Magnetohydrodynamics of a collisionless plasma

S. K. Zhdanov and B. A. Trubnikov

I. V. Kurchatov Institute of Atomic Energy
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We use an expansion of the collisionless kinetic equation for a plasma in a non-uniform magnetic field B to obtain a set of equations for a two-fluid anisotropic magnetohydrodynamics, taking into account terms of order $1/B^2$, in particular, for the magnetic viscosity tensor and the thermal currents.

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1. INTRODUCTION

We evaluate in the present paper corrections to first and second order in the parameter $1/\Omega_B = mc/eB$ to the well known Chew-Goldberger-Low equations for a magnetized anisotropic collisionless plasma.

We showed in[1] that in the framework of the one-fluid magnetohydrodynamics (MHD) one must describe a plasma in a strong magnetic field B in the collisionless approximation by the anisotropic MHD equations with a magnetic (collisionless) viscosity tensor that takes into account effects due to the finite Larmor radius. We retained then only terms of order $1/B$ and ignored "oblique" thermal electron and ion currents which in the one-fluid approximation cancel one another. However, it was shown in[2], where a strongly colliding plasma was considered, that in a number of cases an important role is played in the viscosity tensor by the spatial derivatives of the thermal currents which in the method of[1] are formally contained in the terms of higher order in $1/B$. We do not assume the single-fluid approximation in the present paper and we use an expansion of the kinetic equation taking into account terms of order $1/B^3$ which enables us to make the set of equations of the collisionless anisotropic two-fluid MHD for a plasma in a non-uniform magnetic field more precise; these equations generalize the equations obtained earlier in[3]. Up to first order similar results have been obtained in a recent paper.[4]

2. ZEROTH AND FIRST APPROXIMATION EQUATIONS

The collisionless kinetic equation for charged particles of kind α is of the form

$$\frac{\partial}{\partial t} f_\alpha + (\mathbf{v}\nabla) f_\alpha + \frac{e_\alpha}{m_\alpha}\left(\mathbf{E} + \left[\frac{\mathbf{v}}{c}\times\mathbf{B}\right]\right)\frac{\partial}{\partial \mathbf{v}} f_\alpha = 0. \qquad (2.1)$$

Dropping the index α for the sake of simplicity we put $\mathbf{v} = \mathbf{V} + \mathbf{u}$, where $\mathbf{V}$ is the average and $\mathbf{u}$ the random velocity. Introducing instead of f_α the distribution function $F(t, \mathbf{r}, \mathbf{u})$ and assuming the magnetic field B to be strong we rewrite Eq. (2.1) in operator form:

$$\Omega_B [\mathbf{h}\times\mathbf{u}]\frac{\partial F}{\partial \mathbf{u}} = \Omega_B \frac{\partial}{\partial \varphi} F = \left(\frac{d}{dt} + \mathbf{u}\nabla + (\mathbf{A} - (\mathbf{u}\nabla)\mathbf{V})\frac{\partial}{\partial \mathbf{u}}\right) F, \qquad (2.2)$$

where $\mathbf{A} = (e/m)\mathbf{E} + [\mathbf{V}\times\boldsymbol{\Omega}_B] - \dot{\mathbf{V}}$, $\mathbf{h} = \mathbf{B}/B$, $\boldsymbol{\Omega}_B = (eB/mc)\mathbf{h}$, a point indicates the total derivative $d/dt = \partial/\partial t + (\mathbf{V}\cdot\nabla)$, φ is the azimuthal angle in velocity space with the z-axis directed along $\mathbf{B}$. Assuming Ω_B to be a large quantity, we look for a solution in the form of an expansion $F = F_0 + F_1 + F_2 + \ldots$, where $F_k \sim \Omega_B^{-k}$ (see, e.g.,[5,6]). In zeroth approximation we have $F_0 \equiv F_0(u_\parallel^2, u_\perp^2, \mathbf{r}, t)$, where $u_\parallel = \mathbf{u}\cdot\mathbf{h}$, $\mathbf{u}_\perp = \mathbf{u} - \mathbf{h}u_\parallel$. For the first correction $F_1 = F_0\Phi_1$ we get

$$\Omega_B \frac{\partial}{\partial \varphi}\Phi_1 = L \ln F_0, \quad L = \frac{d}{dt} + (\mathbf{u}\nabla) - (\mathbf{A}^{(0)} - (\mathbf{u}\nabla)\mathbf{V})\frac{\partial}{\partial \mathbf{u}}, \qquad (2.3)$$

and, requiring that the density n, the velocity $\mathbf{V}$, and the pressure tensor components $p_{\alpha\beta} = p_\perp \delta_{\alpha\beta} + (p_\parallel - p_\perp) \times h_\alpha h_\beta$ are completely determined by F_0 alone, we find the MHD equations in the zeroth approximation:

$$\frac{\partial n}{\partial t} + \operatorname{div} n\mathbf{V} = 0, \quad \mathbf{A}^{(0)} = \frac{\nabla\cdot p}{mn}, \qquad (2.4)$$

$$\dot{p}_\parallel^{(0)} = (\dot{p}_\parallel)_{\text{ad}} = -p_\parallel(\operatorname{div}\mathbf{V} + 2\mathbf{h}(\mathbf{h}\nabla)\mathbf{V}), \quad \dot{p}_\perp^{(0)} = (\dot{p}_\perp)_{\text{ad}} = p_\perp(\mathbf{h}(\mathbf{h}\nabla)\mathbf{V} - 2\operatorname{div}\mathbf{V}),$$

thereby determining $A^{(0)}$. For the integration of Eq. (2.3) it is necessary to make a definite choice for F_0. For restrictions on the choice of F_0 follow from the requirement that Φ_1 must be periodic in the angle φ and are found by averaging Eq. (2.3) over φ:

$$\left[\frac{d}{dt} + u_\parallel(\mathbf{h}\nabla) + \left(\mathbf{h}\mathbf{A}^{(0)} + \frac{u_\perp^2}{2}\operatorname{div}\mathbf{h} - u_\parallel \mathbf{h}(\mathbf{h}\nabla)\mathbf{V}\right)\frac{\partial}{\partial u_\parallel} - \frac{u_\perp}{2}(\delta_{\alpha\beta}^\perp \nabla_\alpha V_\beta + u_\parallel \operatorname{div}\mathbf{h})\frac{\partial}{\partial u_\perp}\right] F_0 = 0, \qquad (2.5)$$

where $\delta_{\alpha\beta}^\perp = \delta_{\alpha\beta} - h_\alpha h_\beta$. Solving this equation is not less complicated than integrating the original equation. One can note, however, that Eq. (2.5) can be considerably simplified in the case when the motion of the plasma is independent of the coordinates along the magnetic field. This assumption enables us to use for F_0 a two-temperature anisotropic Maxwell function—the simplest distribution which takes into account the evident anisotropy of a magneto-active plasma without collisions. We note that if we take for F_0 the usual isotropic Maxwell function, we would get from Eq. (2.5) an additional restriction on possible plasma flows:

$$\operatorname{div}\mathbf{V} = 3\mathbf{h}(\mathbf{h}\nabla)\mathbf{V}. \qquad (2.6)$$

We assume in what follows that F_0 has the form of an anisotropic two-temperature Maxwell distribution and we restrict ourselves to plasma flows which are independent of the longitudinal coordinates.

