else when the optical thickness of the medium is small (in the case of collinear observation direction). There are, however, also other possible experimental conditions. If the distance between the active particles of the medium are shorter than the radiation wavelength, then interference causes the coherent part of the scattered radiation, corresponding precisely to the narrow undisplaced line, to propagate collinearly with the exciting radiation. At a large optical thickness the following will take place under these conditions: The external radiation will attenuate as it passes through the medium and will be converted into scattered radiation. The scattered radiation, on the other hand, begins to act as an excitation source with increase of its intensity, i.e., it experiences secondary scattering. Since, however, the spectrum of the scattered radiation differs from the spectrum of the exciting radiation, then a change of the spectrum will take place as a result of the propagation effect. We hope to deal with this interesting question elsewhere.

The author thanks S. G. Rautian for useful discussions and a number of valuable remarks.

1)See also the literature cited in[17].
2)The formulation of the problem in[4] pertains to microwave transitions, i.e., it is assumed that $\Gamma_m = \Gamma$ and the relaxation transition $n \to m$ is additionally taken into account.

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Stationary spectra of high-frequency oscillations of a plasma in a magnetic field

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(Submitted September 14, 1976)
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We consider weak turbulence of Langmuir and electromagnetic waves, the source of which is a beam of relativistic electrons. We take into account the processes of induced scattering of the waves by the ions in the plasma and the damping of the waves due to Coulomb collisions. We find for the case of a weak magnetic field ($\omega_{He} < \omega_{pe}$) stationary turbulence spectra and we calculate the power transferred from the beam to the plasma. We show that the threshold heating power at which a condensation of energy into the long-wavelength part of the spectrum begins increases linearly with increasing magnetic field.

PACS numbers: 52.35.Hr, 52.35.Ra

1. INTRODUCTION

We consider in the present paper within the framework of weak turbulence theory[1-3] the non-linear stage of the instability of a beam of relativistic electrons in a plasma with a weak magnetic field ($\omega_{He} \ll \omega_{pe}$). We are dealing with the excitation by the beam of Langmuir oscillations (l) and of the restriction of their level due to induced scattering by ions. The role of the scattering consists in that it removes the oscillations from resonance with the beam electrons. This occurs, firstly, due to an increase in the phase velocity of the Langmuir oscillations when they are scattered by one another (ll process) and, secondly, due to the transformation of Langmuir into electromagnetic waves (lt process). Secondary waves arising in the scattering can be

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damped, transferring their energy to the plasma particles. We shall assume that their damping is caused by Coulomb collisions.

As a result of the processes enumerated here a stationary turbulence can be maintained in the beam-plasma system. Our aim is to find the turbulence spectrum and to evaluate the power of the plasma heating. The corresponding problem for a plasma without a magnetic field was solved in Refs. 4 and 5. We remind ourselves briefly of the results of those papers and explain why it is of interest to take a magnetic field into account.

If lt scattering is inhibited (e.g., due to a small optical depth) collisions are able to guarantee the dissipation of the whole of the energy given off by the beam only for the case where the instability growth rate γ is not too much larger than the collision frequency ν ($\gamma - \nu \lesssim \nu$). In the case where $\gamma \gg \nu$ the Langmuir oscillations condense into the long-wavelength part of the spectrum which leads to the conditions for the applicability of the weak turbulence approximation being violated and to the appearance of the modulational instability.[6] The development of the modulational instability can be accompanied by a strong heating of a small group of fast electrons ("growth of tails") and an increase of loss of energy from the plasma.[7,8] The threshold value of the heating power P for which such a dangerous situation arises is relatively low:

$$P_0 \sim \frac{\nu^2}{\omega_{pe}} nT \frac{M}{m} \frac{T}{mc^2}. \qquad (1)$$

It is therefore very desirable to find conditions under which $P \gg P_0$ while the modulational instability is nevertheless not present.

One possibility consists in suppressing the electromagnetic waves and to work in a regime where both types of scattering (ll and lt) are allowed.[1)] It was shown in Ref. 5 that in that case the condensation of energy into the long-wavelength part of the spectrum starts at $P \sim P_0 mc^2/T$ rather than at $P = P_0$. The conditions for the applicability of the weak turbulence approximation are then correspondingly weakened. Another possibility is the introduction of a magnetic field. A field changes the dispersion law for the waves and we shall see that this appreciably affects the conditions for the occurrence of the condensate.

In order that a wave excited by the beam reaches the condensate its frequency must be decreased due to scattering by a well-defined amount $\delta\omega$ before the wave is damped through collisions. In the case of a plasma without a magnetic field the following estimate holds for $\delta\omega$: $\delta\omega \sim \omega_{pe} T/mc^2$ (see Refs. 4 and 5). When the field increases, $\delta\omega$ increases: $\delta\omega \sim \omega_{pe} T/mc^2 + \omega_{He}$. Together with $\delta\omega$ the characteristic time for the transfer of waves from the region where they are excited to the condensate also increases. As a result energy dissipation through collisions is made easier and the formation of the condensate starts at a higher heating power than for a plasma without a magnetic field. Of course, the gain is most important when the "magnetic"

