c_2(\partial_s^2\varepsilon-\rho c_1^2/\Gamma)B(b_k-b_{-k}^*).\end{gathered} \tag{11}$$

Taking it into account that the pairs (η,φ) and (ν,ψ) are canonically conjugate variables, and using (9), we find

$$\begin{gathered}A^2=\frac{kc_1}{2\rho^2(\partial_s p)^2(c_1^2-c_2^2)}\left(\rho-\Gamma\frac{\partial_s^2\varepsilon}{c_1^2}\right),\\ B^2=\frac{kc_2}{2\rho^2(\partial_s p)^2(c_1^2-c_2^2)}\left(\Gamma\frac{\partial_s^2\varepsilon}{c_2^2}-\rho\right).\end{gathered} \tag{12}$$

In order to consider the interaction processes of first and second sound, we expand the Hamiltonian to the third power in the small increments of the equilibrium values:

$$H^{(3)}=\int d^3x\left[\frac{1}{2\rho}\eta(\nabla\varphi)^2+\frac{\nu}{\rho}\nabla\varphi\,\nabla\psi\right.$$
$$\left.+\frac{1}{\rho_n}\nu(\nabla\psi)^2+\frac{1}{2}D\frac{1}{\rho_n}(\nabla\psi)^2+\frac{1}{6}D^3\varepsilon\right], \tag{13}$$
$$D=\eta\partial_\rho+\nu\partial_s. \tag{14}$$

By separating the coefficients for the corresponding products of a and b, we can obtain from (13) the vertices corresponding to the processes of interaction of first and second sound.[6] Having this in mind, we shall consider the decay of a wave of first sound into two waves of second sound—we shall designate the corresponding vertex as $U(\omega,\theta)$ (ω is the frequency of the wave of first sound, θ is the angle between the wave vectors of the wave of first sound and one of the wave vectors of the waves of second sound), and also the Cerenkov process of emission of a wave of second sound by a wave of first sound—we designate the corresponding vertex as $V(\omega,\chi)$ (χ is the angle between the wave vectors of the incident and scattered waves of first sound, ω is their frequency).

In the following, we shall also use as variables the specific density $\sigma=s/\rho$, $c=m_3N/\rho$; we denote, corresponding, $\boldsymbol{\sigma}=(\sigma,c)$, $D_0=\boldsymbol{\sigma}_0(\partial/\partial\boldsymbol{\sigma})$, $SD_\rho=\rho_0(\partial/\partial\rho)$ (at constant $\boldsymbol{\sigma}$). In the case considered, there are two characteristic parameters: Γ and $\gamma=D_0p/D_\rho p$. The first parameter, as is seen from the definition, characterizes the value of the concentration or temperature dependence of the thermodynamic variables. For them we have the expression

$$\gamma=-\frac{\sigma}{\rho}\left(\frac{\partial\rho}{\partial\sigma}\right)_{c,p}-\frac{c}{\rho}\left(\frac{\partial\rho}{\partial c}\right)_{\sigma,p}. \tag{15}$$

We consider the case $\Gamma\gamma\ll 1$, which occurs in weak solutions of He^3 and He^4, practically throughout the temperature range up to the λ point. Near T_λ, this smallness is guaranteed by the parameter Γ, at very low temperatures—by the parameter γ. For the velocities, the estimate $(c_2/c_1)^2\sim\Gamma\gamma$ holds, and we can obtain from (8)

$$\rho c_1^2=D_\rho p,\quad \rho c_2^2=\Gamma(D_\sigma^2\varepsilon-\gamma D_\sigma p). \tag{16}$$

Using the formulas (10)–(12), we find

$$\begin{gathered}\eta_k=A_1(a_k+a_{-k}^*)+\gamma B_1(b_k+b_{-k}^*),\\ \nu_k=A_1(a_k+a_{-k}^*)+(\gamma-1)B_1(b_k+b_{-k}^*),\\ ik\varphi_k=\rho c_1(1-\gamma)A_1(a_k-a_{-k}^*)+\Gamma^{-1}\rho c_2B_1(b_k-b_{-k}^*),\\ ik\psi_k=\rho c_1\gamma A_1(a_k-a_{-k}^*)-\Gamma^{-1}\rho c_2B_1(b_k-b_{-k}^*);\end{gathered} \tag{17}$$

where

$$A_1=(\omega_1/2\rho c_1^2)^{1/2},\quad B_1=(\Gamma\omega_2/2\rho c_2^2)^{1/2}.$$

Upon substitution in (13), we obtain the expressions

$$\begin{gathered}U=\frac{\omega^{3/2}}{2^{5/2}\rho^{1/2}c_1}\left[-1-2\cos^2\theta-\frac{\rho}{\Gamma}D_\rho\frac{\Gamma}{\rho}+\frac{\Gamma}{\rho c_2^2}((D_\sigma-\gamma D_\rho)^2p+2\gamma D_\sigma p)\right],\\ V=\frac{\omega^{3/2}\Gamma^{1/2}\gamma}{2\rho^{1/2}c_2}\left[\frac{c_2}{c_1}\sin\frac{\chi}{2}\right]^{1/2}\left(1-\cos\chi-D_\rho\ln\frac{D_\sigma p}{D_\rho p}\right).\end{gathered} \tag{18}$$

It is seen from these expressions that the Cerenkov vertex V has smallness of the order of $(\gamma c_2/c_1)^{1/2}$ in comparison with the decay vertex U.