We look for the solution of Eq. (2.3) in the form $\Phi_1 = \tilde{\Phi}_1(\varphi) + \bar{\Phi}_1$, where $\bar{\Phi}_1$ is independent of φ and is necessary for finding the higher approximations considered in Sec. 3. The function $\tilde{\Phi}_1(\varphi)$ was found in[1] and is equal to $\tilde{\Phi}_1 = \Phi_{th} + \Phi_{vis}$, where

$$\Phi_{th} = \frac{\mathbf{u}}{\Omega_B}\left[\mathbf{h}, \left(1-2\frac{\varepsilon_\parallel}{T_\parallel}\right)\left(\frac{T_\perp - T_\parallel}{T_\perp}(\mathbf{h}\nabla)\mathbf{h} - \frac{1}{2}\nabla \ln T_\parallel\right) + \left(\frac{\varepsilon_\perp}{T_\perp} - 2\right)\nabla \ln T_\perp\right], \quad (2.7)$$

$$\Phi_{vis} = \frac{m u_\alpha u_\beta}{8\Omega_B T_\perp}\left\{ [(\delta_{\alpha\mu} + 3h_\alpha h_\mu)\varepsilon_{\beta\gamma\nu} + (\alpha \rightleftharpoons \beta)]\left(W_{\mu\nu} + u_\parallel \frac{T_\parallel - T_\perp}{T_\parallel} H_{\mu\nu}\right) h_\gamma + 4\frac{T_\perp - T_\parallel}{T_\parallel}[h_\alpha \varepsilon_{\beta\gamma\nu} + (\alpha \rightleftharpoons \beta)] h_\gamma (h_\mu \nabla_\nu V_\mu + u_\parallel (\mathbf{h}\nabla) h_\nu - \dot{h}_\nu)\right\},$$

$$\varepsilon_{\parallel,\perp} = m u^2_{\parallel,\perp}/2.$$

Here $W_{\mu\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\mu - \frac{2}{3}\delta_{\mu\nu}\,\mathrm{div}\mathbf{V}$ is the shear velocity tensor, $H_{\mu\nu}$ an analogous tensor formed from the components of $\mathbf{h}$, and $\varepsilon_{\alpha\beta\gamma}$ is the antisymmetric unit pseudo-tensor.

We now consider the second correction F_2. Substituting F_1 into the right-hand side of (2.2) we get for F_2 the equation

$$\Omega_B \frac{\partial}{\partial\varphi} F_2 = LF_1 + \mathbf{A}^{(1)} \frac{\partial}{\partial \mathbf{u}} F_0, \quad (2.8)$$

and, integrating it over the velocities $\mathbf{u}$, with weights $\mathbf{u}$, $2\varepsilon_\parallel$, and $\varepsilon_\perp$ we find the first corrections to Eqs. (2.4):

$$\dot{p}_\parallel^{(1)} = -\mathrm{div}\,\mathbf{q}_\parallel^{(1)} - 2\pi_*^{(1)} + 2\Pi^{(1)}, \quad \dot{p}_\perp^{(1)} = -\mathrm{div}\,\mathbf{q}_\perp^{(1)} - {}^1/{}_2 \pi_{\alpha\beta}^{(1)} W_{\alpha\beta} + \pi_*^{(1)} - \Pi^{(1)}$$

$$\mathbf{A}^{(1)} = \frac{\nabla \cdot \hat{\pi}^{(1)}}{mn}, \quad \pi_* = h_\gamma \pi_{\gamma\beta}^{(1)} (h_\alpha \nabla_\beta V_\alpha + \dot{h}_\beta). \quad (2.9)$$

Here $\hat{\pi}^{(1)}$ is the magnetic viscosity tensor:

$$\pi_{\alpha\beta}^{(1)} = m\int d^3u\, u_\alpha u_\beta F_1 = \frac{1}{4\Omega_B}[(\delta_{\alpha\mu} + 3h_\alpha h_\mu)\varepsilon_{\beta\gamma\nu} + (\alpha \rightleftharpoons \beta)] h_\gamma I_{\mu\nu}^{(1)},$$

$$I_{\mu\nu}^{(1)} = \dot{p}_{\mu\nu} + p_{\mu\eta}\nabla_\eta V_\nu + p_{\nu\eta}\nabla_\eta V_\mu. \quad (2.10)$$

The thermal currents $\mathbf{q}_{\parallel,\perp}^{(1)}$ are described by the expressions

$$\mathbf{q}_\parallel^{(1)} = 2\int d^3u\, \mathbf{u}\varepsilon_\parallel F_1 = \mathbf{q}_\parallel + \mathbf{h}\tilde{q}_\parallel, \quad \mathbf{q}_\perp^{(1)} = \int d^3u\, \mathbf{u}\varepsilon_\perp F_1 = \mathbf{q}_\perp + \mathbf{h}\tilde{q}_\perp,$$

$$\mathbf{q}_\parallel = T_\parallel \frac{p_\perp}{m\Omega_B}\left[\mathbf{h}, \nabla \ln T_\parallel - 2\frac{T_\perp - T_\parallel}{T_\perp}(\mathbf{h}\nabla)\mathbf{h}\right], \quad \mathbf{q}_\perp = 2\frac{p_\perp}{m\Omega_B}[\mathbf{h}\times\nabla T_\perp],$$

$$\tilde{q}_\parallel = 2\int d^3u\, u_\parallel \varepsilon_\parallel F_0 \Phi_1, \quad \tilde{q}_\perp = \int d^3u\, u_\parallel \varepsilon_\perp F_0 \Phi_1 \quad (2.11)$$

and, finally,

$$\Pi^{(1)} = \frac{1}{2} H_{\alpha\beta}\int d^3u\, u_\alpha u_\beta u_\parallel F_1 = \mathbf{q}_\parallel (\mathbf{h}\nabla)\mathbf{h}. \quad (2.12)$$

Using the definitions (2.10) to (2.12) we can write the first two equations of (2.9) in the form

$$\dot{p}_\parallel^{(1)} = -\mathrm{div}\,\mathbf{q}_\parallel^{(1)} + 2\mathbf{q}_\parallel(\mathbf{h}\nabla)\mathbf{h} - 2\frac{p_\parallel}{\tau_*}, \quad \dot{p}_\perp^{(1)} = -\mathrm{div}\,\mathbf{q}_\perp^{(1)} - \mathbf{q}_\parallel(\mathbf{h}\nabla)\mathbf{h} + \frac{p_\perp}{\tau_*}, \quad (2.13)$$

where

$$\frac{1}{\tau_*} = \frac{1}{\Omega_B}\{\mathbf{h}(\mathbf{h}\times[(\mathbf{h}\cdot\nabla)\mathbf{V} + \dot{\mathbf{h}}]\nabla)\mathbf{V} + [\mathbf{h}\times\dot{\mathbf{h}}](\mathbf{h}\nabla)\mathbf{V}\}.$$

In fact, all results obtained so far are contained in[1], written in the one-fluid approximation.

