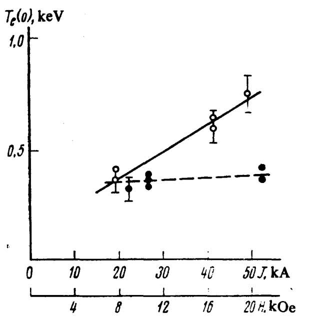

FIG. 7. Dependence of $T_e(0)$ on the magnitude of the discharge current (continuous curve) and on the longitudinal magnetic field (the dashed curve) for fixed $\bar{n}_e \approx 1.4\times 10^{13}$ cm^{-3}.

large number of regimes with a fixed value of one of the parameters. Therefore, the graphs shown in Fig. 7 were constructed for a fixed value of $\bar{n}_e$ $(=1.4\times 10^{13}$ cm$^{-3})$ in some close regimes. The electron temperature at the center of the column does not depend on the longitudinal magnetic field, which is characteristic of the majority of the Tokamak installations, and increases linearly with increasing current in the plasma. The dependence on current is not so strong as in the T-3A installation[2]; by its nature it is close to the data obtained on the ORMAK installation.[4]

The author considers it his pleasant duty to express his thanks to V. V. Alikaev, G. A. Bobrovskiĭ, K. A. Razumova, Yu. A. Sokolov, and D. A. Shcheglov for constant interest and help in the work, as well as to the technical staff of the TM-3 installation for their help in the carrying out of the experiment.

[1]N. J. Peacock, D. C. Robinson, M. J. Forrest, P. D. Wilcock, and V. V. Sannikov, Nature **224**, 448 (1969).
[2]M. J. Forrest, N. J. Peacock, D. C. Robinson, V. V. Sannikov, and P. D. Wilcock, CLM-R107, 1970.
[3]A. M. Anashin, E. P. Gorbunov, D. P. Ivanov, S. E. Lysenko, N. J. Peacock, D. C. Robinson, V. V. Sannikov, and V. S. Strelkov, Zh. Eksp. Teor. Fiz. **60**, 2092 (1971) [Sov. Phys. JETP **33**, 1127 (1971)].
[4]D. L. Dimock, H. P. Eubank, L. C. Johnson, and E. B. Meservey, Nucl. Fusion **13**, 271 (1973).
[5]M. Mukarami, W. Wing, and P. Edmonds, Nucl. Fusion **14**, 779 (1974).
[6]H. P. Furth, Nucl. Fusion **15**, 487 (1975).
[7]A. B. Berlizov, G. E. Notkin, and V. V. Sannikov, Preprint IAÉ-2583, 1975.
[8]A. L. Mikaélyan, M. L. Ter-Mikaélyan, and Yu. G. Turkov, Opticheskie generatory na tverdom tele (Solid-State Lasers), Sovetskoe Radio, 1967, p. 48.
[9]J. Sheffield, Plasma Phys. **14**, 783 (1972).
[10]V. A. Zhuravlev and G. D. Petrov, Opt. Spektrosk. **33**, 36 (1972) [Opt. Spectrosc. (USSR) **33**, 19 (1972)].
[11]L. A. Artsimovich, A. V. Glukhov, and M. P. Petrov, Pis'ma Zh. Eksp. Teor. Fiz. **11**, 449 (1970) [JETP Lett. **11**, 304 (1970)].
[12]V. V. Sannikov and Yu. A. Sokolov, Fiz. Plazmy **2**, 204 (1976) [Sov. J. Plasma Phys. **2**, 112 (1967)].
[13]E. P. Gorbunov, S. V. Mirnov, and V. S. Strelkov, Preprint IAÉ-1681 (1968).

Translated by A. K. Agyei

Spatially non-uniform singular weak turbulence spectra

V. S. L'vov and A. M. Rubenchik

Institute of Automation and Electrometry, Siberian Division USSR Academy of Sciences

(Submitted February 27, 1976; resubmitted July 23, 1976)
Zh. Eksp. Teor. Fiz 72, 127–140 (January 1977)

We consider spatially non-uniform collective oscillations of singular weak turbulence spectra. We study the modulational instability of Langmuir turbulence spectra leading to collapse. We find its maximum growth rate and study the non-linear stage. We formulate equations describing the non-linear stage of the parametric instability in a non-uniform medium. We study the collective oscillations of a system of parametrically excited waves. We consider also the analogous problem of the effect of sample boundaries on the distribution of the oscillations. We estimate the dimensions for which a transition to the spatially non-uniform solution occurs.