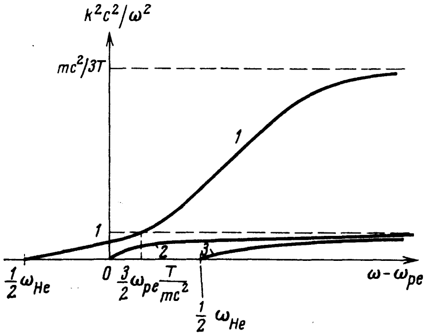

FIG. 1. Dispersion curves of the electronic oscillations of a plasma in a weak ($\omega_{He} \ll \omega_{pe}$) magnetic field. The angle between the direction of the wavevector and the magnetic field is different from zero; 1) slow extra-ordinary branch, 2) ordinary branch, 3) fast extra-ordinary branch (we use the terminology of Ref. 10).

contribution to $\delta\omega$ is large compared to the "thermal" one, i.e., when

$$\omega_{He} \gg \omega_{pe} T/mc^2. \qquad (2)$$

This is just the situation which we shall consider.

The present paper is essentially a continuation of Ref. 9 in which estimates were given for the role of a magnetic field for the case when there was no lt scattering. A difference will consist, firstly, in that we take into consideration all scattering channels and, secondly, that we obtain not only estimates, but also an exact solution of the problem. Section 3 of this paper is devoted to estimates; Secs. 4 and 5 to the exact solution.

2. BASIC EQUATIONS

We show in Fig. 1 typical dispersion curves for the waves of interest to us. The excitation of the waves, their damping, and the induced scattering by the ions in the plasma are described by the following set of equations for the occupation numbers $N_\lambda(\mathbf{k})$ (λ numbers the branches of the oscillations):

$$\frac{\partial}{\partial t} N_\lambda(\mathbf{k}) = \Gamma_\lambda(\mathbf{k}) N_\lambda(\mathbf{k}), \qquad (3)$$

$$\Gamma_\lambda(\mathbf{k}) = 2\gamma_\lambda(\mathbf{k}) + \sum_{\lambda'=1} \int A_{\lambda\lambda'}(\mathbf{k}\cdot\mathbf{k}') N_{\lambda'}(\mathbf{k}') d\mathbf{k}'. \qquad (4)$$

In a weak field the growth rates of the beam instability are very small for branches 2 and 3. We can therefore put

$$\gamma_2(\mathbf{k}) = \gamma_3(\mathbf{k}) = -\nu/2, \quad \gamma_1(\mathbf{k}) = \gamma(\mathbf{k}) - \nu/2,$$

where $\gamma(\mathbf{k})$ is the growth rate for branch 1 and ν of the collision frequency.

If the change in the wave frequency in each scattering process is much smaller than the width of the spectrum (and we assume in what follows that this is just the case) we can apply for the kernel $A_{\lambda\lambda'}$ the differential approximation

$$A_{\lambda\lambda'}(\mathbf{k};\mathbf{k}')=\frac{2\pi^2e^2}{mM}|\mathbf{S}(k;\lambda)\mathbf{S}^*(\mathbf{k}';\lambda')|^2\delta'\left[\frac{\omega_\lambda(\mathbf{k})-\omega_{\lambda'}(\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|}\right]. \quad (5)$$

Here $S(\mathbf{k},\lambda)$ is the polarization vector of the wave and the prime on the δ-function indicates differentiation with respect to its argument.

Bearing in mind the form of the kernel, it is convenient to characterize each wave by its frequency and the quantities $x=\cos\theta$ and φ (θ is the angle between the vectors $\mathbf{k}$ and $\mathbf{H}$, φ is the azimuthal angle). Moreover, it is expedient to change from variables with dimensions to dimensionless ones and to do this to make the following substitutions:

$$t\to\frac{2}{3}\frac{mc^2}{T}\frac{1}{\omega_{pe}}\tau,\quad \mathbf{k}\to\frac{\omega_{pe}}{c}\mathbf{k},\quad \omega-\omega_{pe}\to\frac{3}{2}\omega_{pe}\frac{T}{mc^2}f,$$

$$\gamma_\lambda\to\frac{3}{2}\omega_{pe}\frac{T}{mc^2}\gamma_\lambda,\quad N_\lambda k^2\,dk\,dx\to\frac{27}{8\pi^2}\frac{M}{m}\left(\frac{T}{mc^2}\right)^2\frac{nT}{\omega_{pe}}n_\lambda\,df\,dx.$$

For the sake of simplicity we consider in what follows only axially symmetric spectra, assuming also that the functions $n_\lambda(f;x)$ are independent of the sign of x. Of course, the growth rate must have the same symmetry. On the basis of the results of Ref. 4 we can say that in the stationary state the spectrum must have a jet character, i.e.,

$$n_\lambda(f;x)=\sum_{p=1,2\ldots}n_\lambda^{(p)}(f)\,\delta[x-x_\lambda^{(p)}(f)].$$

The jet parameters $n_\lambda^{(p)}(f)$ and $x_\lambda^{(p)}(f)$ are given by the equations

$$\Gamma_\lambda[f;x_\lambda^{(p)}(f)]=0, \quad (6)$$

$$\frac{\partial}{\partial x}\Gamma_\lambda(f;x)|_{x=x_\lambda^{(p)}(f)}=0,\quad x_\lambda^{(p)}(f)\neq\pm1, \quad (7)$$

and the condition for the instability of the spectrum is $\Gamma_\lambda(f;x)\leq 0$. An exception is the so-called degenerate spectrum. For it $\Gamma_\lambda(f;x)=0$.