3. SECOND APPROXIMATION

To find the second order corrections we must solve Eq. (2.8), determining $\tilde{\Phi}_1$ beforehand. Using the equation for the third approximation

$$\Omega_B \frac{\partial}{\partial\varphi} F_3 = LF_2 + \mathbf{A}^{(1)}\frac{\partial}{\partial\mathbf{u}}F_1 + \mathbf{A}^{(2)}\frac{\partial}{\partial\mathbf{u}}F_0 \quad (3.1)$$

and integrating it over the velocities $\mathbf{u}$ with weights $\mathbf{u}$, $2\varepsilon_\parallel$, $\varepsilon_\perp$, as before, we find the second corrections to the moment Eqs. (2.4). It is clear that they have the same form as Eq. (2.9) which holds also for further approximations. It is only necessary to use in the definitions of $\mathbf{q}_{\parallel,\perp}^{(2)}$, $\hat{\pi}^{(2)}$, $\Pi^{(2)}$ F_2 instead of F_1 in Eqs. (2.10) to (2.12). It all thus reduces to the determination of the function $\tilde{\Phi}_1$. The equation for $\tilde{\Phi}_1$, like the restriction on F_0, follows from the condition that the second approximation be periodic. Integrating Eq. (2.8) over φ and using the independence of the longitudinal coordinates we get

$$\left(\frac{d}{dt} + \frac{\dot{T}_\parallel^{(0)}}{2T_\parallel}u_\parallel\frac{\partial}{\partial u_\parallel} + \frac{\dot{T}_\perp^{(0)}}{2T_\perp}u_\perp\frac{\partial}{\partial u_\perp}\right)\tilde{\Phi}_1(t, \mathbf{r}, u_\parallel, u_\perp) = -\chi, \quad (3.2)$$

where

$$\chi = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\left\{L\Phi_1 + \frac{\nabla\cdot\hat{\pi}^{(1)}}{mn}\frac{\partial}{\partial\mathbf{u}}\ln F_0\right\}. \quad (3.3)$$

We have assumed that initially $\tilde{\Phi}_1 = 0$. We note that owing to the assumed independence of the longitudinal coordinate terms with $\tilde{q}_{\parallel,\perp}$ drop out of Eqs. (2.9) of the first approximation. Therefore, if we had restricted ourselves to terms $\propto 1/\Omega_B$, as was done in[1], we could not evaluate $\tilde{\Phi}_1$. As the corrections $\propto 1/\Omega_B^2$ have formally the same form as Eqs. (2.9), for the same reason we need not consider the symmetric part of Φ_2. But in the second-order correction $\tilde{\Phi}_1$ occurs necessarily.

Taking this remark into account, it is sufficient to determine solely the transverse components of the thermal currents $\mathbf{q}_{\parallel,\perp}^{(2)}$. As for our purposes it is sufficient to calculate only the moments of the functions $\tilde{\Phi}_1$ and F_2, there is no necessity to solve the equations which determine them. We show in the Appendix that the second order corrections in the moment equations are determined by means of the function $\tilde{\Phi}_1$, obtained earlier, and Eq. (3.2) and they can be written in the form

$$\pi_{\alpha\beta}^{(2)} = \frac{1}{4\Omega_B}[(\delta_{\alpha\nu} + 3h_\alpha h_\nu)\varepsilon_{\beta\gamma\nu} + (\alpha \rightleftharpoons \beta)] I_{\mu\nu}^{(2)} h_\gamma, \quad (3.4)$$

$$\mathbf{q}_{\parallel,\perp}^{(2)} = [\mathbf{h}\mathbf{S}_{\parallel,\perp}]/\Omega_B, \quad (3.5)$$

$$\Pi^{(2)} = \mathbf{q}_\parallel^{(2)}(\mathbf{h}\nabla)\mathbf{h} - \frac{1}{2p_\perp}\frac{m}{T_\parallel - T_\perp}D_{\alpha\beta}G_{\alpha\beta}, \quad (3.6)$$

where we have used the notation

$$D_{\alpha\beta} = \frac{p_\perp}{4\Omega_B}\frac{T_\parallel - T_\perp}{m}[\delta_{\alpha\mu} + \varepsilon_{\beta\gamma\nu} + (\alpha \rightleftharpoons \beta)]h_\gamma H_{\mu\nu}. \quad (3.7)$$

Retaining in Eqs. (A.5) only terms which do not drop out identically when substituted into Eqs. (3.4) to (3.6) we find

$$I_{\mu\nu}^{(2)}=\dot{\pi}_{\mu\nu}^{(1)}+\pi_{\mu\nu}^{(1)}\operatorname{div}\mathbf{V}+\pi_{\mu\eta}^{(1)}\nabla_\eta V_\nu+\pi_{\nu\eta}^{(1)}\nabla_\eta V_\mu+{}^1/_2\Xi_{\mu\nu}+\delta I_{\mu\nu},$$
$$\delta I_{\mu\nu}=((\mathbf{q}_\parallel^{(1)}-{}^1/_2\mathbf{q}_\perp^{(+)})\nabla)h_\mu h_\nu+(\mathbf{h}\nabla)(\delta_{\mu\eta}h_\nu+\delta_{\nu\eta}h_\mu)(q_{\parallel\eta}-{}^1/_2 q_\eta^{(+)})$$
$$+\nabla_\eta(h_\eta D_{\mu\nu}+h_\nu D_{\eta\mu}+h_\mu D_{\eta\nu}). \qquad (3.8)$$

Here $\mathbf{q}^{(+)}=\mathbf{q}_\perp^{(1)}+\tilde{q}_\perp\mathbf{h}$, and the tensor $\Xi_{\mu\nu}$ is similar to $W_{\mu\nu}$ and formed from the vector $\mathbf{q}^{(+)}$. This tensor takes into account the contribution from the derivatives of the thermal currents in the viscosity tensor. We have split off in the tensor (3.8) the part $\delta I_{\mu\nu}$ which in an important way depends on the derivatives of the magnetic field. One can show that $\delta I_{\mu\nu}\equiv 0$ for a uniform field. For the vectors $\mathbf{S}_{\perp,\parallel}$ of (3.5) we find

$$\mathbf{S}_{\parallel i}=p_\parallel\frac{d}{dt}\frac{\mathbf{q}_\parallel}{p_\parallel}+(\mathbf{q}_\parallel^{(1)}\nabla)\mathbf{V}+(\tilde{q}_\parallel-2\tilde{q}_\perp)\dot{\mathbf{h}}+2\tilde{q}_\perp h_\alpha\nabla_i V_\alpha$$
$$+\frac{\nabla\Pi^\perp}{m}+\frac{\Pi^\parallel-3\Pi^\perp}{m}(\mathbf{h}\nabla)\mathbf{h}+2D_{i\beta}(h_\alpha\nabla_\beta V_\alpha-\dot{h}_\beta)$$
$$+\frac{T_\parallel}{m}\left[\pi_{i\alpha}^{(1)}\nabla_\alpha\ln T_\parallel+2h_\alpha(\mathbf{h}\nabla)\pi_{i\alpha}^{(1)}+2h_\alpha\pi_{\alpha\beta}^{(1)}\left(\nabla_\beta h_i-\frac{T_\perp}{T_\parallel}H_{i\beta}\right)\right] \qquad (3.9)$$