PACS numbers: 47.25.—c

§1. INTRODUCTION

The traditional way to describe weak wave turbulence in a spatially non-uniform situation is to use the kinetic equation

$$\frac{\partial n_k}{\partial t}+\frac{\partial \omega_k}{\partial \mathbf{k}}\frac{\partial n_k}{\partial \mathbf{r}}-\frac{\partial \omega_k}{\partial \mathbf{r}}\frac{\partial n_k}{\partial \mathbf{k}}=\mathrm{St}\{n_k\}. \qquad (1.1)$$

It describes waves as weakly interacting quasi-particles; its right-hand side is caused by the "collisions of quasi-particles" with a large change in their momentum, while the left-hand side is the total derivative dn_k/dt. The spatial dependence can be caused by the inhomogeneity of the medium and may occur spontaneously due to the non-linear interaction of the oscillations.

For applications of Eq. (1.1) it is necessary to satisfy the adiabaticity condition $kL \gg 1$, which presupposes that the inhomogeneity dimension L is large compared with the wavelength k^{-1}. Moreover, in order that the waves might be considered as quasi-particles with well-defined coordinates it is also necessary that L exceed the coherence length ("size of the quasi-particles") l:

$$L \gg l \sim 1/\Delta k, \tag{1.2}$$

where Δk is the size of the packet in the direction of the inhomogeneity. One must state that Eq. (1.1) may be valid even if the criterion (1.2) is not satisfied. However, for this it is necessary to impose additional restrictions on the form of the solution, on the structure of the matrix elements, and so on, taking into account the specific nature of the problem. Generally speaking, the kinetic equation (1.1) is thus not applicable for describing wavepackets which are narrow in some direction.

In the case when the wavepacket is concentrated near a point k_0 in k-space, i.e., is a quasi-monochromatic wave, there is another way to describe the wave field—using equations for envelopes which have a dynamic character. They have been known for a long time in non-linear optics[1] under the name "reduced equations" for the amplitude of the light field, and have been applied for describing narrow packets of electromagnetic waves in a plasma,[2] spin waves in ferromagnets, and so on.

We wish to draw attention to the fact that for a whole set of physically interesting situations it is necessary to study "singular" wavepackets which are concentrated near a surface or a line in k-space. For instance, the emission spectrum of a powerful laser with a large number of longitudinal modes is concentrated near a line segment in k-space. The turbulence spectra of an isothermal plasma[3–5] are singular, concentrated on lines or surfaces of k-space. A third example is the parametric excitation of waves with a non-decay dispersion law—spin waves in ferrodielectrics, waves on the surface of a liquid, sound in crystals, and so on, when the stationary wave amplitudes are large only near the "resonance surface" $\omega_p = \omega_k + \omega_{-k}$ where ω_p is the frequency of the external action.

To describe such singular distributions in a spatially-non-uniform situation neither the kindetic equation (1.1) for the correlator $n_k(r)$ nor the dynamic equation for the wavepacket envelope is applicable.

In the present paper we propose an intermediate "quasi-dynamic" method for describing singular weak turbulence spectra using, on the one hand, the weakness of the interaction and the randomness of the phases which are due to the extension of the wavepacket in one or two directions and, on the other hand, the narrowness of the packet in the remaining directions: the randomness of the phases enables us in the derivation of the equations (see §2) to decouple the quaternary correlators in terms of the binary ones and to restrict ourselves to the approximation of the self-consistent field. The narrowness of the packets enables us to simplify the equations for $n_{kk'} = \langle a_k^* a_{k'} \rangle$, writing them down in the r-representation, similar to what is done when deriving the equations for the envelopes. As a result we get the relatively simple Eq. (2.5) for the binary correlator of the wavefield. We discuss in §2 the limits of applicability of the obtained Eq. (2.5) and its connection with other ways of describing the system. We use this equation in what follows to solve a number of concrete problems.