In order to sketch in general outline the distribution of the waves over frequencies we turn to the balance equation for waves with frequency f. It is easily obtained from (3):

$$\frac{\partial}{\partial\tau}\sum_\lambda\int_{-1}^{1}n_\lambda(f;x)\,dx=\frac{\partial\Pi}{\partial f}+\sum_\lambda\int_{-1}^{1}2\gamma_\lambda(f;x)\,n_\lambda(f;x)\,dx.$$

Here $\Pi(f)$ is the flux of waves along the spectrum in the direction of low frequencies which is caused by scattering. The second term describes the change in the number of waves due to pumping and damping. It is convenient to consider it as the product of twice the number of waves and some effective growth rate γ_{eff}. We show the function $\gamma_{eff}(f)$ in Fig. 2a. In the resonance region it is positive; in the remaining frequency range $\gamma_{eff}\approx-\nu/2$. The lower limit and width of the resonance region are in Fig. 2 denoted by f_+ and Δf_+. If the waves are excited by a relativistic electron beam

$$f_+\approx1,\quad \Delta f_+\sim1+\beta^{-1},\quad \beta\equiv12\pi nT/H^2. \quad (8)$$

In a stationary state the flux $\Pi(f)$ increases in the resonance region with decreasing frequency, while in

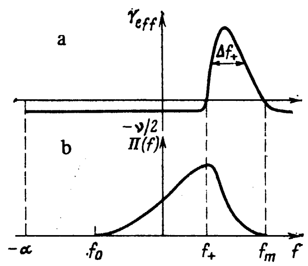

FIG. 2. Sketch of the dependence of the effective growth rate γ_{eff} and the flux $\Pi(f)$ of the number of waves along the spectrum on the dimensionless frequency f. We denote by $-\alpha$ the minimum frequency allowed by the dispersion law (see Eq. (9)).

the non-resonance region it decreases and when there is no condensation it vanishes for $f=f_0>-\alpha$ (see Fig. 2b). We shall assume in what follows that

$$1+\beta^{-1}\ll-f_0\ll\alpha\quad\left(\alpha\equiv\frac{mc^2}{3T}\frac{\omega_{He}}{\omega_{pe}}\right). \quad (9)$$

In this case the role of the magnetic field manifests itself fully and at the same time the solution of the problem turns out to be rather simple. The reason for this is that the spectrum for $-f_0\ll\alpha$ consists of oscillations with wavevectors which are of different orders of magnitude. The Langmuir (l) oscillations have the largest values of k. For them

$$k^2+\beta^{-1}(1-x^2)(1-k^{-2})=f, \quad (10)$$

where $k(f;x)\gg(\omega_{He}/\omega_{pe})^{1/2}$. The other waves (which we shall call electromagnetic t waves) correspond to much smaller values of $k(f;x)$. Taking into account the smallness of the wavevectors of the t waves and the non-potential corrections to the polarization vectors of the l waves we can perform in Eq. (5) the appropriate corrections and, in particular, we can completely neglect the interaction of t waves with one another. We finally get instead of (4):

$$\Gamma_l(f;\ x)=F_1'+x^2F_2'+k^2(f;\ x)[F_3'+x^2F_4']+2\gamma(f;\ x)-\nu,$$
$$\Gamma_t(f;\ S)=F_1'+S^2F_2'-\nu, \quad (11)$$

where[2)]

$$F_1(f)=\int_0^1k^2(f;x)(1-x^2)n^l(f;x)\,dx,\quad F_2(f)=\int_0^1k^2(f;x)(3x^2-1)n^l(f;x)\,dx, \quad (12)$$

$$F_3(f)=\int_0^1(1-x^2)n^l(f;x)\,dx+w_0^t(f),$$

$$F_4(f)=\int_0^1(3x^2-1)n^l(f;x)\,dx+2w_1^t(f)-w_0^t(f);$$

$$w_0^t(f)=\sum_\lambda\int_0^1(1-S^2)n_\lambda^t(f;x)\,dx,\quad w_1^t(f)=\sum_\lambda\int_0^1S^2n_\lambda^t(f;x)\,dx. \quad (13)$$

Here $k(f;x)$ is the wavevector of an l wave given by Eq. (10); S is the projection of the polarization vector of the t wave onto the direction of the magnetic field.

We note that the contribution of the t waves to the functions Γ_l and Γ_t is fully characterized by two quantities: w_0^t and w_1^t. These quantities are both non-vanishing only when the t spectrum is degenerate ($\Gamma_t(f; S) \equiv 0$). In a jet spectrum of t waves, on the other hand, there may be present those for which $S=0$, or those for which $S=1$.

The set of equations which is obtained when we substitute the functions Γ_l and Γ_t into Eqs. (6), (7) will be solved in Secs. 4 and 5.