$$\mathbf{S}_{\perp i}=nT_\perp^{1/2}\frac{d}{dt}\frac{\mathbf{q}_\perp}{nT_\perp^{1/2}}+\frac{1}{2}q_{\perp\alpha}(3\nabla_\alpha V_i+\nabla_i V_\alpha)+2\frac{\Pi^\perp-\Theta}{m}(\mathbf{h}\nabla)\mathbf{h}+2\frac{\nabla\Theta}{m}$$
$$+D_{i\beta}(\dot{h}_\beta+(\mathbf{h}\nabla)V_\beta)+\frac{T_\perp}{m}\left[\delta_{\alpha\beta}{}^\perp\nabla_\beta\pi_{i\alpha}^{(1)}+h_\beta\pi_{\beta\alpha}^{(1)}\nabla_i h_\alpha+\pi_{i\alpha}^{(1)}\nabla_\alpha\ln\frac{T_\perp^2}{n}\right], \qquad (3.10)$$

and for the tensor $G_{\mu\nu}$ we get the following expression:

$$G_{\mu\nu}=nT_\parallel^{1/2}\frac{d}{dt}\frac{D_{\mu\nu}}{nT_\parallel^{1/2}}+\frac{T_\parallel}{m}(\mathbf{h}\nabla)\pi_{\mu\nu}^{(1)}+\tilde{q}_\perp W_{\mu\nu}+\left\{\left[\frac{1}{2}(\dot{h}_\mu+h_\beta\nabla_\mu V_\beta)q_{\perp\nu}\right.\right.$$
$$\left.\left.+q_{\parallel\nu}(\mathbf{h}\nabla)V_\mu+D_{\nu\alpha}\nabla_\alpha V_\mu+\frac{T_\parallel-T}{m}\pi_{\nu\alpha}^{(1)}\nabla_\alpha h_\mu+\frac{p_\perp}{m}h_\alpha\nabla_\nu\frac{\pi_{\mu\alpha}^{(1)}}{n}\right]+[\mu\rightleftarrows\nu]\right\}. \qquad (3.11)$$

The functions $\tilde{q}_{\perp,\parallel}$, $\Pi^{\perp,\parallel}$, and Θ which occur in Eqs. (3.9) to (3.11) are determined in terms of the function $\bar{\Phi}_1$ (see also (2.11))

$$\Pi^{(\perp,\parallel)}=2\int d^3u\,F_0\Phi_1\varepsilon_\parallel\cdot(\varepsilon_\perp,2\varepsilon_\parallel),\qquad \Theta=\frac{1}{2}\int d^3u\,F_0\Phi_1\varepsilon_\perp^2 \qquad (3.12)$$

they satisfy, as one can easily prove by using Eq. (3.2), the relations

$$nT_\parallel^{3/2}\frac{d}{dt}\frac{\tilde{q}_\parallel}{nT_\parallel^{3/2}}=-\mathbf{h}\mathbf{S}_\parallel^I,\qquad n^2\frac{d}{dt}\frac{\tilde{q}_\perp}{n^2}=-\mathbf{h}\mathbf{S}_\perp^I,$$
$$n^2T_\parallel^2\frac{d}{dt}\frac{\Pi^\parallel}{n^2T_\parallel^2}=-\Psi_\parallel,\qquad n^2T_\parallel^{1/2}\frac{d}{dt}\frac{\Pi^\perp}{n^2T_\parallel^{1/2}}=-\Psi_\perp, \qquad (3.13)$$
$$nT_\perp^2\frac{d}{dt}\frac{\Theta}{nT_\perp^2}=-\Psi_\Theta,$$

where the index $(\ldots)^I$ indicates the evaluation of the quantities neglecting $\bar{\Phi}_1$.

Explicitly the right-hand side of the equations can be written in the form

$$\mathbf{h}\mathbf{S}_\parallel^I=3a_\parallel\{4\varkappa(\mathbf{h}\nabla)\mathbf{V}-[(\dot{\mathbf{h}}+(\mathbf{h}\nabla)\mathbf{V})\times\mathbf{h}]\nabla\ln T_\parallel\},$$
$$\mathbf{h}\mathbf{S}_\perp^I=a_\perp\left\{[(\dot{\mathbf{h}}+(\mathbf{h}\nabla)\mathbf{V})\times\mathbf{h}]\left(\nabla\ln\frac{T_\parallel}{T_\perp^2}-3(\mathbf{h}\nabla)\mathbf{h}\right)\right.$$
$$\left.-\frac{T_\perp+2T_\parallel}{T_\parallel}(\mathbf{h}(\varkappa\nabla)\mathbf{V}-\varkappa\dot{\mathbf{h}})\right\}+\frac{p_\perp T_\perp}{m}h_\alpha\nabla_\beta\frac{\pi_{\alpha\beta}^{(1)}}{p_\perp},$$
$$\Psi_\parallel=6T_\parallel\left\{\frac{1}{T_\parallel}\operatorname{div}T_\parallel\mathbf{q}_\parallel-4\mathbf{q}_\parallel(\mathbf{h}\nabla)\mathbf{h}+\frac{p_\parallel}{\tau_*}\right\}, \qquad (3.14)$$
$$\Psi_\perp=T_\perp\left\{\frac{1}{T_\perp}\operatorname{div}(2T_\perp\mathbf{q}_\parallel+T_\parallel\mathbf{q}_\perp)\right.$$
$$\left.-2\left(\frac{2T_\perp-3T_\parallel}{T_\perp}\mathbf{q}_\parallel+\frac{T_\parallel}{T_\perp}\mathbf{q}_\perp\right)(\mathbf{h}\nabla)\mathbf{h}-q_{\parallel\alpha}\frac{\nabla_\beta p_{\alpha\beta}}{p_\perp}+\frac{p_\parallel}{\tau_*}\right\},$$
$$\Psi_\Theta=T_\perp\left\{3\operatorname{div}\mathbf{q}_\perp+3\frac{T_\parallel}{T_\perp}\mathbf{q}_\parallel(\mathbf{h}\nabla)\mathbf{h}\right.$$
$$\left.-q_{\perp\alpha}\frac{\nabla_\beta p_{\alpha\beta}}{p_\perp}+\frac{3}{2}\frac{T_\parallel}{T_\perp}h_\alpha\dot{h}_\beta\pi_{\alpha\beta}^{(1)}+2\frac{p_\parallel-p_\perp}{\tau_*}\right\}.$$