In §3 we study the general properties of the spectrum of the collective oscillations of singular weak wave turbulence spectra and discuss the criteria for their instability. We pay special attention to the modulational instability of Langmuir oscillations. The description of Langmuir packets which are broad in frequency, using various variants of Eq. (1.1), only enables us to determine the fact that they are unstable.[6] In our paper we have succeeded not only in obtaining the maximum growth rate of the instability but also in examining its non-linear stage (see §4); we have also studied the self-focusing of the emission from a single-mode laser. We obtain in §5 the non-linear equations for the decay instability in an inhomogeneous medium, taking into account the anomalous correlators. These equations generalize the basic equations of the S-theory to the spatially non-uniform case.

In §6 we obtain the spectrum of the collective oscillations in a system of parametrically excited waves and we consider the problem of the stability of the "ground state" of a system of parametric waves with respect to the violation of spatial uniformity. We also consider the related problem of the effect of the boundaries of the sample on the distribution of the wave amplitudes.

§2. THE QUASI-DYNAMIC EQUATION

1. *Derivation of the equations.* As usual we start the description of weak turbulence with the canonical equations of motion for the complex amplitude of the wave field[8]:

$$\left(\frac{\partial}{\partial t} + i\omega_k\right) a_k = -i \int T_{k1,23} a_1^* a_2 a_3 \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\, d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3. \tag{2.1}$$

Here ω_k is the dispersion law of the waves, and $T_{12,34}$ are the matrix elements of the four-wave interaction Hamiltonian. We assume that the three-wave processes are forbidden by the conservation laws. It is necessary to add that one can often use Eq. (2.1) to describe waves in a non-conservative medium; in that case ω_k and $T_{12,34}$ acquire anti-Hermitean additions:

$$\begin{gathered} \omega_k \to \omega_k + i\gamma, \quad T_{12,34} \to T_{12,34} - i\eta_{12,34}, \\ T_{12,34} = T^*_{34,12}, \quad \eta_{12,34} = \eta^*_{34,12}. \end{gathered} \tag{2.2}$$

In first order in the interaction, i.e., decoupling the quaternary correlators in terms of the binary ones, we get easily from Eq. (2.1)

$$\begin{gathered} \left[\frac{\partial}{\partial t} + i(\omega_{k'} - \omega_k^*)\right] n_{kk'} + 2i \int d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3 \{T_{k'1,23} n_{k2} n_{13} \\ \times \delta(\mathbf{k}'+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3) - T^*_{k1,23} n_{2k'} n_{31} \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\}. \end{gathered} \tag{2.3}$$

In what follows we assume for the sake of simplicity that the spatial inhomogeneity is one-dimensional (the z-axis is the direction of the inhomogeneity). In that case

$$n_{kk'} = n_{k_\perp}(k_z, k_z')\delta(\mathbf{k}_\perp - \mathbf{k}_\perp'), \quad k_\perp = \{k_x, k_y\}. \tag{2.4}$$

Using the fact that for fixed $k_\perp$ the packet $n_{k_\perp}(k_z, k_z')$ is concentrated in a narrow layer: $\Delta k_z \ll k$, we can expand $\omega(k_\perp, k_z)$ in (2.3) in a series in $k_z - k_z^0$ (k_z^0 is the center of the packet: $k_z^0 = f(k_\perp)$) and neglect the dependence of $T_{12,34}$ on $k_z - k_z^0$. The equations obtained can be simplified after changing to the r-representation in the coordinate z:

$$\left[\frac{\partial}{\partial t} + i(\tilde{\omega}_k(z) - \tilde{\omega}_k^*(z')) + v_k\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial z'}\right) + \frac{i\omega''}{2}\left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial z'^2}\right)\right] n_k(z, z') = 0. \tag{2.5}$$

Here

$$n_k(z, z') = \frac{1}{(2\pi)^3}\int n_{k\perp}(k_z, k_z')\exp i[k_z z - k'z' - k_z^0(z - z')]\,dk_z\,dk_z',$$

$$\tilde{\omega}_k(z) = \omega_k + 2\int T_{kk'}n_{k'}(z, z)\,dk_\perp', \tag{2.6}$$

$$v_k = \partial\omega_k/\partial k_z, \quad \omega'' = \partial^2\omega/\partial k_z^2, \quad T_{kk'} = T_{kk', kk'}.$$

We must understand in Eqs. (2.5) and (2.6) the vector $\mathbf{k}$ to be $\{k_\perp, k_z^0 = f(k_\perp)\}$. In deriving (2.5) we assumed that the condition that the interaction is small was satisfied, which guaranteed the randomness of the phases:

$$\int T_{kk'}n_{k'}\,dk_\perp' \ll \{v_k(\Delta k)_{max}, \omega''(\Delta k)_{max}\}, \tag{2.7}$$

where $(\Delta k)_{max}$ is the maximum dimension of the packet.