Second sound in weak He^3-He^4 solutions in the temperature range 0–0.6 K can be considered as the excitation of a gas of impurity quasiparticles, the mass of which we shall denote as m_*. The energy density ε is composed of the principal part $\varepsilon(\varphi)$ and the correction u due to the gas of quasiparticles.

We shall assume that the solution is degenerate. Then the formula $c_2=v_F/\sqrt{3}$ is valid, so that $(c_2/c_1)^2\ll 1$, i.e., the formulas given above are applicable to weak He^3-He^4 solutions at low temperatures, in contrast with the pure He^4, in which, at low temperatures, $c_2=c_1/\sqrt{3}$. The smallness parameter in weak degenerate solutions is the concentration c, in particular, $\Gamma\sim 1/c$, $c_2/c_1\sim c^{1/3}$, $\gamma\sim c^{5/3}$, whence $V\sim cU$. We find from Eq. (18), by using $u\propto N^{5/3}m_*$, that

$$U=\frac{\omega^{3/2}}{2^{3/2}\rho^{1/2}c_1}\left(\frac{1}{3}-\cos^2\theta\right),$$
$$V=\frac{\omega^{3/2}}{2\rho^{1/2}c_1}\left(\frac{m_*c}{m_3}\right)^{1/2}\left(\frac{c_2}{c_1}\right)^{3/2}\sin^{1/2}\frac{\chi}{2}\left[\left(\frac{D_\rho^2p}{D_\rho p}-\frac{2}{3}-\cos\chi\right)\times\right.$$

$$\times\left(1+\frac{3}{2}m_*D_\rho\frac{1}{m_*}\right)-\frac{3}{2}m_*D_\rho^2\frac{1}{m_*}-\frac{5}{2}m_*D_\rho\frac{1}{m_*}\Bigg]. \qquad (19)$$

We now assume that the impurity excitations can be considered to be a Boltzmann gas. Then

$$u\propto e^{2s/3N}N^{5/3},\quad c_2=(5T/3m_*)^{1/2},$$

whence $c_2/c_1 \lesssim \frac{1}{8}$. We also have

$$\Gamma=\frac{\rho}{m_*N},\quad \gamma=c\frac{m_*}{m_3}\left(\frac{c_2}{c_1}\right)^2,\quad V\sim c^{1/2}\left(\frac{c_2}{c_1}\right)^{3/2}U.$$

Substituting in (18), we find the formulas which differ from (19) only by the expressions in the square brackets:

$$[U]=\frac{1}{3}-\cos^2\theta-\frac{1}{2}m_*D_\rho\frac{1}{m_*},\quad [V]=\frac{D_\rho^2 p}{D_\rho p}-\frac{2}{3}-\cos\chi. \qquad (20)$$

We now consider the case of concentrated solutions of He^3 in He^4, where $\Gamma\gamma\sim 1$. As is known,[6] in concentrated solutions also $(c_2/c_1)^2 \ll 1$; therefore, $D_\sigma^2 \varepsilon D_\rho p/(D_\sigma p)^2-1\ll 1$ follows from (8). Keeping this condition in mind, and using (10)–(12), we find:

$$\begin{gathered}\eta_k=A_2(a_k+a_{-k}^*)+B_2(b_k+b_{-k}^*),\\ v_k=(1+\Gamma\gamma)A_2(a_k+a_{-k}^*)+(1-\gamma^{-1})B_2(b_k+b_{-k}^*),\\ ik\varphi_k=\rho c_1(1-\gamma)A_2(a_k-a_{-k}^*),\quad ik\psi_k=\rho c_1\gamma A_2(a_k-a_{-k}^*),\end{gathered} \qquad (21)$$

where

$$A_2=(0.5\omega_1 D_\rho p)^{1/2}/\rho c_1^2,\quad B_2=\gamma(0.5\omega_2\Gamma D_\rho p)^{1/2}/\rho c_1 c_2.$$