To simplify the notation we have in Eqs. (3.14) introduced the notation: $\varkappa=[\mathbf{h}\times(\mathbf{h}\nabla)\mathbf{h}]$, $a_\parallel=T_\parallel p_\parallel/m\Omega_B$, $a_\perp=T_\perp p_\parallel/m\Omega_B$; τ_* is given by Eq. (2.13). In deriving Eqs. (3.14) we have used Eqs. (2.10), (2.11) which determine $\pi_{\alpha\beta}^{(1)}$, $\mathbf{q}_{\parallel,\perp}$ and the identities

$$W_{\alpha\beta}\pi_{\alpha\beta}^{(1)}=-2\frac{T_\parallel-T_\perp}{T_\perp}(\dot{h}_\alpha+(\mathbf{h}\nabla)V_\alpha)h_\beta\pi_{\alpha\beta}^{(1)}=2\frac{p_\parallel-p_\perp}{\tau_*},$$
$$D_{\alpha\beta}\pi_{\alpha\beta}^{(1)}=-2\frac{T_\parallel-T_\perp}{m}\pi_{\alpha\beta}^{(1)}\delta_{\alpha\gamma}{}^\perp\nabla_\gamma h_\beta. \qquad (3.15)$$

We note moreover that in Eqs. (3.9) to (3.15) we must use for the calculation of the total derivatives of $T_{\parallel,\perp}$ the moment Eqs. (2.4) in the zeroth approximation. Moreover, bearing in mind the remark made for Eq. (3.2), we must determine the solutions of Eqs. (3.13) with the zeroth initial conditions. This means that if for some reason or other the functions (3.14) are identically zero, the corresponding moments are also equal to zero.

The complete set of MHD equations is obtained by summing the corrections of all orders obtained above; together with the Maxwell equations it describes the ion and electron components of the plasma taking into account terms $\propto 1/\Omega_B^2$ in the case where collisions may be neglected. The only restriction which allows us to obtain the simplest solution which is explicitly expressed in most terms in terms of the first moments of the distribution function consists in the requirement that the plasma flow be independent of the longitudinal coordinate.

To compare the equations obtained here with the results of[1] in which the collisionless MHD equations were obtained by the moment method with a consistent assumption about the symmetry of the higher moments we consider the particular case $T_\parallel=\text{const}$, $B=\text{const}$, $V_\parallel=0$. The complete set of equations then simplifies considerably and becomes

$$\dot{n}=-n\operatorname{div}\mathbf{V},\quad mn\mathrm{A}_\beta=\nabla_\beta p_\perp+\nabla_\alpha\pi_{\alpha\beta},$$
$$\dot{p}_\perp=-\operatorname{div}\mathbf{q}-\pi_{\alpha\beta}\nabla_\alpha V_\beta+2p_\perp\dot{n}/n, \qquad (3.16)$$

where

$$\pi_{\alpha\beta}=[\delta_{\alpha\mu}e_{\beta\gamma\nu}+(\alpha\rightleftarrows\beta)]h_\gamma I_{\mu\nu}/4\Omega_B,$$
$$I_{\mu\nu}=p_\perp W_{\mu\nu}+n\frac{d}{dt}\frac{\pi_{\mu\nu}^{(1)}}{n}+\pi_{\alpha\mu}^{(1)}\nabla_\alpha V_\nu+\pi_{\alpha\nu}^{(1)}\nabla_\alpha V_\mu+\frac{1}{2}\Xi_{\mu\nu}|_{\tilde{q}_\perp=0},$$
$$\mathbf{q}=[\mathbf{h}\times\mathbf{S}]/\Omega_B, \qquad (3.17)$$
$$\mathbf{S}_i=\dot{\mathbf{q}}_\perp+\frac{3}{2}\left\{(\mathbf{q}_\perp\nabla)\mathbf{V}+\mathbf{q}_\perp\operatorname{div}\mathbf{V}+\frac{1}{3}q_{\perp\alpha}\nabla_i V_\alpha\right\}$$
$$+2\frac{p_\perp}{m}\nabla T_\perp+\frac{n}{mT_\perp}\nabla_\alpha\frac{T_\perp^2}{n}\pi_{i\alpha}^{(1)}.$$

We have omitted in Eqs. (3.17) terms with $\nabla\Theta$ and $\nabla\Pi^\perp$, which drop out of Eqs. (3.16). One sees easily that up to terms $\propto 1/\Omega_B^3$ Eqs. (3.16), (3.17) are the same as the corresponding equations of [3].

4. ONE-FLUID APPROXIMATION

In the one-fluid approximation the set of MHD equations simplifies. Putting $e_i = -e_e = e_e$, $n_e = n_i = n$, $\hat{p}_e = \hat{p}_i = \frac{1}{2}\hat{p}$ we consider a single-charged neutral plasma with equal ion and electron pressures. Changing in the electron equation of motion to the limit $m_e \to 0$, we find Ohm's law:

$$\mathbf{E}+\left[\frac{\mathbf{V}}{c}\times\mathbf{B}\right]=-\frac{1}{ne}\left\{\frac{\nabla\cdot\hat{p}}{2}-\frac{1}{c}\,[\mathbf{j}\times\mathbf{B}]\right\},\mathbf{V}=\mathbf{V}_i,\quad \mathbf{j}=en(\mathbf{V}_i-\mathbf{V}_e), \tag{4.1}$$

which takes into account the anisotropic pressure and the Hall effect. It is clear from Eqs. (2.10) and (2.11) which determine the viscosity tensor and the thermal currents in first order that $\mathbf{q}_{\parallel,\perp}^{(e)} = -\mathbf{q}_{\parallel,\perp}^{(i)}$ and $\hat{\pi}_1^{(e)} \approx 0$ as $m_e \to 0$. Adding the ion and electron equations of motion we get therefore

$$nM\dot{\mathbf{V}}=-\nabla\cdot(\hat{p}+\hat{\pi})+[\mathbf{j}\times\mathbf{B}]/c+Mn\mathbf{g}\,, \tag{4.2}$$

where $\hat{\pi} \equiv \hat{\pi}^{(i)} = \hat{\pi}^{(1)} + \hat{\pi}^{(2)}$ is the total ion viscosity tensor. We have added to the right-hand side of Eq. (4.2) a term taking into account the gravitational acceleration $\mathbf{g}$. We can derive similarly the equations for the longitudinal and transverse pressure components. We assume that the relative velocity of the plasma components is small compared to the mass velocity and then

$$\dot{p}_\parallel=(\dot{p}_\parallel)_{\mathrm{ad}}-\operatorname{div}\mathbf{q}_\parallel{}^\Sigma-2\pi_*+2\Pi_\Sigma. \tag{4.3}$$

$$\dot{p}_\perp=(\dot{p}_\perp)_{\mathrm{ad}}-\operatorname{div}\mathbf{q}_\perp{}^\Sigma-{}^1/_2\pi_{\alpha\beta}W_{\alpha\beta}+\pi_*-\Pi_\Sigma,$$

where

$$\mathbf{q}_{\parallel,\perp}^{\Sigma}=\sum_{i,e}\mathbf{q}_{\parallel,\perp}^{(2)},\qquad \Pi_\Sigma=\sum_{i,e}\Pi^{(2)},\qquad \pi_*=h_\alpha\pi_{\alpha\beta}(h_\gamma\nabla_\beta V_\gamma-\dot{h}_\beta).$$

We shall not write down expressions for the right-hand sides of (4.3) as the summation does not lead to an appreciable simplification of the notation. We note merely that the currents $\mathbf{q}^\Sigma$ are given by Eqs. (3.5), (3.9), (3.10) in which we need take into account only the ion terms and instead of $\Pi^{\perp,\parallel}$ and Θ use the difference between the ion and electron contributions to these functions.