Equation (2.5) obtained by us is essentially an equation with a self-consistent field, since it differs from the linear equation for the correlator $n_k(z, z')$ only by the renormalization of the frequency (2.6) by the interaction. Of principal importance is the neglect of the collision term. The criterion (2.7) which guarantees that this term is small and which enables us to evaluate it in second order of perturbation theory in H_{int} is only necessary. The determination of a sufficient criterion requires taking into account the concrete specific nature of the problem and is therefore irrelevant here. We note merely that in our Eqs. (2.5) we retained terms of first order in H_{int} and there may therefore easily occur many situations when one can indeed neglect the collision term.

2. *Connection with other methods of description.* A. For a broad packet in a medium with a slow inhomogeneity, $n_k(z, z')$ depends more weakly on $z - z'$ than on $z + z'$ by a factor equal to the parameter of (1.2) $(L\Delta k)^{-1}$. This enables us to expand $\tilde{\omega}_k(z) - \tilde{\omega}_k(z')$ in (2.5) in terms of $z - z'$ and after changing to the momentum representation in terms of $z - z'$ to write (2.5) in the form

$$\left[\frac{\partial}{\partial t} + \frac{\partial\tilde{\omega}_k}{\partial k}\frac{\partial}{\partial z} - \frac{\partial\tilde{\omega}_k}{\partial z}\frac{\partial}{\partial k}\right] n_k(z) = 0. \tag{2.8}$$

One of us obtained Eq. (2.8) in[9]; it is analogous to Eq. (1.1) in the case when the spatial inhomogeneity is not given, but determined by the distribution of the oscillations.

B. Establishing the connection between Eq. (2.5) and the equation for the envelopes we note that (2.5) has an exact solution in factorized form $n_k(z, z') = A_k(z)A_k^*(z')$ where $A_k(z)$ satisfies the equation

$$\left[\frac{\partial}{\partial t} + i\tilde{\omega}_k(z) + v_k\frac{\partial}{\partial z} + i\frac{\omega''}{2}\frac{\partial^2}{\partial z^2}\right] A_k(z) = 0, \tag{2.9}$$

$$\tilde{\omega}_k = \omega_k - i\gamma_k + 2\int T_{kk'}|A_{k'}(z)|^2\,dk_\perp'.$$

This equation is a generalization of the equation for the envelope of a monochromatic wave to the case of an extended packet: $A_k(z, t)$ is a function of $k_\perp$. If we substitute $A_k \propto \delta^{1/2}(\mathbf{k}_\perp - \mathbf{k}_0)$ in (2.9), we get for $A_{k_0}(z, t)$ an equation that differs from the equation for the envelope solely by the coefficient 2 in front of $T_{k_0 k_0}$; this factor two is, clearly, due to the randomness of the phases.

3. *Possible generalizations of the quasi-dynamic Eq.* (2.5). A. If the medium is inhomogeneous and when there are no waves, this can be taken into account in the quadratic Hamiltonian:

$$H^{(2)} = \int \omega_{kk'}a_k^* a_{k'}\,d\mathbf{k}\,d\mathbf{k}'. \tag{2.10}$$

Using this fact and repeating the whole discussion and assuming that the inhomogeneity is slow we are again led to Eqs. (2.5) and (2.6) in which ω_k will now depend on r:

$$\omega_k \to \omega_k(r) = \frac{1}{(2\pi)^{3/2}}\int \omega\left(\mathbf{k} + \frac{\varkappa}{2}, \mathbf{k} - \frac{\varkappa}{2}\right)e^{i\varkappa r}\,d^3\varkappa.$$

