The vacuum charge distribution near super-charged nuclei

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The distribution of the vacuum charge has been determined for supercritical ($Ze^2 \gg 1$) nuclei. The calculation is carried out within the framework of the Thomas-Fermi method generalized to the relativistic case. A characteristic parameter in this problem is $Ze^3 \approx Z/1600$. For $Ze^3 \sim 1$ the total charge of the vacuum cloud becomes comparable to the charge Z of the nucleus. The relativistic Thomas-Fermi equation for the vacuum shell of a supercritical atom has been solved analytically in the two limiting cases $Ze^3 \ll 1$ and $Ze^3 \gg 1$, and numerically for the intermediate region $Ze^3 \sim 1$. For $Ze^3 \gtrsim 1$ the electron shell penetrates inside the nucleus and almost completely screens its charge. For $Ze^3 \gg 1$ a supercharged nucleus represents an electrically neutral plasma with equal concentration of electrons, protons, and neutrons (for $N = Z$). Inside the nucleus the potential takes on a constant value equal to $-V_0 = -(3\pi^2 n_p)^{1/3} \simeq -1.94 m_\pi c^2$. Near the edge of the nucleus there is a transition layer with a width independent of Z in which the electric field and surface charge are concentrated. On account of the screening the Coulomb energy E_Q, which prevents its stability, decreases sharply: the dependence $E_Q \infty Z^{5/3}$ is replaced by a $Z^{2/3}$ dependence. We are also considering the Thomas-Fermi equation for a neutral atom (in which not only the vacuum shell but all external electron shells are filled completely), as well as the equation which takes into account the exchange and correlation corrections (the relativistic generalization of the Thomas-Fermi-Dirac equation).

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1. INTRODUCTION

The Dirac equation for the electron in the field of a point charge Ze loses its meaning for $Ze^2 > 1$, since the ground state energy becomes imaginary

$$\varepsilon_0 = [1-(Ze^2)^2]^{1/2}.$$

If one takes into account the finite size of the nucleus one can remove this difficulty.[1] When a finite radius R is introduced the ground level $1s_{1/2}$ is lowered to the boundary of the lower continuum $\varepsilon = -1$ for $Z = Z_{cr} \simeq 170$ (cf.[2,3]). For $Z = Z_{cr}$ the total energy for the production of an e^+e^- pair vanishes, and the vacuum becomes unstable with respect to the production of electron–positron pairs. On account of the Pauli principle the number of produced pairs is determined by the number of discrete levels which have descended into the lower continuum. Passing through the Coulomb barrier the positrons go off to infinity,[4] and the electrons remain near the nucleus, partially screening its charge. Thus, a naked nucleus of supercritical charge $Z > Z_{cr}$ will envelop itself with an electron shell created out of the vacuum; in the sequel we shall call this shell the vacuum shell.

A correct description of the vacuum shell and of its physical properties was given in[5] (in the one-particle approximation, i.e., taking into account the interaction between the electrons and the nucleus to order Ze^2, but neglecting the interaction $\sim e^2$ between the electrons and positrons). Numerical calculations[6–9] of the charge distribution of the vacuum shell, of its mean radius, a and of other quantities confirm the conclusions of[5]. Recently a systematic investigation of this problem was carried out in the framework of field theory, taking into account the electron–positron interaction.[10] For $Z > Z_{cr}$ the positron has a longlived quasistationary state with a wave function close to the wave function of the K electron.[5,11] As $Z > Z_{cr}$ is reached there appear e^+e^- pairs consisting of an electron on the K shell and a positron in this quasistationary state. After the e^+e^- interaction is switched on the energies of the states with different numbers of pairs (0, 1, 2 pairs) move apart by a distance of the order 10–30 keV. This has the consequence that the energy spectrum of the positrons emitted in the process of bringing two uranium nuclei together may have several maxima separated by 10–30 keV.[10]

We note that in these papers the case of not too large Z was considered (namely, $Z - Z_{cr} \lesssim Z_{cr}$), when there are still few electrons in the vacuum shell. In the present paper we consider the opposite case $Z \gg Z_{cr}$. An investigation of these questions was stimulated by the papers[21,13] dedicated to the problem of vacuum stability and the production of Bose-particles in critical fields. In these papers it was shown that a phase transition is possible with the formation of a pion condensate in ordinary nuclei,[12] as well as the existence of superdense, neutronic and supercharged ($Ze^3 \gtrsim 1$) nuclei.[12,13]

It is interesting to consider the situation $Ze^2 \gg 1$ also for Fermi-particles. In this case the vacuum shell contains many electrons; as a consequence of this one may apply the relativistic Thomas–Fermi equation for the calculation of the electron density. This equation can be obtained from the following intuitive considerations.

Let $V(r)$ be the self-consistent potential for an electron, taking into account both the field of the nucleus and the average field created by the other electrons of the vacuum shell. If $Ze^2 \gg 1$ the quasiclassical approximation is applicable and the spin effects are inessential, since only large angular momenta play a role (cf. the Appendix). The quasiclassical momentum of the electron is

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$$p(r)=[(\varepsilon-V(r))^2-1]^{1/2}, \tag{1}$$

defining three regions (cf. Fig. 1): I) $\varepsilon > \varepsilon_+ = V(r)+1$; II) $\varepsilon < \varepsilon_- = V(r)-1$; III) the classically forbidden region $\varepsilon_- < \varepsilon < \varepsilon_+$. In the regions I, II the square of the momentum is positive, $p^2(r) > 0$; the region I corresponds to the upper continuum, the region II corresponds to the lower one.

We make the following clarification. Let $V(r) = \text{const}$ throughout the region of r values. Then the lower continuum should be understood as the region between the curves $\varepsilon_- = V-1$ and $V-W$, as $W \to \infty$. Similarly, if $V(r)$ is a smooth function of r, one should understand by lower continuum the region between $V(r)-1$ and $V(r) - W$ (the dotted line in Fig. 1) with subsequent taking of the limit $W \to \infty$. As it should be, the charge density in the lower continuum at each point r does not change on account of adding the potential $V(r)$.

When $V(r)$ becomes smaller than -2 the discrete levels go over into the lower continuum. If these levels were not occupied by electrons (naked nucleus) there appears the possibility of a tunneling transition from the lower continuum into the upper one.[1)] The barrier to be penetrated (corresponding to the classically forbidden region III with $p^2(r) < 0$) has an exponentially small penetrability. The electrons of the vacuum shell represent a degenerate relativistic Fermi gas and fill all the cells of phase space with momenta $p \le p_{\max} = (V^2 + 2V)^{1/2}$. This value of $p_{\max}$ follows from (1) for $\varepsilon = \varepsilon_{\max} = -1$. The electron density $n_e(r)$ of the vacuum shell is related to the maximal momentum by the well known relation

$$n_e(r)=\frac{p_{max}^3}{3\pi^2}=\frac{1}{3\pi^2}(V^2+2V)^{3/2}. \tag{2}$$

The spatial distribution of electrons is determined by

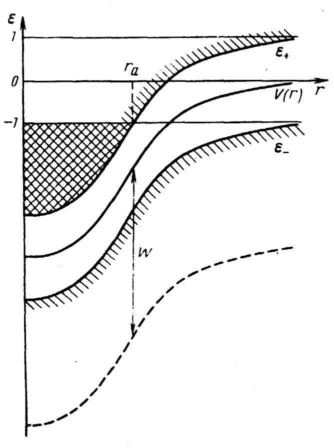

FIG. 1. The deformation of the upper and lower continua in a strong external field (the boundaries of the continua are shaded). The electrons belonging to the vacuum shell of a supercritical atom fill the cross-hatched region. The states below the curve $\varepsilon_-(r) = V(r) - 1$ form the unobservable Dirac sea. The quantity W has the meaning of a cutoff energy, the introduction of which is necessary in order to give meaning to the difference between two divergent integrals for the charge density. All energies are measured in units of $m_e c^2$.