If we neglect displacement currents, we must add to the set (4.2), (4.3) the equations

$$\operatorname{rot}\mathbf{B}=\frac{4\pi\mathbf{j}}{c},\qquad \operatorname{div}\mathbf{B}=0. \tag{4.4}$$

Hence, when we neglect the Hall effect and the contribution from the pressure gradient we get the freezing in equation:

$$\dot{\mathbf{B}}=-\mathbf{B}\operatorname{div}\mathbf{V}+(\mathbf{B}\nabla)\mathbf{V}. \tag{4.5}$$

Using the continuity equation

$$\dot{n}=-n\operatorname{div}\mathbf{V}, \tag{4.6}$$

one can show that the energy conservation law is satisfied in the set of Eqs. (4.2), (4.3), (4.5), and (4.6) ($g \equiv 0$);

$$\frac{\partial}{\partial t}\int d^3\mathbf{r}\left(Mn\frac{V^2}{2}+\frac{p_\parallel+2p_\perp}{2}+\frac{B^2}{8\pi}\right)$$
$$=-\oint dS_\alpha\left(\frac{B^2}{8\pi}\mathbf{V}-\frac{B^2}{4\pi}\mathbf{h}(\mathbf{hV})+V_\beta(\pi_{\alpha\beta}+p_{\alpha\beta})+\mathbf{q}_\perp{}^\Sigma+\frac{1}{2}\mathbf{q}_\parallel{}^\Sigma\right). \tag{4.7}$$

The integral is here over the volume occupied by the plasma.

We show that the set of Eqs. (4.2), (4.3), (4.5), and (4.6) satisfies the entropy conservation law

$$s=-\int d^3u\,F\ln F. \tag{4.8}$$

Let

$$s_0=-\int d^3u\,F_0\ln F_0$$

be the entropy density in zeroth approximation. Expanding F in a series in the corrections, we find $s = s_0 - \delta s$, where

$$\delta s=\frac{1}{2}\int d^3u\,F_0\Phi_1{}^2>0. \tag{4.9}$$

It follows from Eq. (4.9) that the correction δs is of second order in $1/\Omega_B$ and we must therefore in the system of moments, taking into account the first approximation, retain s_0, as in that case

$$\partial s_0/\partial t+\operatorname{div}s_0\mathbf{V}=0. \tag{4.10}$$

We consider second order terms. Evaluating the total time derivative of δs and using the equations for the functions $F_{1,2}$, we find

$$\frac{d\delta s}{dt}+\delta s\cdot\operatorname{div}\mathbf{V}=-\operatorname{div}\boldsymbol{\lambda}-I, \tag{4.11}$$

where

$$\boldsymbol{\lambda}=\int d^3u\,\mathbf{u}F_0\Phi_1{}^2,\qquad I=\int d^3u\,\Omega_B F_2\frac{\partial}{\partial\varphi}\Phi_1. \tag{4.12}$$

Using Eqs. (2.3), (2.8) we can express the integral I in terms of $\pi_{\alpha\beta}^{(2)}$, $\mathbf{q}^{(2)}$, and $\Pi^{(2)}$. Using then Eqs. (4.3) and (4.10) one can easily show that the combination obtained can be brought to the form

$$-I=\frac{\partial}{\partial t}s_0+\operatorname{div}\left(s_0\mathbf{V}+\frac{2}{T_\perp}\mathbf{q}_\perp{}^\Sigma+\frac{1}{T_\parallel}\mathbf{q}_\parallel{}^\Sigma\right) \tag{4.13}$$

independent of the actual expression for $\pi_{\alpha\beta}^{(2)}$, $\mathbf{q}^{(2)}$, and $\Pi^{(2)}$. Combining Eqs. (4.11) and (4.13) we find the entropy conservation law:

$$\frac{\partial}{\partial t}s+\operatorname{div}\left(s\mathbf{V}+\frac{2}{T_\perp}\mathbf{q}_\perp{}^\Sigma+\frac{1}{T_\parallel}\mathbf{q}_\parallel{}^\Sigma-\boldsymbol{\lambda}\right)=0. \tag{4.14}$$

The vector $\boldsymbol{\lambda}$ in Eq. (14) describes the total contribution from the ions and electrons. By using the identities (A.1) and Eq. (2.3) we can find the actual expression for $\boldsymbol{\lambda}$ for each of the components.

In conclusion we use the set (4.2), (4.3), (4.5), and

(4.6) to consider drift waves of a plane plasma layer. Let in the equilibrium state $T_{\parallel}^0 = T_{\perp}^0 = T(x)$, $\dot{n} = n(x)$, $\mathbf{B}^0 = B(x)e_y$, $\mathbf{g} = g\mathbf{e}_e$, $\Theta^0 = \Pi^0_{\parallel,\perp} = q^0_{\parallel,\perp} = 0$. Using $p^i = \frac{1}{2}p$ for the ion thermal currents (2.11) we then have

$$\mathbf{q}_{\parallel}{}^0 = {}^1/_2\mathbf{q}_{\perp}{}^0 = \mathbf{e}_z q, \quad q = -\eta T'(x)/m, \tag{4.15}$$

where $\eta = p/4\Omega_{Bi}$ is the magnetic viscosity coefficient.

From Eqs. (4.2) and (4.5) we find the equilibrium condition

$$\frac{d}{dx}\left(P + \frac{1}{2}\frac{1}{\Omega_{Bi}}\frac{d}{dx}q\right) = \rho g, \quad \rho = Mn, \quad P = p + \frac{B^2}{8\pi} \tag{4.16}$$

If we now consider small perturbations of the form $f(x)\exp(-i\omega t + ikz)$, we can show that for slow waves, satisfying the condition $\omega^2/k^2 \ll 2P/\rho$ (see[1]) it follows from Eqs. (4.2), (4.3), (4.5), and (4.6) (using the fact that $\omega/\Omega_{Bi} \ll 1$) that

$$\frac{d}{dx}\left\{\rho\left[1 - 2a + \frac{k^2}{\omega^2}\frac{q}{\rho}\left(\frac{1}{\Omega_{Bi}}\right)'\right]V_x'\right\}$$
$$-k^2\rho V_x\left[1 - 2a + \frac{k^2}{\omega^2}\frac{q}{\rho}\left(\frac{1}{\Omega_B}\right)' - \frac{g}{\omega^2}\frac{\rho'}{\rho}\right] = 0, \tag{4.17}$$

where we have put $a = k\eta'/\omega\rho$. If $T' \approx 0$, the current terms in (4.17) as should be expected, drop out and we arrive at the well known equation from the theory of drift waves.[7] As the current terms occur in combination with derivatives of the magnetic field, they will clearly be unimportant when $\beta = 8\pi p/B^2 \ll 1$.