3. ESTIMATES OF THE STATIONARY SPECTRUM PARAMETERS

When giving estimates we shall assume that inequality (9) is satisfied. The total width of the spectrum is then large compared to the width of the resonance region and approximately equal to $|f_0|$ (see Fig. 2). The characteristic value of $|f|$ in the spectrum is also of the order of $|f_0|$.

We denote by N^l the number of l waves per unit volume and by N^t the number of t waves:

$$N^l = \int df \int_{-1}^{1} n^l(f; x)\, dx, \quad N^t = 2\int df (w_0^t + w_1^t).$$

One can easily express the characteristic times for ll-, lt-, and tl-processes in the non-resonance region in terms of these quantities:

$$\tau^{ll} \sim \tau^{lt} \sim f_0^2/k^2 N^l, \quad \tau^{tl} \sim f_0^2/k^2 N^t.$$

Here k is a characteristic value of the wavenumber of the non-resonance l waves which is equal to $|\beta f_0|^{-1/2}$ (see (10)).

In the stationary state each of the quantities τ^{ll}, τ^{lt}, and τ^{tl} must equal the time for the damping of the waves, ν^{-1}. Hence we get the following relation:

$$N^l \sim N^t \sim \nu\beta |f_0|^3. \tag{14}$$

At the same time the characteristic time for the excitation of l waves must be equal to the time of their pumping from the resonance region due to scattering, τ_R. If we assume that the pumping is mainly caused by lt scattering we must give for an estimate of τ_R the number of t waves in the resonance frequency band. We assume that there are here per unit frequency range approximately just as many t waves as in the rest of the spectrum, that is, $N^t/|f_0|$. This will be confirmed by the exact solution of the problem. Assuming that the resonance region has a width $1+\beta^{-1}$ (see (8)) and that the wavevector in it is of the order of magnitude of unity we have

$$\tau_R \sim \frac{|f_0|}{N^t}\frac{1+\beta}{\beta}.$$

The inverse pumping time τ_R^{-1} is clearly the same as the beam instability growth rate γ, i.e.,

$$N^t \sim \gamma |f_0| (1+\beta)/\beta. \tag{15}$$

Combining Eqs. (14) and (15) we find

$$N^l \sim N^t \sim \gamma^{3/2}(1+\beta)^{3/2}/\nu^{1/2}\beta^2, \tag{16}$$

$$|f_0| \sim \gamma^{1/2}(1+\beta)^{1/2}/\nu^{1/2}\beta. \tag{17}$$

We now estimate the power of heating the plasma P and the number of l waves interacting with the beam, N_R^l. The power dissipated due to collisions differs from the total wave energy only by a factor ν, i.e., in dimensionless units

$$P \sim \nu^{1/2}\gamma^{3/2}\beta^{-2}(1+\beta)^{3/2}. \tag{18}$$

In the stationary state this quantity is equal to the power which the beam loses. The latter, on the other hand, equals, apart from a numerical factor, γN_R^l. Therefore

$$N_R^l \sim \nu^{1/2}\gamma^{1/2}\beta^{-2}(1+\beta)^{3/2}.$$

We note that when $\gamma \gg \nu$ the spectral density of the l waves in the resonance region turns out to be small compared to their average spectral density $N^l/|f_0|$. It is clear from Eq. (18) that the efficiency of plasma heating increases with increasing magnetic field. When $\beta \gg 1$ the power is proportional to the first power of the field strength, and for $\beta \ll 1$ to the fourth power. To show how the power depends on the other parameters we rewrite Eq. (18) in variables with dimensions:

$$P \sim \nu n T \frac{\gamma}{\omega_{pe}} \frac{M}{m} \frac{T}{mc^2} \left(\frac{\gamma}{\nu}\right)^{1/2} \frac{(1+\beta)^{3/2}}{\beta^2}. \tag{19}$$

For a comparison we remind ourselves that when there is no magnetic field the expression for P has the following form (see Ref. 5):

$$P \sim P_* \equiv \nu n T \frac{\gamma}{\omega_{pe}} \frac{M}{m} \frac{T}{mc^2}, \quad 1 \ll \frac{\gamma}{\nu} \ll \frac{mc^2}{T}.$$

The criterion for the applicability of Eq. (19) is given by inequalities (9). Substituting into these the exact expression for $|f_0|$ (see (17)) we get

$$(1+\beta)^2 \ll \frac{\gamma}{\nu}(1+\beta) \ll \beta^2\alpha^2, \quad \alpha \equiv \frac{mc^2}{3T}\frac{\omega_{He}}{\omega_{pe}}. \tag{20}$$

We note that according to condition (2) $\alpha \gg 1$. The left-hand inequality (20) means that $P \gg P_*$. If it is not satisfied the heating power will be close to P_*, and the correction connected with the magnetic field will be small. If, however, the right-hand inequality is violated, the width of the spectrum, formally found from (17) turns out to be larger than α and this means that there occurs an energy sink at the point $f = -\alpha$ (see Fig. 2). The maximum heating power, for which the sink will already be absent is equal to

$$P_{max} \sim \frac{\nu^2}{\omega_{pe}} n T \frac{M}{m} \frac{\omega_{He}}{\omega_{pe}} \frac{mc^2}{T}.$$

It is larger by a factor α then the corresponding quantity for a plasma without a magnetic field. The inclusion of the magnetic field thus enables us to increase substantially the power of plasma heating and appreciably widens the conditions under which the energy given off by the beam can be absorbed due to Coulomb collisions without the "growth of tails."