the relativistic Thomas–Fermi equation

$$\Delta V=-4\pi e^2\left\{\frac{1}{3\pi^2}(V^2+2V)^{3/2}-n_p(r)\right\}, \tag{3}$$

where $p_p(r)$ is the proton density. In the sequel we assume $n_p(r) = n_p\theta(R-r)$, where $n_p = 3Z/4\pi R^3 = Zn_0/A \simeq 0.25m^3$, $Z/A \sim 0.5$; n_0 is the usual nuclear density: $n_0 = 3/4\pi r_0^3$; $R = r_0 A^{1/3}$ is the nuclear radius $r_0 = 1.1\ F$. As can be seen from (2), $n_e(r)$ is nonzero only in the region of space where $V(r) < -2$. Therefore the vacuum shell has a finite radius $r = r_a$. The boundary conditions for the equation (3) are the following:

$$|V(0)|<\infty, \quad V(r_a)=-r_aV'(r_a)=-2. \tag{4}$$

The latter condition follows from the fact that $V(r) = -Z_1 e^2/r$ for $r \ge r_a$; $Z_1 = Z - N_e$ is the atomic charge for an external observer.

We note that retaining the term $2V$ together with V^2 in the expressions (2) and (3) is legitimate in all regions of r where it represents a correction larger than ζ^{-1} relative to V^2 (cf. the Appendix for details).

In the next section we describe a more detailed derivation of Eq. (3) which allows one to obtain the distribution of the electrons of the shell with respect to angular momenta. In Sec. 3 it is shown that for $Ze^2 \gg 1$ the contribution of vacuum polarization does not change equation (3). Further we consider the properties of solutions of this equation: in Sec. 4 for $Ze^3 \ll 1$ (weak screening), in Sec. 5, for $Ze^3 \gg 1$ (supercharged nucleus, extreme screening). This situation can be discussed analytically. In Sec. 6 we list the results of numerical calculations in the intermediate region $Ze^3 \sim 1$, which allows us to join the two limiting cases. Section 7 contains a generalization and refinement of the results.

In this paper we use a system of units with $\hbar = c = m_e = 1$, $e^2 = 1/137$, and we introduce the notations: $\zeta = Ze^2$, where Z is the charge of the naked nucleus, N_e is the number of electrons in the vacuum shell, $Z_1 = Z - N_e$ is the charge of the system at large ($r > r_a$) distances, Z' denotes the total charge situated inside the nucleus.

2. DERIVATION OF THE RELATIVISTIC THOMAS–FERMI EQUATION FROM THE DIRAC EQUATION

The Dirac equation for the radical functions G and F

$$\frac{dG}{dr}=-\frac{\varkappa}{r}G+(1+\varepsilon-V)F, \quad \frac{dF}{dr}=(1-\varepsilon+V)G+\frac{\varkappa}{r}F \tag{5}$$

can be reduced by means of the substitution $G = (1+\varepsilon - V)^{1/2}\chi$[3] to a form analogous to the Schrödinger equation:

$$\chi''+2(E-U)\chi=0. \tag{6}$$

Here $E = (\varepsilon^2 - 1)/2$, ε is the electron energy, $U = U(r, \varepsilon)$ is the effective potential[3–5]:

$$U=\varepsilon V-\frac{1}{2}V^2+\frac{\varkappa^2}{2r^2}-\frac{\varkappa}{r}w+\frac{1}{2}\left(w'+w^2+\frac{w}{r}\right), \tag{7}$$
$$w=\frac{1}{2}\left(\frac{V'}{1+\varepsilon-V}-\frac{1}{r}\right),$$

$\varkappa = \mp(j+\frac{1}{2})$ for states of angular momentum $j = l \pm \frac{1}{2}$ and parity $(-1)^l$. The terms in (7) which contain the function w are due to the electron spin. For $|V(r)| \gg 1$ they are small compared to the first three terms,[2)] therefore the expression for the effective potential simplifies and takes the same form as for a scalar particle (the Klein–Gordon–Fock equation):

$$U = -\frac{1}{2}V^2 + \frac{\varkappa^2}{2r^2} + \varepsilon V. \qquad (8)$$

For $\varepsilon < -1$ and $V(r) < 0$ this potential exhibits a barrier (Fig. 2). For $V(r) = -\zeta/r$ the penetrability of the barrier is

$$\gamma(\varepsilon, \varkappa) = \gamma_0 \exp\left\{-2\pi\left[\frac{\zeta}{(1-\varepsilon^{-2})^{1/2}} - (\zeta^2 - \varkappa^2)^{1/2}\right]\right\}, \qquad (9)$$

where γ_0 is a constant of the order of unity. This formula determines[4,5] the width of the positron quasistationary state for $Z > Z_{cr}$ as well as the width of the s smearing $\Delta\varepsilon$ of the wave function of the electron in the vacuum shell over the functions of the lower continuum.

For $\gamma \ll 1$ the one-particle approximation can be used for the description of the vacuum electrons. This is always true for the levels near the boundary of the lower continuum, where[4]

$$\gamma(\varepsilon, \varkappa) = \text{const}\cdot\exp(-2\pi\zeta/k), \quad k = (\varepsilon^2 - 1)^{1/2} \to 0. \qquad (9')$$

In the case $\zeta \gg 1$ the exponential smallness of γ is retained also for levels which have descended deeply into the lower continuum:

$$\gamma(\varepsilon, \varkappa) = \gamma_0 \exp\{-\pi(\varkappa^2/\zeta + \zeta/\varepsilon^2)\}, \qquad (9'')$$

if $|\varkappa| \ll \zeta$ and $|\varepsilon| \gg 1$. For $|\varkappa| \gtrsim \varkappa_0 = (\zeta/\pi)^{1/2}$ the exponential smallness of holds independently of the value of the energy ε.

The maximal angular momentum of levels which have descended into the lower continuum is $\varkappa_{max} \sim \zeta$. Since $\varkappa_0 \ll \varkappa_{max}$, the number of dangerous levels, for which the exponential in (9″) is of the order of unity, and the level "melts" into the lower continuum, is for $\zeta \gg 1$ negligibly small compared to the total number of states in the vacuum shell (for a quantitative estimate, cf. the Appendix). On account of this the electron density $n_e(r)$ can be obtained by direct summation over the one-particle states localized in the region $r_1 < r < r_2$ (cf. Fig. 2):

$$n_e(r) = \sum_{n,\varkappa,M} |\psi_{n\varkappa M}(r)|^2 \approx \sum_{n,\varkappa} \frac{(2j+1)A_{n\varkappa}^2}{4\pi r^2 p(r)}[\varepsilon_{n\varkappa} - V(r)].$$

Here we have made use of the Pauli principle and of the quasiclassical formulas (A.4) for the solutions of the Dirac equation (with the rapidly oscillating functions $\sin^2\theta$ and $\sin^2(\theta+\eta)$ replaced by $\frac{1}{2}$.