B. If we drop the assumption that the inhomogeneity is one-dimensional Eq. (2.5) retains its form, but the expression for the renormalized frequency $\tilde{\omega}_k$ will no longer have the simple local form (2.6). If the distribution is almost one-dimensional (arbitrary line), but the inhomogeneity is two-dimensional, we can from (2.3) obtain for the quantities

$$n_{k_z, k_\perp^0}(r_\perp, r_\perp') = \frac{1}{(2\pi)^6}\int dk_\perp\,dk_\perp'\,n_{k_z}(k_\perp, k_\perp') \times \exp i[k_\perp r_\perp - k_\perp' r_\perp' - k_\perp^0(r_\perp - r_\perp')] \tag{2.11}$$

the simple equations

$$\left[\frac{\partial}{\partial t} + v\left(\frac{\partial}{\partial r_\perp} + \frac{\partial}{\partial r_\perp'}\right) + i(\tilde{\omega}_k(r_\perp) - \tilde{\omega}_k(r_\perp'))\right] n_k(r_\perp, r_\perp') = 0,$$

$$\tilde{\omega}_k(r_\perp) = \omega_k + 2\int T_{kk'}n_{k'}(r_\perp, r_\perp')\,dk_z'. \tag{2.12}$$

For simplicity we have not written out here the diffraction term proportional to ω''. We can obtain a local expression for $\tilde{\omega}_k$ for any inhomogeneity, if $T_{kk'} = \text{const}$, and in some other special cases.

C. The generalization of (2.5) to the case when it is necessary to take into account the anomalous correlators $\langle a_k a_{-k}\rangle$ is, as we noted already earlier, given in §5 in connection with the consideration of the non-linear stage of the parametric instability in an inhomogeneous medium.

§3. COLLECTIVE OSCILLATIONS OF JET DISTRIBUTIONS AND THEIR INSTABILITY

1. *Dispersion relations.* First of all we make some remarks about the nature of the spatially uniform solutions of Eqs. (2.9). We look for them in the form

$$A_k(z,t)=A_k^0\exp(-i\delta_k t),\quad \delta_k=\mathrm{Re}\,\tilde\omega_k=\omega_k+2\int T_{kk'}|A_{k'}^0|^2\,dk', \qquad (3.1)$$

δ_k is the frequency of the waves including the interaction and A_k^0 is found from the equation

$$\Gamma_k A_k^0=0,\quad \Gamma_k=\gamma_k+2\int \eta_{kk'}|A_{k'}^0|^2\,dk'. \qquad (3.2)$$

Hence it follows that in a transparent medium the distribution of waves can be arbitrary; it is determined by the conditions of excitation. The structure of the stationary spectra in an absorbing medium has been studied in detail in a number of papers[3-5] using the Langmuir turbulence of an isothermal plasma as an example. In this case the oscillations are, as a rule, concentrated on lines and surfaces in k-space. Their location and the intensity of the oscillations are determined from (3.2) and the condition $\partial\Gamma_k/\partial k=0$ which is necessary for the stability with respect to the excitation of waves outside these surfaces.

Equations (2.9) make it in principle possible to solve the problem of the stability, including stability with respect to a violation of the spatial uniformity. For this it is necessary to linearize them above the background of the stationary spectrum A_k^0: $\delta A_k=b_k\exp(-i\delta_k t)$. Putting $b_k(t,z)\propto\exp-i(\Omega t-\varkappa z)$ and introducing new variables $\Psi^\pm=(A_k^0 b_k^*\pm\mathrm{c.c.})$ we get

$$\begin{aligned}&-{}^1/_2\omega''\varkappa^2\Psi^+-(\Omega-\varkappa v)\Psi^-=4|A_k^0|^2\int T_{kk'}\Psi_{k'}^+\,dk_\perp',\\ &(\Omega-\varkappa v)\Psi^+-{}^1/_2\omega''\varkappa^2\Psi^-=-4i|A_k^0|^2\int \eta_{kk'}\Psi_{k'}^+\,dk_\perp'.\end{aligned} \qquad (3.3)$$

For $\varkappa=0$ Eqs. (3.3) split up. The first of them gives $\Omega=0$ which reflects the indifferent equilibrium solutions of Eq. (3.2) relative to a change in the phase A_k^0. From the second we get a dispersion relation for Ψ^+ describing the uniform change in the amplitude of the oscillations along the jet. These oscillations are sound propagating without damping along the stationary spectrum in k-space.[10] For us it is important that both types of modes in Eq. (3.3) are indifferently stable for $\varkappa=0$ so that of special interest is the study of collective oscillations with small, but finite $\varkappa$ which may become unstable.