4. STATIONARY SPECTRA IN THE NON-RESONANCE REGION

The construction of a stationary spectrum reduces to finding different solutions of Eqs. (6), (7), the selection of stable solutions, and the joining up of them with one another. When applied to the non-resonance region, where $\gamma(f;x)=0$, this problem possesses a well-defined generality as there is in Eqs. (6), (7) no dependence on the actual form of the source of the waves. It thus has sense to give a complete classification of the "non-resonance" spectra and we shall do this in what follows.

It is convenient to start the classification, starting by considering the frequency range

$$-f \gg \beta^{-1}+\beta^{-1/2}, \tag{21}$$

in which the dispersion law (10) is close to a power-law one[3]:

$$k^2(f;\,x)=-(1-x^2)/\beta f. \tag{22}$$

Such a relation between k and f gives a self-similar spectrum as the quantities Γ_l and Γ_t (see (11)) turn out to be invariant under a replacement of f by cf, n^l by c^2n^l, where c is an arbitrary constant. In the self-similarity region Γ_l is, as function of x, a polynomial which has no more than two maxima in the interval $0\leqslant x\leqslant 1$. Hence it follows that the number of l jets in the non-degenerate spectrum is also not more than two (we are dealing here with the range $0\leqslant x\leqslant 1$). A simple sorting out of all possible variants of the arrangement of l and t jets shows that the total number of different solutions which can make up a self-similar spectrum is equal to six. Two of them are single-jet, three two-jet, and one solution is degenerate. We consider now each of these solutions separately and analyze the conditions of their joining up.

4.1 Jet of l waves moving across the field

This solution is noteworthy as it is realized near the lower limit of the spectrum f_0. Here there are no t waves, i.e., $w_0^t=w_1^t=0$ and it is convenient to write the spectral function of the l waves in the following form:

$$n^l(f;x)=\frac{\nu\beta f^2}{Z(f)}\delta(x-0). \tag{23}$$

Using the self-similarity of the spectrum we introduced instead of the intensity of the jet the function $Z(f)$ which does not change under a similarity transformation. We shall make similar substitutions also in the other cases. The function $Z(f)$ is determined from the equation $\Gamma_l(f;0)=0$ and the boundary condition $Z(f_0)=\infty$:

$$Z(f)=3[(f_0/f_1)^{1/2}-1]^{-1}. \tag{24}$$

Direct evaluation of Γ_l and Γ_t shows that the solution (23), (24) is stable only when $Z(f)\geqslant 1$, i.e., in the region

$$f\leqslant 4^{-1/3}f_0. \tag{25}$$

4.2. Jets of l and t waves polarized across the magnetic field

The spectral functions of the waves are determined by the conditions $\Gamma_l(f;0)=\Gamma_t(f;0)=0$ and have the following form:

$$n^l(f;x)=\frac{\nu(f-A)}{k^2(f;0)}\delta(x-0),\quad w_0{}^t(f)=B-\frac{\nu(f-A)}{k^2(f;0)}. \tag{26}$$

Here $k^2(f;0)$ is evaluated from Eq. (10) in which we must put $x=0$. The constants of integration A and B are determined from the condition that the solution (26) must join up with the jet (23) at its lowest limit. In the point of joining $w_0^t=0$ and $dw_0^t/df\geqslant 0$. This last condition is equivalent to the inequality $Z\leqslant 1$ which is the opposite of (25). Hence it follows that joining up is possible only when $Z=1$. Using this fact and the continuity of the spectral functions we have

$$A=2^{-2/3}f_0,\quad B=2^{-1/3}\nu\beta f_0^2.$$

The solution which we have found is stable in the whole of the non-resonance region up to its upper limit.

4.3. Jets of oblique l waves

In this spectrum

$$n^l(f;x)=\frac{\nu\beta f^2}{Z(f)}\delta(x-\sqrt{\eta(f)}),\quad w_0{}^t=w_1{}^t=0. \tag{27}$$

We obtain the equations for the functions $Z(f)$ and $\eta(f)$ directly from the conditions (6) and (7):

$$D(\eta)\frac{dZ}{d\ln|f|}=4Z^2(2-3\eta)+2ZD(\eta)+Z(1-\eta)\times[(3\eta-1)(3\eta^2-2\eta+1)+4\eta-4], \tag{28}$$

$$D(\eta)\frac{d\eta}{d\ln|f|}=-Z(9\eta^2-10\eta+3)+(1-\eta)(3\eta^2-2\eta+1)^2. \tag{29}$$