The remaining calculations do not differ from the derivation of nonrelativistic Thomas–Fermi equation in[15]. Making use of the equation (A.7) of the Appendix we pass from summation over n to integration over the energy in the interval.

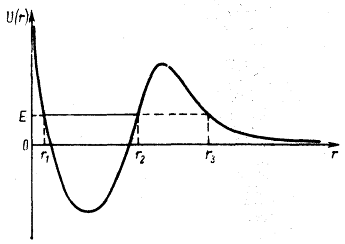

FIG. 2. The shape of the effective potential (8) for states with a definite value of the angular momentum j and energy $\varepsilon < -1$.

$$\varepsilon_+(r) < \varepsilon < -1, \quad \varepsilon_+(r) = V(r) + (1 + \varkappa^2/r^2)^{1/2}. \qquad (10)$$

For $\varepsilon = \varepsilon_+(r)$ the point r coincides with the turning point r_2. If $\varepsilon_+(r)$, the point r is either in the subbarrier region $r_2 < r < r_3$, where the electron wave function decays exponentially, or in the region $r > r_3$ corresponding to the unobservable Dirac background (sea); such states do not contribute to $n_e(r)$. As a result we obtain the spatial distribution of electrons with angular momentum j:

$$n_j(r) = \frac{2j+1}{2\pi^2 r^2}\left[V^2 + 2V - \frac{(j+1/2)^2}{r^2}\right]^{1/2}. \qquad (11)$$

Summing over j we arrive at (2) from the Poisson equation $\Delta\varphi = -4\pi e(n_p - n_e)$ taking into account $V = -e\varphi$ $(e > 0$, φ is the electrostatic potential) we obtain the fundamental equation (3).

In addition to the vacuum shell, the electrons may fill also the external shells of a supercritical atom. The expression for the density of electrons in the neutral atom follows from Eq. (1) for $\varepsilon_{max} = 1$:

$$\tilde{n}_e(r) = (V^2 - 2V)^{3/2}/3\pi^2. \qquad (12)$$

In this case the electron density differs from zero for all $V(r) < 0$, and in the limit $|V(r)| \ll 1$ Eq. (3) goes over into the usual Thomas–Fermi equation. The boundary condition for the neutral atom has the usual form $\lim rV(r) = 0$ as $r \to \infty$.

As is well known,[15] the domain of applicability of the nonrelativistic Thomas–Fermi equation is restricted by the condition: $Z^{-1} \ll r \lesssim 1$ (in atomic units). In distinction from this, in the case $\zeta \gg 1$ there is no restriction on the side of small r (cf. the Appendix), and Eq. (3) with the density (12) is applicable for all $0 < r \lesssim 1$.

A relativistic Thomas–Fermi equation with the density (12) has been considered previously in the papers of Vallarta, Rosen and Jensen.[16,17] They have restricted their attention to the region $\zeta < 1$, and this led only to the introduction of small corrections to the nonrelativistic Thomas–Fermi model, corrections which did not present essential interest. In those papers the region $\zeta \gtrsim 1$ was not considered.

3. AN ESTIMATE OF THE CONTRIBUTION OF VACUUM POLARIZATION

We now discuss the role of vacuum polarization. The charge density induced by the nuclear charge in the vacuum is[18]

$$\rho_{vac}(r)=-\frac{e}{2}\left\{\sum_{(+)}|\psi_{\varepsilon\varkappa M}(r)|^2-\sum_{(-)}|\psi_{\varepsilon\varkappa M}(r)|^2\right\}, \quad (13)$$

where $\psi_{\varepsilon\varkappa M}$ are the exact solutions of the Dirac equation in the Coulomb field of the nucleus. Equation (13) is the starting point for the description of the K shell of a supercritical atom,[5] as well as for numerical calculations of the vacuum polarization and of the charge distribution in the K shell for atoms with large Z.[8,9,19] This density can be split into two parts:

$$\rho_{vac}(r)=\rho(r)+\rho_1(r), \quad (14)$$

where $\rho(r)=-en_e(r)$ is the charge density in the vacuum shell, $\rho_1(r)$ is the contribution of the vacuum polarization proper, which has not been taken into account above. Making use of the analogy with dielectrics, one may call $\rho(r)$ the density of free charges and $\rho_1(r)$ the density of bound charges:

$$\rho_1=-\operatorname{div}\mathbf{P}, \quad \int\rho_1 d^3r=0.$$

We note that for $Z<Z_{cr}\simeq 170$ the density $\rho(r)\equiv 0$, but $\rho_1(r)$ is different from zero. A numerical calculation[8,19] shows that for $Z\gtrsim Z_{cr}$ the density ρ and ρ_1 are quantities of the same order of magnitude. We show that in the case $\zeta=Ze^2\gg 1$ which interests us, the density ρ_1 becomes negligibly small compared to ρ, which justifies the neglect of the contribution of $\rho_1(r)$ in Eq. (3).

We make use of the property of localization of the Green's function and of the polarization operator $\Pi(\mathbf{r}, \mathbf{r}', \omega)$ in strong fields. As was shown in[20], the motion of particles in a strong electric field is characterized by the correlation radius $R_c=(eE)^{-1/2}$. This assertion has a particularly intuitive meaning in the case of a magnetic field H when the orbital radius equals $r=pc/eH$ (the Larmor radius). Since on account of the uncertainty relation $pr\sim 1$, the correlation radius is $R_c\sim(eH)^{-1/2}$. When the distance over which the field changes substantially becomes much larger than R_c the field may be considered as homogeneous. On this account $\mathbf{P}$ and $\mathbf{E}$ are related by the local relation

$$\mathbf{P}=\frac{\varepsilon-1}{4\pi}\mathbf{E}, \quad \varepsilon=1-\frac{e^2}{3\pi}\ln eE, \quad (15)$$

which has been obtained before[21] for the case of the homogeneous field (here ε is the dielectric constant of the vacuum). Hence

$$\rho_1=-[(\mathbf{E}\nabla\varepsilon)/4\pi e+(1-\varepsilon^{-1})\rho],$$

and taking into account $eE(r)=V'(r)$ we obtain:

$$\frac{\rho_1(r)}{\rho(r)}\approx\frac{V''}{4V^3}+\frac{e^2}{3\pi}\ln V'\approx\frac{1}{2\zeta^2}\ll 1 \quad (16)$$

(for $V(r)=-\zeta/r$ and $\zeta\gg 1$). It can be seen from this that taking into account the density $\rho_1(r)$ in the case $\zeta\gg 1$ one exceeds the accuracy of the quasiclassical method used by us.

We now solve Eq. (3). We first consider the cases $Ze^3\ll 1$ (weak screening), and $Ze^3\gg 1$, when the solutions can be obtained in analytic form.