Substituting these expressions in (13), we find

$$\begin{gathered}U=\frac{\omega^{3/2}(D_\rho p)^{-1/2}\gamma^2\Gamma}{2^{5/2}\rho^3c_1^4c_2^2}(\partial_\rho+\partial_s+\Gamma\gamma\partial_s)(\partial_\rho+\partial_s-\gamma^{-1}\partial_s)^2\varepsilon,\\ V=\frac{\omega^{3/2}(D_\rho p)^{3/2}\gamma\Gamma^{1/2}}{2\rho^2c_1^3c_2}\left(\frac{c_2}{c_1}\sin\frac{\chi}{2}\right)^{1/2}\Bigg[\Bigg[\gamma^2\rho\left(D_\rho-\frac{D_\sigma}{\gamma}\right)\frac{1}{\rho_n}\\ +(\gamma-1)(1+\gamma(1+2\Gamma))\Bigg]\cos\chi+\frac{1}{\rho c_1^2}(\partial_\rho+\partial_s+\Gamma\gamma\partial_s)^2\left(\partial_\rho+\partial_s-\frac{\partial_s}{\gamma}\right)\varepsilon\Bigg].\end{gathered} \qquad (22)$$

We note that the decay vertex U is isotropic, while the vertex V has the smallness $(c_2/c_1)^{3/2}$ in relation to it.

In all the obtained formulas, the vertex V has the factor $\sin^{1/2}(\chi/2)$, just as in pure He^4, which is natural since it is connected only with the laws of conservation and smallness of c_2/c_1. Thus the Cerenkov process should dominate with little change in the wave vector of first sound.

The vertices U and V can be used for the description of the nonlinear process of excitation of second sound by first. If we consider a stationary process with the frequency of the exciting wave of first sound ω, then we have for the "contracted" amplitudes the equations for the decay process

$$\begin{gathered}c_1(\mathbf{n}\nabla+\alpha_1)a=-iUb_1b_2,\\ c_2(\mathbf{n}_1\nabla+\alpha_2)b_1=iUab_2^*,\quad c_2(\mathbf{n}_2\nabla+\alpha_2)b_2=iUab_1^*,\end{gathered} \qquad (23)$$

and the Cerenkov process

$$\begin{gathered}c_1(\mathbf{n}\nabla+\alpha_1)a=-iVa'b,\\ c_1(\mathbf{n}'\nabla+\alpha_1)a'=iVab^*,\quad c_2(\mathbf{n}_b\nabla+\alpha_2)b=iVaa'^*,\end{gathered} \qquad (24)$$

where n represents the unit vector along the direction of propagation of the corresponding waves, α_1 and α_2 are the attenuations of first and second sound.

On the basis of what has been said above on the smallness of the vertex V in comparison with the vertex U in terms of the parameter c_2/c_1, we can draw the conclusion that the decay process will predominate in above-threshold region. However, the quantity c_2/c_1 is numerically ~ 0.1, which is not very small; therefore, in specific cases a finer analysis is required. In concentrated solutions ρ, c_1 are close in value to the same quantities in weak solutions.[6] Therefore, the quantity U for weak solutions has, in comparison with its value in concentrated solutions, a smallness of the order of $(c_2/c_1)^2$, i.e., a linking of first and second sound in concentrated solutions is stronger.

The process of parametric excitation of second sound by first sound has the threshold $a_p=\alpha_2c_2/U$ in the decay channel and $a_r=(\alpha_1\alpha_2c_1c_2)^{1/2}V$ in the Cerenkov channel. To obtain the threshold energy flux density Q in the incident wave, we can use the expression $Q=c_1\omega|a|^2$.

The question arises as to which of the processes of excitation of second sound "turns on" first. For weak degenerate solutions, the expressions $\alpha_2=2\omega^2\eta_3/3\rho_nc_2^3$ (see Ref. 5, Eq. (67.11)). A similar expression can be used for estimate of α_1 with accuracy to replacement of ρ_n by ρ and c_2 by c_1 (see Ref. 6, Eq. (8.114)). Thus $\alpha_1(\omega)\sim c^2\alpha_2(\omega)$ jointly with the estimate $V\sim cU$ gives $a_r/a_p\sim c^{1/6}$, which is a small quantity only at very low temperatures. There are data in the literature[7] for the damping of second sound in weak solutions in the temperature range 0.2–1 K. Here $c_2/c_1\sim 0.1$, $\alpha_2/\alpha_1\sim 10^{11}$–$10^{15}$ (see Refs. 6 and 7); consequently, the threshold of excitation of the Cerenkov process will be lower than that of the decay process.

On the basis of the analysis given here, we can conclude that for the experimental study of parametric excitation of second sound by first,[2)] concentrations of He^3-He^4 are most suitable, since in them, on the one hand, there is strong linking of the sounds through the vertex, and on the other, there is less damping, which guarantees a lower excitation threshold.

In conclusion, the author thanks I. M. Khalatnikov for proposing the problem and useful discussions.

[1)] Generally speaking, the two independent functions β and ξ are not sufficient for the descripton of j (they would be sufficient in a three-component solution); it is therefore necessary to introduce the Clebsch variables. However, they are eliminated as in the paper of Prokrovskiĭ and Khalatnikov.[3]

[2)] Experimental researches devoted to this process are not known to the author.

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