Here

$$D(\eta)=8(1-\eta)^2-(3\eta^2-2\eta+1)^2.$$

The set (28) and (29) allows us to express the quantities Γ_l and Γ_t and with them the requirement for the stability of the spectrum in terms of Z and η. Finally, the criterions for stability reduce to the following two inequalities:

$$\begin{gathered}D(\eta)[Z(1+6\eta-9\eta^2)+2(1-\eta)(1-6\eta+3\eta^2)]\geqslant 0,\\ D(\eta)[5Z(1-\eta)^2-Z\eta^2(3\eta-1)^2-2\eta(1-\eta)^2(3\eta^2-2\eta+1)-4(1-\eta)^4]\geqslant 0.\end{gathered} \tag{30}$$

Equations (28) and (29) can only be integrated numerically. The results are given in Fig. 3. The solutions for which $Z|_{\eta=0}\geqslant 1$ are extensions of the spectrum considered in Sec. 4.1. The other solutions can for small values of f be joined to the spectrum of Sec. 4.5 (see

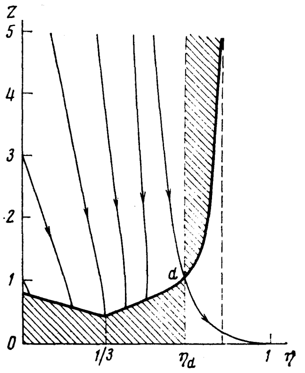

FIG. 3. Integral curves of the set (28), (29). The arrows indicate the direction of change of Z and η when the frequency increases. The instability zone of the spectrum (27) is hatched. The solutions for which $Z\,|_{\eta=0} < 1$ are shown in the insert of Fig. 4.

below). When the frequency increases all curves reach the limit of the stability region and after that the nature of the solution changes.

The family of curves in Fig. 3 is bounded on the right by the solution going through the singular point ($\eta = \eta_d \approx 0.655$, $Z = Z_d \approx 1.060$). In that point $D(\eta) = 0$. At the same time the right-hand sides of Eqs. (28), (29) also vanish. As $f \to 0$ the singular solution has a simple asymptotic behavior:

$$Z = 2.20(f/f_0)^2, \quad 1-\eta = 0.909\, f/f_0,$$

which shows that the jet reaches the longitudinal direction. In fact, η becomes unity not when $f = 0$, but for a small positive value of f which is connected with the fact that the exact dispersion law (10) differs from the power law (22). Using Eq. (10) one can show that

$$Z = 2.20(f/f_0)^2,$$
$$1-\eta = \frac{0.909}{|f_0|}\left(\frac{0.909}{\beta|f_0|} - f\right).$$

When $f = 0.909/(\beta|f_0|)$ this solution changes to the spectrum consisting of l and t waves, polarized along the field. This can be found easily from the relations $\Gamma_l(f;1) = \Gamma_t(f;1) = 0$. It turns out to be stable and can be continued up to the resonance region of frequencies.

4.4. Jets of oblique *l* waves and *l* waves moving across the magnetic field

In this case

$$n^l(f;x) = \frac{\nu\beta f^2}{Z(f)}\delta(x - \sqrt{\eta(f)}) + \frac{\nu\beta f^2}{Z_0(f)}\delta(x - 0), \quad w_0{}^l = w_1{}^l = 0. \tag{31}$$

In the region of the self-similarity the equations $\Gamma_l(f;0) = 0$, $\Gamma_l(f;\sqrt{\eta}) = 0$, $\partial\Gamma_l(f;\sqrt{\eta})/\partial\eta = 0$ can easily be integrated and give the following relations:

$$\frac{3\eta-1}{Z} - \frac{1}{Z_0} = \frac{A}{f^2}, \quad \frac{2\eta(1-\eta)}{Z} = \frac{B}{f} - 1,$$
$$\frac{1-\eta}{Z} + \frac{1}{Z_0} = \left(\frac{C}{f}\right)^{3/2} - \frac{1}{2}. \tag{32}$$

The constants A, B and C are determined from the condition that the solution (31) should at its lower limit be joined to the jet (27).[4)]

The requirement that the function Z_0 be positive leads to the following limit on the quantities Z and η in the point of joining:

$$2(1-\eta)(-3\eta^2 + 6\eta - 1) + Z(9\eta^2 - 6\eta - 1) \geqslant 0. \tag{33}$$

Comparison of this inequality with (30) shows that joining is possible when $\frac{1}{3} \leqslant \eta \leqslant \eta_d$ at the limit of stability of the jet (27) and when $\eta > \eta_d$. In the latter case (31) joins up with the jet going through the singular point. We note that the value of the quantity η in the point of joining uniquely (apart from similarity) determines the form of the solution (31).

The condition of stability of the spectrum (31) reduces to the inequality

$$(1-\eta)/Z + 1/Z_0 \leqslant 3/2. \tag{34}$$

The limit of the stability region in the Z, η plane is shown in Fig. 4 by the line *ce*. In the same figure we show typical integral curves $Z(\eta)$. When the curve reaches the stability limit the spectrum (31) changes into the degenerate one (see below).