4. WEAK SCREENING: $Ze^3\ll 1$

Going over in (3) to the function $\psi=-V-1$ (with $\psi\geq 1$) and setting $x=r/r_a$, $\mu=4e^2r_a^2/3\pi$, we have for $r>R$:

$$\psi''+2x^{-1}\psi'=\mu(\psi^2-1)^{3/2}, \quad \psi(1)=1, \quad \psi'(1)=-2. \quad (17)$$

It follows from (4) that the atomic radius is $r_a=Z_1e^2/2$, where $Z_1=Z-N_e$ is the external charge of the supercritical atom. Hence $\mu=(Z_1e^3)^2/3\pi$, and in (17) there appears the small parameter $\mu\ll 1$. On account of this Eq. (17) can be solved according to perturbation theory:

$$\psi(x,\mu)=\psi_0(x)+\mu\psi_1(x)+\mu^2\psi_2(x)+\ldots \quad (18)$$

For ψ_n we obtain the chain of equations:

$$\Delta\psi_0=0, \quad \Delta\psi_1=(\psi_0^2-1)^{3/2}, \quad \Delta\psi_2=3(\psi_0^2-1)^{1/2}\psi_0\psi_1,\ldots,$$

where $\Delta\psi\equiv\psi''+2x^{-1}\psi'$. The first two functions can be calculated explicitly:

$$\psi_0(x)=2x^{-1}-1,$$
$$\psi_1(x)=8x^{-1}[(3x+2)\operatorname{Arth}(1-x)^{1/2}-5(1-x)^{1/2}+{}^4/_3(1-x)^{3/2}],$$

where

$$\psi_1(x)=\begin{cases}8x^{-1}[-\ln x+(2\ln 2-{}^{11}/_3)+\ldots], & x\to 0\\ {}^{32}/_{35}(1-x)^{7/2}, & x\to 1\end{cases}.$$

We note that at the edge of the nucleus $x=2R/\zeta\simeq 0.04\zeta^{-2/3}\ll 1$.

In the region $x\ll 1$ the function $\psi(x,\mu)$ simplifies:

$$\psi(x,\mu)=2x^{-1}[1+4\mu(-\ln x+c_0)+O(x,\mu^2)]; \quad (19)$$

$c_0=2\ln 2-11/3$. Inside the nucleus, for $Ze^3\ll 1$ the potential $V(r)$ is still close to the potential of the naked nucleus; setting $\psi=\zeta y(\xi)/R$, $\xi=r/R$, we obtain:

$$\frac{d^2y}{d\xi^2}+\frac{2}{\xi}\frac{dy}{d\xi}=-3+\nu y^3,$$

where

$$0<\xi<1, \quad \nu=4\zeta^2e^2/3\pi=(2Ze^3)^2/3\pi.$$

Considering the nonlinear term νy^3 as a perturbation ($\nu\ll 1$) we find

$$y=y_0(\xi)+\nu y_1(\xi)+\nu^2y_2(\xi)+\ldots, \quad (20$$
$$y_0(\xi)={}^1/_2(3-\xi^2), \quad y_1(\xi)=C+\xi^{-1}\int_0^\xi y_0^3(x)x(\xi-x)\,dx.$$

Joining the solutions (19) and (20) at the edge of the nucleus determines the integration constant

$$C=-1-\int_0^1 y_0^3(x)x\,dx$$

and yields a relation between the charge Z of the nucleus and the external charge Z_1:

$$Z_1=Z\left\{1-\frac{4}{3\pi}(Ze^3)^2\left(\ln\frac{\zeta}{R}+c_1\right)+\ldots\right\},\qquad(21)$$
$$c_1=\ln 2-{}^8/_3+\int_0^1 y_0^3(x)x^2\,dx=-1{,}38.$$

The total number of electrons in the vacuum shell equals

$$N_e=Z-Z_1=\frac{4}{3\pi}\zeta^3\left(\ln\frac{\zeta}{R}+c_1\right).\qquad(22)$$

Let us consider the charge distribution in this shell. Let $eQ(r)$ be the total charge situated inside a sphere of radius r; according to Gauss's theory $V'(r)=eE=Q(r)e^2/r^2$. The number of electrons of the vacuum shell inside a sphere of radius r equals

$$N_e(r)=Z-Q(r)=\frac{4\zeta^3}{3\pi}\left\{\ln\frac{\zeta}{R}+c_1-\int_{2r/\zeta}^{1}\frac{dx}{x}(1-x)^{3/2}\right\}\qquad(23)$$

$(R<r<r_a)$. For $r\ll r_a$ the expression simplifies to:

$$N_e(r)\approx\frac{4\zeta^3}{3\pi}\ln\frac{2r}{R}\Big/\ln\frac{\zeta}{R}.$$

Inside the nucleus is contained a small fraction of the electron cloud

$$\frac{N_e(R)}{N_e}\approx\frac{0{,}7}{[\ln(\zeta/R)]^2}\ll 1.$$

In conclusion of this section we list formulas for the total charge Z' of the nucleus (taking into account that part of the vacuum shell which is situated at $r<R$), of the potential $V(0)$ at the center of the nucleus and of the electric field strength at the edge of the nucleus:

$$Z'=Z-N_e(R)=Z\left[1-c_2\frac{(Ze^3)^2}{\ln(\zeta/R)}+\ldots\right],\qquad(24)$$
$$\frac{V(0)}{V_0}=-c_3(Ze^3)^{2/3}[1-c_4(Ze^3)^2+\ldots],\qquad(25)$$
$$\frac{E(r=R)}{E_{max}}=c_5(Ze^3)^{1/3}\left[1-c_2\frac{(Ze^3)^2}{\ln\zeta/R}+\ldots\right].\qquad(26)$$

Here $V_0=(3\pi^2 n_p)^{1/3}$ and E_{max} are the limiting values of the potential and field strength in the case $Ze^3\gg 1$ (cf. the next section), and c_n are numerical constants:

$$c_2=\frac{4\ln 2}{3\pi}=0.294,\quad c_3=\left(\frac{3}{2\pi}\right)^{1/3}=0.782,$$
$$c_4=\frac{8}{9\pi}\left(1+\int_0^1 y_0^3(x)x\,dx\right)=0.570,\quad c_5=\frac{32}{27}\left(\frac{3}{2\pi}\right)^{1/6}=1.05.$$

These expansions are valid for $Ze^3\ll 1$. Corrections to the unit in the square brackets come from taking into account the screening effect of the potential of the bare nucleus by the vacuum shell. It can be seen directly that for $Ze^3\sim 1$ these corrections stop being small, i. e., the screening changes the total potential $V(r)$ substantially.

5. THE CASE $Ze^3 \gg 1$ (EXTREME SCREENING)

In this case the electric field is pushed out from the volume of the nucleus onto its surface and is concentrated in a transition layer near $r=R$ of thickness of the order $\lambda=12-15\ F$. In the region $Ze^3\gg 1$ the condition $R\gg\lambda$ holds[3)], hence one may neglect the curvature of the edge of the nucleus and the geometry reduces to flat geometry. In the variables χ and $x=(r-R)/\lambda$ we have

$$V=-(3\pi^2 n_p)^{1/3}\chi,\quad \lambda^{-2}=4e^2(\pi/3)^{1/3}n_p^{2/3}\approx 0.03n_p^{2/3},\qquad(27)$$

and we transform (3) into the one-dimensional equation:

$$\chi''=\chi^3-\theta(-x).\qquad(28)$$

In the sequel we set: $A=2Z$, $n_p=0.5n_0\approx 0.25m_\pi^3$, where n_0 is the usual nuclear density. For these values of the parameters $\lambda=9.2m_\pi^{-1}\approx 13\ F$, which exceeds by an order the diffuseness of the edge of the nucleus and justifies the selection of $n_p(r)$ in the form of a step function. Equation (28) must be solved with the boundary conditions: 1) $\chi\to 1$, $\chi'\to 0$ as $x\to-\infty$ (corresponding to the center of the nucleus, since $R\gg\lambda$); 2) χ and $\chi'\to 0$ as $x\to\infty$; 3) continuity of χ and χ' at the edge of the nucleus $(x=0)$.