2. *Spatially non-uniform types of oscillations.* Equations (3.3) make it possible in principle to study spatially non-uniform oscillations of singular spectra. However, it is impossible to carry out such an analysis without simplifications corresponding to the concrete physical situation. In the present section we restrict ourselves therefore to a study of the spectra of oscillations with a quadratic dispersion law which makes it possible to assume that ω'' is independent of k. Such a form of the spectrum corresponds to a very wide range of physical phenomena—spin waves, optical phonons, Langmuir oscillations. Moreover, a narrow wavepacket in an arbitrary medium is also described by a quadratic dispersion law. We consider a few concrete examples.

A. A narrow packet in a transparent medium: $\eta_{kk'}=0$, $T_{kk'}=T$. We get in this case easily from (3.3)

$$\Omega=\varkappa v\pm\left[\frac{\omega''\varkappa^2}{2}\left(\frac{\omega''\varkappa^2}{2}+4T\int|A_k^0|^2\,dk\right)\right]. \qquad (3.4)$$

It is clear that for $\omega''T<0$ there occurs a spontaneous breaking of the spatial uniformity which is a generalization of the modulational instability of a monochromatic wave.

B. In the opposite limiting case when the scale of the distribution considerably exceeds the dimension of the kernels η and T we put $A_k^0=A^0$ and, assuming the kernels to be to depend on the difference, we change to the q-representation

$$\Psi_q=\frac{1}{\sqrt{2\pi}}\int\Psi_k e^{-iqk}\,dk. \qquad (3.5)$$

For the sake of simplicity we shall additionally assume that $\varkappa\perp\mathbf{v}$. It then easily follows from (3.3) that

$$\Omega_q(\varkappa)=2i|A^0|^2\eta_q\pm\left[4|A^0|^4\eta_q^2+\frac{\omega''\varkappa^2}{2}\left(\frac{\omega''\varkappa^2}{2}+4|A^0|^2T_q\right)\right]^{1/2}, \qquad (3.6)$$

where

$$\eta_q=\int\eta_k e^{-iqk}\,dk,\quad T_q=\int T_k e^{-iqk}\,dk. \qquad (3.7)$$

Bearing in mind that η_k and T_k are real and that $\eta_k=-\eta_{-k}$, $T_k=T_{-k}$ we find that η_q is an odd, purely imaginary function of q and T_q a real even function of q. When $\varkappa=0$ there follows a "sound spectrum" from (3.6): $\Omega_q=2|A^0|^2|\eta_q|\propto q$ as $q\to 0$. The instability when $\varkappa\neq 0$ can arise only when $\omega''T_q<0$, and for fixed q the least stable perturbations are those with $\varkappa=\varkappa_0$, $\omega''\varkappa_0^2=4T_q|A^0|^2$, for which

$$\Omega_q(\varkappa_0)=2i|A^0|^2\eta_q\pm 2|A^0|^2(|\eta_q|^2-|T_q|^2)^{1/2}.$$

For the occurrence of an instability it is thus necessary that

$$\omega''T_q<0,\quad |T_q|^2>|\eta_q|^2. \qquad (3.8)$$

By virtue of the antisymmetry $\eta(0)=0$. Therefore, if $T(0)\neq 0$ the condition $\omega''T(0)<0$ is necessary for instability.

C. We consider in more detail Langmuir oscillations in an isothermal plasma. Their non-linear interaction is caused by induced scattering by ions, and the matrix elements are determined by the expression[3,5]

$$T_{kk'}+i\eta_{kk'}=\frac{\omega_{pe}^2}{4nT_e}\frac{(\mathbf{kk}')^2}{k^2k'^2}f\left(\frac{\omega_k-\omega_{k'}}{|\mathbf{k}-\mathbf{k}'|v_{Ti}}\right)=T_0f(x). \qquad (3.9)$$