4.5. Jets of oblique *l* and *t* waves polarized across the magnetic field

It is convenient to write the spectral functions $n^l f; x)$ and $w_0^t(f)$ in the following form:

$$n^l(f;x) = \frac{\nu\beta f^2}{Z(f)}\delta(x - \sqrt{\eta(f)}), \quad w_0{}^t(f) = \frac{\nu\beta f^2}{u(f)}. \tag{35}$$

The quantities $Z(f)$, $\eta(f)$, and $u(f)$ are determined by the equations $\Gamma_t(f;0) = 0$, $\Gamma_l(f;\sqrt{\eta}) = 0$, and $\partial\Gamma_l(f;\sqrt{\eta})/\partial\eta = 0$, which leads to the following set:

$$\frac{(1-\eta)^2}{Z} = \frac{A}{f} - 1, \quad \frac{\partial\eta}{\partial\ln|f|} = -\frac{\eta(1-\eta)}{2(1+\eta)} + Z\frac{3\eta^2 + 4\eta - 1}{4(1-\eta^2)},$$

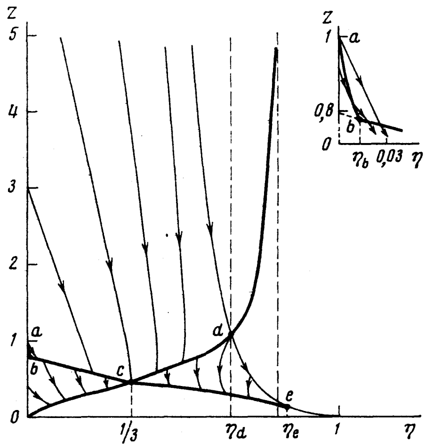

FIG. 4. Integral curves $Z(\eta)$ and boundaries of the stability regions of the self-similar solutions of Secs. 4.3 to 4.5. In the region lying above the line *abcd* the solution has the form of Sec. 4.3; in the region 0*abc* that of Sec. 4.4, and in the region *cde* that of Sec. 4.5. The direction of the arrows corresponds to increasing frequency. In the insert with a larger scale we show the vicinity of the line *ab*.

$$\frac{\partial u}{\partial \ln|f|}=2u+\frac{u^2}{4Z(1+\eta)(1-\eta)^3}$$
$$\times\{2\eta(1-\eta)^2(3\eta^2-2\eta+1)+4(1-\eta)^4+Z\eta^2(3\eta-1)^2-5Z(1-\eta)^2\}. \quad (36)$$

Here A is integration constant.

The condition of stability for the spectrum (35) is given by the inequality

$$Z(3\eta^2-1)+2\eta(1-\eta)^2\leq 0.$$

The limit of the stability region is shown in Fig. 4 (line $0c$). We have also drawn the sections of the integral curves $Z(\eta)$ which have a physical meaning. They all lie within the tetragon $0abc$. If the integral curve starts on the side bc the solution (35) joins at its lower limit to the jet (27). Z and η are then continuous, and $u^{-1}=0$. The condition that the quantity u is positive ($du^{-1}/df\geq 0$) gives for Z and η in the point of joining the inequality

$$2\eta(1-\eta)^2(3\eta^2-2\eta+1)+4(1-\eta)^4+Z\eta^2(3\eta-1)^2-5Z(1-\eta)^2\geq 0,$$

which in conjunction with (30) shows that joining is possible only at the limit of stability of the jet (27) when $\eta\leq\frac{1}{3}$. If, however, the curve $Z(\eta)$ starts on the side $0a$, the solution (35) is a continuation of (23) and to each value $Z(0)$ there corresponds a single (apart from similarity) solution of the set (36). All these solutions lie in the region $\eta\ll 1$ which enables us to integrate the set (36) analytically. It turns out that for $0.893\leq Z(0)\leq 1$ the intensity of the t waves vanishes with increasing frequency and the spectrum becomes a single jet one. The values of η and Z in the points where the t jet terminates cover the line

$$\eta\approx(1-Z)^2/4Z$$

(line ab in Fig. 4). The point b has the coordinates $\eta_b\approx 0.013$, $Z_b\approx 0.785$.

We note finally that as in the case 4.4 when the curve $Z(\eta)$ reaches the stability limit this corresponds to a transition of the jet spectrum to the degenerate one (see below).

4.6. Degenerate spectrum

In that case there are only integral restrictions on the spectral functions:

$$F_1=\nu f+\text{const},\quad F_2=\text{const},\quad F_3=\text{const},\quad F_4=\text{const} \quad (37)$$

(the functions F_1, F_2, F_3, and F_4 are given by Eq. (12)).

We note that the degenerate solution conserves essentially the positive quantity $3F_3+F_4$, and it can therefore certainly not be extended to the point f_0 where $3F_3+F_4=0$. At its lowest limit which we denote by f_* the spectrum (37) must thus join to one of the jet solutions. A comparison of the condition that the spectral functions are positive to the right of the point of joining ($f=f_*+0$) with the condition for the stability of the jet spectrum shows that joining is possible either with the solution (27) when $\eta(f_*)\equiv\eta_*\gtrsim\eta_e=0.831$ or with the solutions (31) and (35) at the limit of their stability. In principle the degenerate spectrum can be continued from the point of joining to the side of high frequencies up to the resonance region. It is merely necessary that the relations between the quantities F_1, F_2, F_3, and F_4 do not violate the requirement that the spectral functions must be positive. Simple estimates enable us to establish that for this the value of the quantity η_* must not be too close to zer or unity ($\eta_*\gg\beta^{-1}+\beta^{-1/2})/|f_0|$; $1-\eta_*\gg(1+\beta)/\beta^2 f_0^2$). We shall not consider here the situation when one of these conditions is violated as it is *a priori* clear that then the shape of the spectrum is close to the one found in the limit $\eta_*=0$ or $\eta_*=1$.