Taking the boundary conditions into account Eq. (28) has the first integral:

$$2\chi'^2=\begin{cases}\chi^4-4\chi+3=(\chi-1)^2(\chi^2+2\chi+3) & \text{for } x<0\\ \chi^4, & \text{for } x>0\end{cases}$$

It follows that at the edge of the nucleus

$$\chi'(0)=-2^{-1/2}\chi^2(0).\qquad(29)$$

Finally we obtain

$$\chi(x)=\begin{cases}1-3[1+2^{-1/2}\,\mathrm{sh}(a-x\sqrt{3})]^{-1}, & x<0\\ 2^{1/2}(x+b)^{-1}, & x>0\end{cases}\qquad(30)$$

(cf. the curve 1 on Fig. 3). The integration constants a and b are: $\sinh a=11\sqrt{2}$, $a=3.439$; $b=(4/3)\sqrt{2}=1.886$; then $\chi(0)=3/4$, $\chi'(0)=-0.398$.

It is interesting to note that inside the nucleus one can indicate a simple approximate solution[4)] of Eq. (28):

$$\chi(x)=1-C'\exp(x\sqrt{3})\quad\text{for } x<0,\quad C'=0.2374,\qquad(31)$$

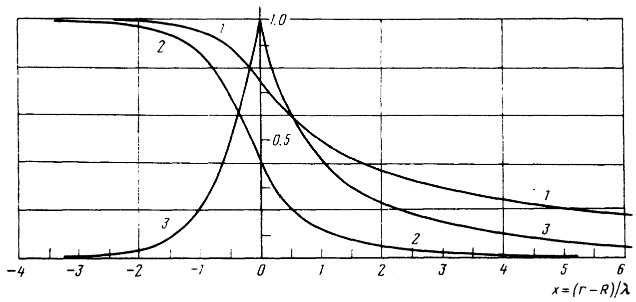

FIG. 3. The variation of the potential, the electron density of the vacuum shell and of the electric field strength near the edge of the supercharged nucleus for $Ze^3\gg 1$. The curve 1 represents the function $\chi=-V(r)/V_0$; the curve 2 represents the ratio $n_e(r)/n_e(0)$, where $n_e(0)$ is the electron density at the center of the nucleus; curve 3 represents the ratio $E(r)/E_{max}$.

which agrees with the exact solution to an accuracy of ~ 1.5%. The asymptotic behavior of the exact solution (30) for $x \to -\infty$ also has the form (31), but with different values of the constant in front of the exponential function: $C'' = 6(9\sqrt{6} - 22) = 0.2724$.

Let us discuss the physical meaning of the solution. Inside the nucleus the densities $n_e \simeq V_0^3/3\pi^2$ and n_p are equal and the potential is practically constant: $V(r) = -V_0$ for $R - r \gg \lambda$, where $V_0 = (3\pi^2 n_p)^{1/3} \simeq 1.94 m_\pi$. The quantities V_0 and λ do not depend on the nuclear charge Z and on its radius R. Near the edge of the nucleus there is a transition layer, the structure of which is exhibited in Fig. 3. The curve 2 represents the variation of the electron density. Here $n_e(R) = 0.42 n_e(0)$, and in the exterior region $r > R$ the density falls rapidly as one goes away from the edge of the nucleus.

In the region $R - r \gg \lambda$ the electric field is practically absent, i.e., the charge of the protons is completely compensated by the electron cloud and the system is electrically neutral. The maximal electric field strength is attained at the edge of the nucleus:

$$E_{max} = {}^9/_{16}\pi\sqrt{2}(3/\pi)^{1/6} n_p{}^{2/3} = 0.084 m_\pi{}^2/e, \tag{32}$$

$$\frac{E(x)}{E_{max}} = -\frac{16\sqrt{2}}{9}\chi'(x) = \begin{cases} k\exp(x\sqrt{3}), & x \to -\infty \\ (b/x)^2, & x \to \infty \end{cases}, \tag{33}$$

where $k = (8/3)^{3/2} C''$.

A plot of the function $E(x)/E_{max}$ is shown in Fig. 3 (curve 3). The energy density of the electric field equals

$$W = E^2/8\pi = {}^1/_2\pi(3/\pi)^{1/3} n_p{}^{4/3}\chi'^2(x).$$

A simple calculation shows that approximately 2/3 of the energy of the field is contained in the region $r > R$, and 1/3 is inside the nucleus.

Allowance for screening leads to the result that the field at the edge of the nucleus remains bounded: $E(R) \to E_{max}$ for $Z \to \infty$. In ordinary units the limiting field $E_{max} = 8.2 \times 10^{19}$ V/cm, which exceeds by a factor of about 6000 the characteristic field strength $E_0 = m_e^2 c^3/e = 1.3 \times 10^{16}$ V/cm in quantum electrodynamics (for $E \sim E_0$ nonlinear effects of quantum electrodynamics become essential: vacuum polarization and the production of e^+e^- pairs in a homogeneous electric field).

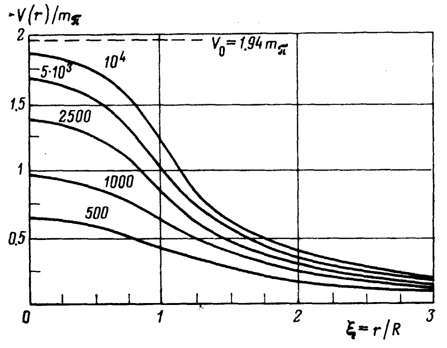

FIG. 4. The quantity $V(r)$ in pion units as a function of $\xi = r/R$. The numbers near the curves indicate the charge of the nucleus Z. The curves are the result of numerical integration of Eq. (35).

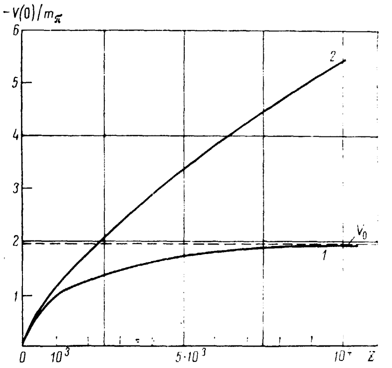

FIG. 5. The potential at the center of the nucleus in pionic units. The curve 1 represents the solution of Eq. (35), the curve 2 is the potential for the naked nucleus (not taking into account the effect of screening). The dotted line indicates the limiting value of the potential $V_0 = (3\pi^2 n_p)^{1/2}$.

The total charge Z' situated inside the nucleus is easily determined in terms of $\chi'(0)$ by means of the Gauss theorem. We obtain

$$Z'/Z = c_6(Ze^3)^{-1/3}, \quad c_6 = {}^{27}/_{32}({}^2/_3\pi)^{1/6} = 0.95. \tag{34}$$

A comparison of this formula with (24) shows how important screening becomes in the case $Ze^3 \gg 1$. In this case an electrically neutral plasma is formed inside the supercharged nucleus. The uncompensated charge is situated in a layer of finite thickness ~ λ near the edge of the nucleus.