Here $\omega_k=\omega_{pe}(1+\frac{3}{2}k^2r_d^2)$ is the frequency of the Langmuir waves ($\omega''>0$), ω_{pe} the plasma frequency, and r_d the Debye radius. For long waves $kr_d<\sqrt{(m/M)}$, $f\approx-1$ and in accordance with the criterion (3.8) the singular distributions are always unstable. When $kr_d>\sqrt{(m/M)}$ the structure of the function f is well described by the expression

$$f(x)=[(x+i\gamma_s/\Omega_s)^2-1]^{-1},\quad x=(\omega_k-\omega_{k'})/\Omega_s, \qquad (3.10)$$

in which γ_s and Ω_s are the damping rate and frequency of the ion-sound oscillations. When $T_i\ll T_e$ Eq. (3.10) is exact. When the ion temperature increases γ_s increases and when $T_i\sim T_e$ we shall have $\gamma_s\sim\Omega_s$. The characteristic size of the function f is $k_{\mathrm{diff}}=r_d^{-1}\sqrt{(m/M)}$—

the spacing of the diffusive energy transfer along the spectrum. For spectral distributions which change smoothly over a distance k_{diff} we can use the general Eq. (3.6) in which T_q and η_q are calculated from Eqs. (3.7), (3.9), and (3.10):

$$\eta_q=-i\pi T_0\sin(qk_{diff})\exp\left(-\frac{\gamma_*}{\Omega_*}|q|k_{diff}\right),$$
$$T_q=-\pi T_0\sin(|q|k_{diff})\exp\left(-\frac{\gamma_*}{\Omega_*}|q|k_{diff}\right). \quad (3.11)$$

It is clear that $\omega''T_q<0$ for $qk_{diff}<\frac{1}{2}\pi$, but $|T_q|=|\eta_q|$ and the criterion (3.8) for the occurrence of the instability is not satisfied. It turns out that in our approximation the least stable perturbation reaches the boundary of stability. An instability can occur if a more exact approximation is made of the matrix element $T_{kk'}$ but its growth rate will have an additional small factor k_{diff}/k.

It was shown in[5] that for temperatures $T_i \lesssim T_e$ jet spectra turn out to be strongly cut up—with a deep modulation of order unity and scale length k_{diff}. We show that this fact leads to a strong spatially non-uniform instability. To do this we consider an extremely strongly cut-up spectrum (satellite approximation[5]):

$$|A_k^0|^2=\sum_n |A^0|^2\delta(k-nk_{diff}). \quad (3.12)$$

Writing $\Psi^{\pm}$ in the form

$$\Psi_n^{\pm}=\frac{1}{\sqrt{Z}}\sum_q \Psi_q^{\pm}\exp\{iqnk_{diff}\}$$

(Z is the number of satellites) we find that the solution of the dispersion relation (3.3) has the form (3.6). However, in this formula $|A^0|^2$ is the amplitude in one satellite (see (3.12)), while η_q and T_q in contrast to (3.7) are given by the formulae

$$\eta_q=\sum_n \eta_{k+nk_{diff},k}\exp\{-iqnk_{diff}\},$$
$$T_q=\sum_n T_{k+nk_{diff},k}\exp\{-iqnk_{diff}\}. \quad (3.13)$$

Now we have $|\eta_q|\neq|T_q|$ and an instability occurs. When $q\ll\pi/k_{diff}$ its growth rate is small, but when $qk_{diff}=\pi$ we have $\eta_q=0$, $T_q\approx 0.5\,T_0$ for $T_i\ll T_e$ and the instability develops with a growth rate

$$\operatorname{Im}\Omega=[{}^1/_2\omega''\varkappa^2({}^1/_2\omega''\varkappa^2-2.2|A^0|^2T_0)]^{1/2}.$$

Its characteristic scale (in coordinate space) is $l^2\sim\omega''/T_0|A^0|^2$. In momentum space the development of this instability means in accordance with (3.13) the increase of the amplitude of every other one of the satellites.

§4. THE NON-LINEAR STAGE OF THE DEVELOPMENT OF THE INSTABILITY OF THE COLLECTIVE OSCILLATIONS IN A TRANSPARENT MEDIUM

It is necessary to emphasize here once more that the quasi-dynamic equations generalize the equations for the envelopes of a narrow wavepacket and that the instability found by us is similar to the modulational instability. However, the development of the modulational instability of a quasi-monochromatic wave was studied in detail in a number of papers both analytically and by means of a computer.[11-13] In particular, the conditions were elucidated under which a singularity is formed after a finite time in the distribution of the waves—the oscillations show self-focusing (collapse). We generalize below these results to the case of wavepackets which are broad in one or two directions.