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APPENDIX: THE DERIVATION OF THE SECOND ORDER CORRECTIONS FOR THE VISCOSITY TENSOR AND THE THERMAL CURRENTS

For the derivation of the second-order corrections it is convenient to use the following directly verifiable identities for arbitrary velocity-independent tensors C_α, $M_{\alpha\beta}$, $P_{\alpha\beta\gamma}$:

$$u_\alpha{}^\perp C_\alpha = \frac{\partial}{\partial\varphi}\mathbf{u}\,[\mathbf{h}\times\mathbf{C}],$$
$$u_{\alpha\beta}{}^\perp M_{\alpha\beta} = \frac{1}{4}\frac{\partial}{\partial\varphi}u_\alpha u_\beta(\delta_{\alpha\mu}{}^\perp\varepsilon_{\beta\gamma\nu} + \delta_{\alpha\nu}{}^\perp\varepsilon_{\beta\gamma\mu})h_\gamma M_{\mu\nu},$$
$$P_{\alpha\beta\gamma}u_\alpha{}^\perp u_\beta{}^\perp u_\gamma{}^\perp = \frac{1}{6}\frac{\partial}{\partial\varphi}\{u_\alpha u_\beta u_\gamma{}^\perp(\delta_{\alpha\mu}{}^\perp\varepsilon_{\beta\eta\nu} + \delta_{\alpha\nu}{}^\perp\varepsilon_{\beta\eta\mu})h_\eta P_{\mu\nu\gamma}$$
$$+ u_\perp{}^2 u_\mu\varepsilon_{\mu\eta\alpha}h_\eta(\delta_{\gamma\beta}{}^\perp P_{\alpha\gamma\beta} + \delta_{\gamma\beta}{}^\perp P_{\beta\alpha\gamma})\},$$
$$u_\mu{}^\perp u_\nu{}^\perp + [\mathbf{h}\times\mathbf{u}]_\mu[\mathbf{h}\times\mathbf{u}]_\nu = u_\perp^2\delta^\perp_{\mu\nu}, \tag{A.1}$$

where $u^\perp_{\alpha\beta} = u^\perp_\alpha u^\perp_\beta - \frac{1}{2}\delta^\perp_{\alpha\beta}u^2_\perp$. When evaluating $\mathbf{q}^{(2)}_{\parallel,\perp}$, $\pi^{(2)}_{\alpha\beta}$, $\Pi^{(2)}$ it is necessary to consider integrals which are similar to Eqs. (2.10) to (2.12) with the function F_2. Using Eq. (2.8) for F_2 we can reduce them to combinations of integrals of the function F_1. For instance, for the calculation of the transverse components $\mathbf{q}_\perp^{(2)}$ we use the transformation

$$\mathbf{C}\mathbf{q}_\perp^{(2)} = \int d^3u\,\varepsilon_\perp\mathbf{C}u^\perp F_2 = \int d^3u\,\varepsilon_\perp F_2\frac{\partial}{\partial\varphi}\mathbf{u}\,[\mathbf{h}\times\mathbf{C}] = \mathbf{C}\int d^3u\,\varepsilon_\perp[\mathbf{h}\times\mathbf{u}]\frac{\partial}{\partial\varphi}F_2. \tag{A.2}$$

Consider Eq. (2.8) we find then

$$\mathbf{q}_\perp^{(2)} = [\mathbf{h}\times\mathbf{S}_\perp]/\Omega_B, \tag{A.3}$$

where

$$\mathbf{S}_\perp = \int d^3u\,\mathbf{u}\varepsilon_\perp\left[LF_1 + \frac{\nabla\cdot\pi^{(1)}}{mn}\frac{\partial}{\partial\mathbf{u}}F_0\right]. \tag{A.4}$$

Using similar transformations for $\mathbf{q}_\parallel^{(2)}$, $\pi^{(2)}_{\alpha\beta}$, $\Pi^{(2)}$ we can write down the second corrections to Eqs. (3.4) to (3.7). Transforming integrals such as (A.4) we find expressions for the functions occurring in them:

$$I^{(2)}_{\mu\nu} = \dot{\pi}^{(1)}_{\mu\nu} + \pi^{(1)}_{\mu\nu}\operatorname{div}\mathbf{V} + \pi^{(1)}_{\alpha\mu}\nabla_\alpha V_\nu + \pi^{(1)}_{\alpha\nu}\nabla_\alpha V_\mu + \nabla_\alpha\Sigma_{\nu\mu\alpha},$$
$$S_{\parallel i} = -\dot{\mathbf{q}}_\parallel^{(1)} + 2h_\alpha\dot{Q}_{\alpha i} - \nabla_\alpha\Pi^\parallel_{i\alpha} + 2h_\alpha\nabla_\beta P_{i\alpha\beta} + (\mathbf{q}_\parallel^{(1)}\nabla)\mathbf{V}$$
$$+ \mathbf{q}_\parallel^{(1)}\operatorname{div}\mathbf{V} + 2Q_{i\beta}h_\alpha\nabla_\beta V_\alpha - \frac{T_\parallel}{m}\nabla_\alpha\pi^{(1)}_{i\alpha},$$
$$\mathbf{S}_{\perp i} = \frac{d}{dt}(\mathbf{q}_\perp^{(1)} + \mathbf{q}_\parallel^{(1)}) - h_\alpha\dot{Q}_{\alpha i} - \frac{1}{mn}\pi^{(1)}_{i\alpha}\nabla_\beta p_{\alpha\beta} + \nabla_\alpha(\Pi_{i\alpha}{}^\perp + \Pi_{i\alpha}{}^\parallel)$$
$$-h_\alpha\nabla_\beta P_{i\alpha\beta} + (\mathbf{q}_\perp^{(1)}\nabla)\mathbf{V} + \mathbf{q}_\perp^{(1)}\operatorname{div}\mathbf{V} + (\Sigma_{i\alpha\beta} - h_\beta Q_{i\alpha})\nabla_\alpha V_\beta - 2\frac{T_\perp}{m}\nabla_\alpha\pi^{(1)}_{i\alpha},$$
$$G_{\mu\nu} = h_\alpha\dot{\Sigma}_{\mu\nu\alpha} + \Sigma_{\mu\nu\alpha}h_\beta\nabla_\alpha V_\beta + Q_{\mu\nu}\operatorname{div}\mathbf{V} + h_\alpha\nabla_\beta\Theta_{\mu\nu\alpha\beta}$$
$$-\frac{1}{mn}[(\delta_{\mu\alpha}h_\beta\pi^{(1)}_{\nu\beta} + (\mu\rightleftarrows\nu))(\nabla\cdot\hat{p})_\alpha + (\delta_{\mu\alpha}h_\beta p_{\beta\nu} + (\mu\rightleftarrows\nu))(\nabla\cdot\pi^{(1)})_\alpha]$$
$$+[\delta_{\beta\mu}Q_{\nu\alpha} + (\nu\rightleftarrows\mu)]\nabla_\alpha V_\beta. \tag{A.5}$$