6. RESULTS OF NUMERICAL CALCULATIONS

We have considered above the limiting cases $Ze^3 \ll 1$ and $Ze^3 \gg 1$. In order to join these two regions, Eq. (3) has been solved numerically for $Z = 500 - 10\,000$. We make some remarks on the numerical analysis.

It is convenient to pass to the variable $\xi = r/R$ and to set $V(r) = -v/R$. Neglecting in (3) the term $2V$ compared to V^2, which is valid for a wide range of r, except the edge of the atom[5)], we obtain the equation

$$\frac{d^2v}{d\xi^2} = \frac{4(Ze^3)^2}{3\pi\xi^2}v^3 - 3\xi\theta(1-\xi), \tag{35}$$

which does not involve the radius R of the nucleus. Therefore $v = v(\xi, Z)$.

The results of the numerical calculation are represented in Figs. 4 and 5. The dependence of $|V(r)|$ on ξ for several values of Z is shown in Fig. 4 (for $n_p = 0.25 m_\pi^3$). The corresponding curves for a nucleus with a different value of n_p can be obtained from Fig. 4 by multiplication by $(n_p/0.25 m_\pi^3)^{1/3}$. Figure 5 shows the dependence of the potential at the center of the nucleus, $V(0)$ on Z (curve 1). The initial portion of this curve is described by Eq. (25). For comparison the same picture shows the function $|V(0)| = 3Ze^2/2R = c_3 V_0(Ze^3)^{2/3}$ for the naked nucleus (curve 2). The influence of screening of the nuclear potential by the vacuum shell begins to show for $Z \sim 1000$. With further growth of Z the potential at the center of the nucleus takes on the constant value: $V(0) = -V_0 = -1.94 m_\pi$.

7. CONCLUSION

We conclude with several remarks.

1. In Eq. (3) only the Coulomb interaction between the electrons has been taken into account. In order to improve the accuracy of the statistical method one should take into account the exchange interaction of the electrons with parallel spins as well as the repulsion between electrons with antiparallel spins (the correlation effect). This problem is discussed in the Appendix, where it is shown that in the region $Ze^2 \gg 1$ the introduction of exchange and correlation corrections does not lead to a substantial change of Eq. (3); cf. the equations (A.13) and (A.14).

2. The formation of an electrically neutral plasma inside the nucleus for $Ze^3 \gg 1$ strongly diminishes the Coulomb energy of the nucleus:

$$E_Q = \frac{3(Ze)^2}{5R} = 0{,}61 (Ze^3)^{2/3} Z m_\pi, \tag{36}$$

which impedes the stability of supercharged nuclei. Since the electric field is concentrated in the transition region near the edge of the nucleus, the electrostatic energy reduces to surface energy:

$$E_S = -\frac{1}{8\pi e^2} \int V \Delta V \, d^3 r = \frac{V_0^2 R^2}{2\lambda e^2} \int_{-\infty}^{\infty} \chi'^2 \, dx = 0{,}35 (Ze^3)^{-1/3} Z m_\pi. \tag{37}$$

Thus, on account of the screening effect, the Coulomb energy of the nucleus for $Ze^3 \gg 1$ is reduced by a factor of $1.7 Ze^3$. There remains however the kinetic energy of the degenerate electron gas:

$$E_e = \frac{p_F^4}{4\pi^2} \frac{4\pi R^3}{3} = \frac{3}{4} Z p_F \approx 1.5 Z m_\pi. \tag{38}$$

3. At $Z \gtrsim 10^3$ the depth of the potential in the center of the nucleus, $|V(0)|$ exceeds the value m_μ, where m_μ is the muon mass. In this region of Z the filling of negative muon levels with energies $\varepsilon < 0$ is possible on account of the process $n \to p + \mu^- + \tilde{\nu}_\mu$, as a result of which there appears a supplementary charge density $n_\mu(r)$, due to negative muons. The condition for equilibrium of this process with the usual beta-process $n \to p + e^- + \tilde{\nu}$ is the equality of the chemical potentials $\mu_e = \mu_{\mu^-}$, hence

$$\frac{\tilde{n}_\mu(r)}{n_e(r)} = \left[\frac{2}{m_\mu} (|V| - m_\mu) \right]^{3/2} \tag{39}$$

(for $|V| - m_\mu \ll m_\mu$). For $Z \sim 5 \times 10^3$ the depth $|V(0)|$ of the well reaches the value $2m_\mu$ after which the muonic vacuum shell begins to fill up.[6)] Denoting the corresponding charge density by $n_\mu(r)$, we obtain

$$\frac{n_\mu}{n_e} = \left[\frac{|V(r)| - 2m_\mu}{|V(r)| - 2m_e} \right]^{3/2}. \tag{40}$$

In this case inside the nucleus we have $|V(r)| \approx V_0 \sim 2m_\pi$, hence the screening by negative vacuum muons leads only to an insignificant change of the results obtained above:

$$n_\mu / n_e \approx (1 - m_\mu / m_\pi)^{3/2} \approx 0.1.$$

4. Without taking into account the possibility of formation of a pion condensate the supercharged nuclei are unstable. Since $V_0 < 2m$ the $\pi^+\pi^-$ condensation does not yet set in apparently, but for $Ze^3 \gtrsim 1$ the formation of a π^- condensate is possible.[23] The presence of a π^- condensate changes the distribution of electrons in the nucleus substantially. The problem of stability of supercharged nuclei with consideration of the π^- condensate requires the solution of a self-consistent problem of the distribution of the electronic, muonic and pionic charges inside the nucleus and will be considered in the sequel.

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APPENDIX

Here we obtain formulas for the quasiclassical approximation to the solutions of the Dirac equation and we introduce the exchange correction to the relativistic Thomas–Fermi equation. Limiting ourselves to the class of potentials with central symmetry we set in (5)

$$G = ae^{iS}, \quad F = be^{iS}, \tag{A.1}$$

where $a(r)$ and $b(r)$ are functions which vary slowly compared to the exponential function. Neglecting the derivatives a' and b' compared to S' we obtain (cf. also[22]):

$$S = \int^r p \, dr, \quad p = [(\varepsilon - V(r))^2 - 1 - \varkappa^2/r^2]^{1/2}, \tag{A.2}$$

$$\frac{b(r)}{a(r)} = \frac{iS' + \varkappa/r}{1 + \varepsilon - V} = Ne^{i\eta}, \tag{A.3}$$

where $p = p(r)$ is the radial momentum,

$$N(r) = [(\varepsilon - V - 1)/(\varepsilon - V + 1)]^{1/2},$$
$$\eta(r) = \arcsin\{p(r)[(\varepsilon - V(r))^2 - 1]^{-1/2}\}.$$

In the classically allowed region $r_1 < r < r_2$ (cf. Fig. 2) we have $p^2(r) \gg 0$, $\varepsilon - V - 1 > 0$, on account of which the quantities N and η are real. Since $\chi = (1 + \varepsilon - V)^{-1/2} G$ satisfies Eq. (6), the quasiclassical asymptotic behavior for $\chi(r)$ has the usual form. Taking (A.3) into account we obtain the quasiclassical formulas for the radial functions G and F:

$$G = A \left[\frac{\varepsilon - V + 1}{p(r)} \right]^{1/2} \sin \theta(r), \quad F = A \left[\frac{\varepsilon - V - 1}{p(r)} \right]^{1/2} \sin(\theta + \eta), \tag{A.4}$$

where

$$\theta(r) = \int_{r_1}^{r} p \, dr + \pi/4.$$

The normalization condition to one particle localized in the region $r_1 < r < r_2$

$$\int_{r_1}^{r_2} (G^2 + F^2) \, dr = 2 \int_{r_1}^{r_2} (\varepsilon - V) \chi^2 \, dr = 1 \tag{A.5}$$

determines the normalization constant A:

$$A\equiv A_{n\varkappa}=\left\{\int_{r_1}^{r_2}\frac{\varepsilon_{n\varkappa}-V(r)}{p(r)}dr\right\}^{-1/2}. \qquad (A.6)$$

Here $\varepsilon_{n\varkappa}$ is the energy of the level with quantum numbers n and $\varkappa$ ($n=0, 1, \ldots$ is the radial quantum number).