In conclusion we give a list of all possible variants of constructing a stationary spectrum from the solutions described in sections 4.1 to 4.6. The family of spectra is characterized by two parameters for which it is convenient to choose f_0 and η_*. The spectrum can be constructed in the seven following ways:

1) $4.1\to4.2\to R$,
2) $4.1\to4.5\to4.6\to R$,
3) $4.1\to4.5\to4.3\to4.5\to4.6\to R$,
4) $4.1\to4.3\to4.5\to4.6\to R$, (38)
5) $4.1\to4.3\to4.4\to4.6\to R$,
6) $4.1\to4.3\to4.6\to R$,
7) $4.1\to4.3\to L\to R$.

Each solution in this list is denoted by the number of the appropriate section. The symbol R denotes the solution in the resonance region and the symbol L the two jet solution consisting of l and t waves polarized along the magnetic field (see Sec. 4.3). The arrows indicate how the type of the solution changes when the frequency increases. The variants are numbered in order of increasing parameter η_*.

5. RESONANCE REGION. EVALUATION OF THE HEATING POWER

The distribution of the waves in the resonance range of frequencies depends significantly on the actual form of the source. Below we assume that the oscillations are driven by two identical counterstreaming relativistic electron beams each of which has a small angular spread $\Delta\theta\ll 1$. In that case the growth rate $\gamma(f;x)$ is in the region $x>0$ non-vanishing in the vicinity of the line $xk(f;x)=1$ and for each fixed value of f has a sharp maximum in x (see Ref. 11).[5] If we denote the position of the maximum by $x_0(f)$ we may with good accuracy assume that $x_0\,k(f;x_0)=1$. In the maximum the growth rate is given by the following formula[6]:

$$\gamma=\omega_{pe}\frac{n'}{n}\frac{mc^2}{\mathscr{E}}\frac{1}{\Delta\theta^2}x_0^2.$$

Here n' is the beam density, $\mathscr{E}$ the energy of the relativistic electrons and $\Delta\theta$ their angular spread.

To construct the resonance spectrum we need to carry out a procedure analogous to the one given in Sec. 4 with the sole difference that we must include the growth rate $\gamma(f;x)$ in the equations. After that it is necessary to join the solution found to the non-resonance spectrum. The corresponding calculations are simple in principle but rather cumbersome. They show that join-

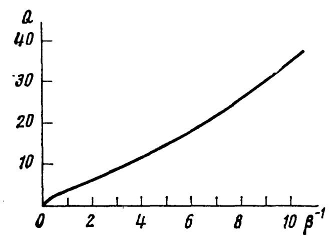

FIG. 5. The dependence of the heating power on the magnetic field (see Eq. (39)) $\beta \equiv 12\pi mT/H^2$.

ing is possible only when the non-resonance spectrum consists of the solutions 4.1, 4.3, 4.4, and 4.6, i.e., when it refers to number 5) of the scheme (38). The spectrum obtained is described in detail in Ref. 12. Its most important feature is that the number of l waves in it is larger than in the analogous spectrum in a plasma without a magnetic field. This, naturally, leads to an increase in the power transferred from the beam per unit plasma volume. The result of the calculations of the power turns out to be the following[7]:

$$P=\nu nT\frac{18}{\pi\Delta\theta^2}\frac{M}{m}\frac{n'}{n}\frac{T}{\mathscr{E}}\left[\ln\left(\frac{\omega_{pe}}{\nu}\frac{n'}{n}\frac{mc^2}{\mathscr{E}\Delta\theta^2}\right)+\left(\frac{2\omega_{pe}}{\nu}\frac{n'}{n}\frac{mc^2}{\mathscr{E}\Delta\theta^2}\right)^{1/2}Q(\beta)\right] \qquad (39)$$

(the function $Q(\beta)$ is shown in Fig. 5). The first term in Eq. (39) is the same as the heating power when there is no magnetic field (see Ref. 5), and the second one describes the increase in dissipation caused by including the field. Its dependence of the beam and plasma parameters corresponds completely to the estimates given in Sec. 3.

[1]For the suppression of the electromagnetic waves it is sufficient that the density of the plasma through which the beam passes be somewhat below the density of the neighboring plasma. The required drop in density $\delta n/n$ is as to order of magnitude equal to T/mc^2.

[2]Using the fact that the spectral functions are even, we take the interval in which the variable x changes to be the section (0, 1).

[3]The lower limit of the spectrum considered f_0 lies just in this region (see (9)).

[4]It is impossible to join (31) with other solutions at the lower limit.

[5]The function $k(f; \dot{x})$ is given by Eq. (10).

[6]Here and in what follows we use variables with dimensions.

[7]To avoid confusion we point out that Eq. (39) was obtained for the case where the instability growth rate was well above the collision frequency. The number standing under the logarithm sign is thus appreciably larger than unity.

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Translated by D. ter Haar.