In Eqs. (A.5) we have written

$$\Theta_{\alpha\beta\gamma\nu} = m\int d^3u\,u_\alpha u_\beta u_\gamma u_\nu F_1, \quad P_{\alpha\beta\gamma} = h_\nu\Theta_{\alpha\beta\gamma\nu},$$
$$\Pi_{\alpha\beta}{}^\parallel = h_\gamma P_{\alpha\beta\gamma}, \quad \Pi_{\alpha\beta}{}^\perp = \frac{1}{2}\delta_{\gamma\nu}{}^\perp\Theta_{\alpha\beta\gamma\nu},$$
$$\Sigma_{\alpha\beta\gamma} = m\int d^3u\,u_\alpha u_\beta u_\gamma F_1, \quad Q_{\alpha\beta} = h_\gamma\Sigma_{\alpha\beta\gamma}. \tag{A.6}$$

As the periodic part of F_1 is known (see (2.7)) the contribution from it to the tensors (A.6) can be completely calculated as it reduces to moments of the function F_0. When evaluating these integrals we use the identities

$$\Delta^{(2n)} = h_{i_1}\frac{T_\parallel - T_\perp}{T_\parallel}\Delta^{(2n)}_{\parallel,i_2,\ldots,i_{2n}} + \frac{T_\perp}{m}\sum_s\delta_{i_1 i_s}\Delta^{2(n-1)}_{i_2,\ldots,i_{s-1},i_{s+1},\ldots,i_{2n}},$$
$$\Delta_\parallel^{(2n)} = h_{i_2}\left(\frac{T_\parallel - T_\perp}{T_\parallel}A\frac{2T_\parallel^2}{m}\frac{\partial}{\partial T_\parallel}\frac{1}{A} + \frac{T_\perp}{m}\right)\Delta^{2(n-1)}_{i_3,\ldots,i_{2n}}$$
$$+\frac{T_\perp}{m}\sum_s\delta_{i_2 i_s}\Delta^{2(n-1)}_{\parallel,i_3,\ldots,i_{s-1},i_{s+1},\ldots,i_{2n}},$$
$$\Delta^{(2n+1)} \equiv 0, \tag{A.7}$$

where we have put

$$\Delta^{(n)} = \langle u_{i_1}\cdot\ldots\cdot u_{i_n}\rangle, \quad \Delta_\parallel^{(n)} = \langle u_\parallel\cdot u_{i_2}\cdot\ldots\cdot u_{i_n}\rangle,$$
$$\langle(\ldots)\rangle = \int d^3u(\ldots)F_0, \quad A = 1/T_\perp\sqrt{T_\parallel}.$$

We then have

$$\Delta^{(4)}_{\parallel\alpha\beta\gamma} = \frac{T_\perp p_\parallel}{m^2}\left[\delta_{\alpha\beta}h_\gamma + \delta_{\alpha\gamma}h_\beta + \delta_{\beta\gamma}h_\alpha + 3\frac{T_\parallel - T_\perp}{T_\perp}h_\alpha h_\beta h_\gamma\right], \tag{A.8}$$
$$\Delta^{(4)}_{\alpha\beta\gamma\eta} = h_\eta\frac{T_\parallel - T_\perp}{T_\parallel}\Delta_{\parallel\alpha\beta\gamma} + \frac{T_\perp}{m^2}(\delta_{\eta\alpha}p_{\beta\gamma} + \delta_{\eta\beta}p_{\alpha\gamma} + \delta_{\eta\gamma}p_{\alpha\beta}).$$

The components of the tensors (A.6), determined in terms of the function $F_0\bar{\Phi}_1$, reduce to moments of that function satisfying the equations given in the text.

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Interaction between a beam with a supercritical initial velocity and finite-amplitude waves

A. S. Bakaĭ, S. S. Krivulya, I. N. Onishchenko, and V. D. Fedorchenko

Physico-technical Institute, Ukrainian Academy of Sciences
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Results are presented of an investigation of the instability of finite-amplitude waves in a system consisting of an electron beam with a supercritical velocity and a bounded plasma. It is shown that the beam with the supercritical velocity ($v_b > v_c \approx \omega_p/k_\perp$) can transfer up to 60% of its energy to the wave. Results are presented of a computer simulation of the amplification process and compared with the results of experiment.

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The instability of waves of finite amplitude in a system consisting of an electron beam with a supercritical velocity and a bounded plasma was discovered theoretically in[1] and then observed experimentally in[2]. The theoretical analysis shows that beams with supercritical velocities, in the course of their interaction with an amplifiable wave, transfer to the wave a considerably higher energy than do beams of the same density, but with subcritical velocities, as a result of which beams with supercritical velocities are more efficient means of wave amplification. Therefore, the investigation of the development of the instability of waves with finite amplitudes in such systems is of interest both for plasma physics and for practical application.

The present report contains the results of a further study of the process of interaction of finite-amplitude waves with beams whose initial velocities exceed the critical velocity, which is equal to the maximum phase velocity of the wave in the bounded plasma.

1. It is well known that waves with infinitely small amplitudes are not intensified by a beam in a bounded plasma if the initial velocity of the beam exceeds the maximum possible phase velocity of the plasma waves. This can easily be seen from the dispersion equation for the waves. In the case of magnetized electrons ($\omega_H > \omega_p$, where ω_H is the electron cyclotron, and ω_p the electron plasma, frequency), the electron motion can be assumed in the description of the potential oscillations to be one-dimensional. In this case the dispersion equation has the following form (see[3]):

$$\frac{\omega_b^2}{(\omega - k_\parallel v_b)^2} + \frac{\omega_p^2}{\omega^2} = 1 + \frac{k_\perp^2}{k_\parallel^2}, \qquad (1)$$

where v_b is the beam velocity; $\omega_b^2 = 4\pi e^2 n_b/m$, n_b being the beam density; e, m are the electron charge and mass; $k_\parallel$ and $k_\perp = \pi/d$ are the longitudinal and transverse components of the wave vector, d denoting the transverse dimensions of the plasma.

This equation does not possess complex solutions when $v_b > v_c \approx \omega_p/k_\perp$. This means that waves with infinitely small amplitudes are not intensified by a beam in a bounded plasma if the initial velocity of the beam exceeds the maximum possible phase velocity, v_c, of the plasma waves.

As shown in[1], the system, which is stable in the linear approximation, loses its stability if the amplitude of an initial perturbation exceeds some critical value at which the wave begins to capture the beam electrons. The value of the critical amplitude depends on the velocity of propagation of the bare wave in the system.

As is well known, the characteristic time, τ, during which the velocity of a particle can change significantly under the influence of the field of a wave with amplitude φ_0 is equal in order of magnitude to

$$\tau \sim k_\parallel^{-1} (e\varphi_0/m)^{-1/2}, \qquad (2)$$

if the initial particle velocity is not very far from the phase velocity of the wave. Therefore, if the amplitude of the bare wave in the course of the wave's excitation increases for a period of time much shorter than τ, then the particle velocities do not have time to change appreciably during this period, and the beam can be considered to be monoenergetic. Since the velocities of the beam particles at the initial moment exceed the

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