If the barrier penetrability γ is exponentially small (cf. (9)) $\varepsilon_{n\varkappa}$ is determined by the Bohr–Sommerfield quantization condition:

$$\int_{r_1}^{r_2} p\,dr=(n+{}^1/_2)\pi.$$

Differentiating this with respect to n, we obtain

$$A_{n\varkappa}=\left[\frac{1}{\pi}\frac{\partial\varepsilon_{n\varkappa}}{\partial n}\right]^{1/2}, \qquad (A.7)$$

which differs only by the factor $\sqrt{2}$ from the corresponding expression for the nonrelativistic case.[15] The equations (A.4)–(A.7) have been used in Sec. 2 for the derivation of the relativistic Thomas–Fermi equation.

Let us discuss the accuracy of this approximation. In the derivation of Eq. (3) we have made a series of approximations, the accuracy of which is discussed below.

1) The quasiclassical approximation stops being applicable at the boundary of the vacuum shell. The contribution to $n_e(r)$ for $r\to r_a$ come from states with $\varepsilon\simeq-1$ and $|\varkappa|\ll\zeta$ for which $p(r)=[-2U(r)]^{1/2}=\zeta r^{-1}(1-r/r_a)^{1/2}$. The condition of applicability of the quasiclassical approximation

$$\frac{d}{dr}\left(\frac{1}{p(r)}\right)=\frac{1-r/2r_a}{\zeta(1-r/r_a)^{3/2}}\ll 1 \qquad (A.8)$$

reduces to the inequality $r_a-r\gg r_a(2\zeta)^{-2/3}$ and is verified the better the larger $\zeta=Ze^2$ is. We note that here there are no restrictions on the side of small r, in distinction from the nonrelativistic Coulomb problem.[15]

2) In the derivation of the quasiclassical formulas we omit the term

$$A''/p^2A\sim(pr)^{-2}\sim\zeta^{-2}(1-r/r_a)^{-1}, \qquad (A.9)$$

where A is the pre-exponential factor (cf.,[15] p. 131). Therefore the quasiclassical approximation for the wave function of the electron contained in the vacuum shell has an accuracy of the order ζ^{-2}.

3) Let us estimate the number of electrons in the vacuum shell for which the one-particle approximation of Sec. 2 is incorrect. For this we determine the fraction of the levels with energy $\varepsilon<-1$ for which the width γ is not exponentially small.

Integrating (11) over the volume we find the number of electrons in the vacuum shell which have the angular momentum j:

$$N_j=\frac{2}{\pi}(2j+1)\int\left(V^2+2V-\frac{\varkappa^2}{r^2}\right)^{1/2}dr. \qquad (A.10)$$

Let $1\ll\varkappa\ll\varkappa_{max}$. Summing (A.10) with respect to j we determine the fraction of electrons with $j\leqslant\varkappa-\frac{1}{2}$:

$$\delta(\varkappa)=\frac{1}{N_e}\sum_{j=1/2}^{\varkappa-1/2}N_j=c\varkappa^2; \qquad (A.11)$$

here $c=3I_1/2I_2$

$$I_1=\int(V^2+2V)^{1/2}dr,\quad I_2=\int(V^2+2V)^{3/2}r^2\,dr.$$

In particular, for $V(r)=-\zeta/r$ we have to logarithmic accuracy[7]:

$$I_1=\zeta\ln(\zeta/R),\quad I_2=\zeta^3\ln(\zeta/R),\quad c=3/2\zeta^2.$$

For $\zeta\gg1$ and $\varkappa_0=(\zeta/\pi)^{1/2}$ (cf. Eq. (9'')) we obtain

$$\delta(\varkappa_0)=3/2\pi\zeta\ll1. \qquad (A.12)$$

Thus, the results of the one-particle approximation have an accuracy of the order ζ^{-1}. This refers, in particular, to Eqs. (2) and (11) for the electron density.

4) In the expressions (2) and (3) we have retained, together with V^2, the correction term $2V$. Retaining this term in the density $n_e(r)$ is legitimate only in those regions of r where its ratio to V^2 exceeds ζ^{-1}. Thus, in the case $Ze^3\ll1$ we have $2V/V^2\sim r/\zeta$ and the radius of the vacuum shell is $r_a\simeq\zeta/2$. Therefore in a wide region $1\ll r<r_a$ the corrections which we have not calculated, and which are related to the incertitude of the one-particle approximation for the electrons with small angular momenta $j<(\zeta/\pi)^{1/2}$, is definitely smaller than the term $2V(r)$. On the other hand, at small distances $r\lesssim1$ this term should be omitted and one should use the formula $n_e(r)=|V(r)|^3/3\pi^2$, as we did in Secs. 5 and 6.

We now take into account the exchange and correlation corrections in the relativistic Thomas–Fermi equation. This is conveniently done by means of a variational method analogous to the derivation of the nonrelativistic Thomas–Fermi–Dirac equation.[24] The difference consists in the fact that one must vary the relativistic expression for the kinetic energy of the electron gas.

As a result of this we obtain in place of (2)

$$n_e(r)=\frac{1}{3\pi^2}\left[\frac{(V^2+2V+\nu^2)^{1/2}-\nu(V+1)}{1-\nu^2}\right]^3\approx\frac{1}{3\pi^2}[(V^2+2V)^{1/2}-\nu(V+1)]^3, \qquad (A.13)$$

where $\nu=e^2/\pi$. As long as $Ze^3\ll1$ one may neglect the exchange correlation (compared to the correction $2V$ in the term $(V^2+2V)^{3/2}$). If however $Ze^2\gtrsim1$ and $|V(r)|\gg1$, then $(V^2+2V)^{3/2}\simeq-(V+1)$ with accuracy ζ^{-2} and (A.13) reduces to

$$n_e(r)=(1+\nu)^3(V^2+2V)^{3/2}/3\pi^2.$$

Therefore we again obtain Eq. (3) with the renormalized constant:

$$e^2\to e^2(1+\nu)^3\approx e^2(1+3e^2/\pi).$$

The electron density of the neutral atom ($\varepsilon_{max}=1$) has a similar form:

$$\tilde{n}_e(r)=\frac{1}{3\pi^2}\left[\frac{(V^2-2V+\nu^2)^{1/2}+\nu(1-V)}{1-\nu^2}\right]^3. \qquad \text{(A. 14)}$$

The corresponding equation in the nonrelativistic case $Ze^2\ll 1$, $|V(r)|\ll 1$ coincides (after a transition of atomic units) with the Thomas–Fermi–Dirac equation[24]:

$$\psi''=x[(\psi/x)^{1/2}+\beta_0]^3, \qquad \text{(A. 15)}$$

where

$$x=r/\mu,\quad \mu=0.885Z^{-1/3},\quad \beta_0=(3/32\pi^2)^{1/2}Z^{-2/3},\quad \psi=rZ^{-1}(1/2\pi^2-V).$$

Taking account of the correlation reduces in some approximation,[24] to replacing the constant $\beta_0=0.2118Z^{-2/3}$ in the Thomas–Fermi–Dirac equation by $\beta_0'=0.2394Z^{-2/3}$. The appropriate change in (A. 13) has the form: $\nu\to\nu'=1.13e^2/\pi$. It can be seen from this that the consideration of the exchange and correlation corrections in the region $\zeta=Ze^2\gg 1$ does not lead to a substantial change of Eq. (3).

In conclusion we note that Eqs. (2) and (12) for the electron density and the equation (3) remain valid also in the case of a noncentral potential $V(r)$ which is sufficiently smooth so that the quasiclassical approximation be applicable. As a possible application of this equation we indicate the problem of calculation of the critical distance R_{cr} for the collision of two heavy ions, account being taken of the screening influence of the electron shell (the account of screening for the model problem of a spherical superheavy nucleus was carried out in[25]). A rough estimate shows that account of screening is important for a comparison of the theory of spontaneous positron production with experiment.

1) From the analogous idea of tunneling of the electron through the gap between the continua is derived the classical method of calculation of the probability of formation of an e^+e^- pair in variable electric fields, the so-called imaginary-time method.[14]

2) Thus, for the case of a Coulomb field $V(r)=-\zeta/r$ we have $|w/V^2|\sim\zeta^{-2}r\ll 1$ for $\zeta\gg 1$ and $r\lesssim 1$.

3) We note that $R>\lambda$ for $Ze^3>\frac{1}{2}(\pi/3)^{1/2}=0.51$ (this number does not depend on the adopted value of the proton density n_p).

4) This solution is obtained if $\chi=1-\varphi$ is substituted into (28), which is then linearized with respect to φ and the constant C' determined from the boundary condition (29).

5) We note that $|2V(r)|\ll V^2(r)$ for $r\ll r_a$. A numerical calculation was carried out in the region $r\sim R\ll r_a$, in which this equality is definitely valid.

6) Since $R\gg m_\mu^{-1}$, the nucleus represents a wide well for muons. Therefore the critical value of the potential for which spontaneous creation of $\mu^+\mu^-$-pairs starts is only somewhat above $2m_\mu$: $V_{cr}=2m_\mu+\pi^2/2m_\mu R^2$ (cf.[4,22]).

7) The divergence of the integrals I_1 and I_2 for $R\to 0$ is related to the fact that in the case of the Coulomb field the spatial distribution of electrons with angular momentum j is of the form

$$n_j(r)=\frac{(2j+1)g}{2\pi^2r^3}\left(1-\frac{r}{r_0}\right)^{1/2}\theta(r_0-r),$$

where $g=(\zeta^2-\varkappa^2)^{1/2}$, $r_0=g^2/2\zeta$. For $r\ll r_0$ the density is $n_j\sim r^{-3}$. In such an increase of the electron density would remain in force up to $r=0$, the number of electrons N_e would become infinite. Therefore the point-nucleus approximation becomes inapplicable here, and a physically acceptable solution of Eq. (3) exists only for a finite radius R of the nucleus. The connection of this result with the "falling into the center" of an individual electron in the Coulomb field of a point-charge Ze for $\zeta>j+\frac{1}{2}$ should be obvious.

1 I. Pomeranchuk and Ya. Smorodinsky, J. Phys. USSR **9**, 97 (1945).

2 W. Pieper and W. Greiner, Z. Physik **218**, 327 (1969).

3 V. S. Popov, ZhETF Pis. Red. 11, 254 (1970) [JETP Lett. **11**, 162 (1970)]; Yad. Fiz. **12**, 429 (1970) [Sov. J. Nucl. Phys. **12**, 235 (1971)].

4 V. S. Popov, Zh. Eksp. Teor. Fiz. **59**, 965 (1970) [Sov. Phys. JETP **32**, 526 (1971)].

5 Ya. B. Zel'dovich and V. S. Popov, Usp. Fiz. Nauk **105**, 403 (1971) [Sov. Phys. Uspekhi **14**, 673 (1971/72)].

6 B. Müller, J. Rafelski, and W. Greiner, Z. Physik **257**, 183 (1972).

7 J. Rafelski, B. Müller, and W. Greiner, Nucl. Phys. **B68**, 585 (1974).

8 M. Gyulassy, Phys. Rev. Lett. **33**, 921 (1974); Nucl. Phys. **A244**, 497 (1975).

9 G. A. Rinker and L. Wilets, Phys. Rev. **A12**, 748 (1975).

10 A. B. Migdal, Zh. Eksp. Teor. Fiz. **70**, 411 (1976) [Sov. Phys. JETP **43**, 211 (1976)].

11 A. B. Migdal, A. M. Perelomov and V. S. Popov, Yad. Fiz. **14**, 874 (1971) [Sov. J. Nucl. Phys. **14**, 488 (1972)].

12 A. B. Migdal, Zh. Eksp. Teor. Fiz. **61**, 2209 (1971); **63**, 1993 (1972) [Sov. Phys. JETP **34**, 1184 (1972); **36**, 1052 (1973)].

13 A. B. Migdal, Phys. Lett. **52B**, 182 (1974).

14 V. S. Popov, Zh. Eksp. Teor. Fiz. **61**, 1334 (1971) [Sov. Phys. JETP **34**, 709 (1972)]; Yad. Fiz. **19**, 1140 (1974) [Sov. J. Nucl. Phys. **19**, 584 (1974)]. M. S. Marinov and V. S. Popov, *ibid.* **15**, 1271 (1972) [*ibid.* **15**, 702 (1972)].

15 A. B. Migdal, Kachestvennye metody v kvantovoĭ teorii (Qualitative methods in quantum theory), Nauka, Moscow, 1975.

16 M. S. Vallarta and N. Rosen, Phys. Rev. **41**, 708 (1932).

17 H. Jensen, Z. Physik **82**, 794 (1933).

18 P. A. M. Dirac, Proc. Cambr. Phil. Soc. **30**, 150 (1934).

19 L. S. Brown, R. N. Cahn, and L. D. McLerran, Phys. Rev. **D12**, 581, 596, 609 (1975).

20 A. B. Migdal, Zh. Eksp. Teor. Fiz. **62**, 1621 (1972) [Sov. Phys. JETP **35**, 845 (1972)]; Nucl. Phys. **A210**, 421 (1973).

21 W. Heisenberg and H. Euler, Z. Physik, **98**, 714 (1936).

22 V. S. Popov and V. D. Mur, Yad. Fiz. **18**, 684 (1973) [Sov. J. Nucl. Phys. **18**, 350 (1974)].

23 A. B. Migdal, Rev. Mod. Phys. 1977 (in press).

24 P. Gambas, Die Statistische Theorie des Atoms und ihre Anwendungen, Springer-Wien, 1949 (Russian Transl., IIL 1951).

25 M. S. Marinov and V. S. Popov, ZhETF Pis. Red. **17**, 511 (1973) [JETP Lett. **17**, 368 (1973)]. Zh. Eksp. Teor. Fiz. **65**, 2141 (1973) [Sov. Phys. JETP **38**, 1069 (1974)].

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