In a transparent medium Eq. (2.9) is Hamiltonian in nature. For transverse perturbations ($\varkappa\cdot\mathbf{r}=0$) the Hamiltonian has the form

$$H=\int d\mathbf{k}_\parallel\, dr_\perp\left\{\frac{\omega''}{2}|\nabla A_k|^2+{}^1/_2\int T_{kk'}|A_k|^2|A_{k'}|^2\,d\mathbf{k}_\parallel'\right\} \quad (4.1)$$

and is, clearly, an integral of motion. Moreover, Eq. (4.1) conserves also the total "number of waves" for each k: $N_k=\int|A_k|^2dr_\perp$. Here $\mathbf{k}_\parallel$ is the wavevector along the surface (or line), and $r_\perp$ are the spatial coordinates in the transverse direction.

Following the ideas of[12], we consider the essentially positive quantity $R=\int r_\perp^2|A_k|^2dr_\perp d\mathbf{k}_\parallel$ and by direct calculation we find

$$\frac{1}{2}\frac{d^2R}{dt^2}=\int dr_\perp dk_\parallel\left[n\int T_{kk'}|A_k|^2|A_{k'}|^2dk'+2\omega''|\nabla A_k|^2\right], \quad (4.2)$$

where n is the dimensionality of the inhomogeneity. In the case when the oscillations are distributed along a line in k-space, $n=2$ and we have from (4.2) $d^2R/dt^2=8H$ or $R=4Ht^2+\alpha t+\beta$, where α and β are constants of integration. It is clear that when $H<0$ the quantity R becomes negative after sufficiently long time which is in contradiction to its definition. The fact is that in deriving (4.2) we integrated by parts, i.e., we assumed that the solution was differentiable. The contradiction obtained thus means the formation of a singularity in the solution.

We describe qualitatively how this proceeds. Initially the modulational instability of the mode with the largest growth rate develops. When $\eta=0$ this is the mode (3.5) with $q=q_0$ corresponding to the maximum T_q. When $T_{kk'}$ is close to a constant $q=0$ and the A_k increase simultaneously along the whole line. If, however, the kernel $T_{kk'}$ is well localized, i.e., decreases fast when $k-k'>\Delta k$ (in the example with a plasma considered above $\Delta k=k_{diff}$), we have $q_0\approx\pi/\Delta k$ and in the process of the growth of the wave along the line groups (spots) develop with a characteristic size along z of the order of Δk. Simultaneously the uniformity of the distribution in the transverse (x,y) direction in coordinate space is broken and humps are formed in the amplitude $A(x,y)$ with a scale of the order l_0: $l_0\approx\omega''/\Delta\omega_{nl}$. In the non-linear stage of the development of the modulational instability the scale of the humps l_0 decreases fast (in the framework of the idealized equation) to zero after a finite time t_0 ($t_0^{1/2}\sim l_0/\sqrt{\omega''}$) and by virtue of the conservation of N_k the amplitude at the center of the humps increases as $|A(0,t)|^2\propto l^{-2}(t)$.

When $T_{kk'}=T_0$ this process goes on until the limit of applicability of Eq. (2.5) according to the criterion

(2.7), i.e., until $l \sim \lambda = 2\pi/k$. The further fate of the filament obtained which is long in the z-direction and strongly focused in the transverse direction depends on the actual specific features of the problem (in particular, on the dispersion law of ω_k). When $\omega_k = \omega_0 + \alpha k^2$ and $T_{kk'} = T_0 < 0$ in this stage there occurs a breaking of the spatial uniformity along z with a scale k^{-1} and the collapse of the envelopes changes to a collapse of the separate waves: "luminous filaments will flare up" in space at distances of the order of l_0 along x and y, will disintegrate afterwards into separate "points" of size k^{-1}, and then collapse.

In the case of a localized kernel $T_{kk'}$ the contraction of a filament which is uniform along z will proceed only down to a scale $l \sim (\Delta k)^{-1}$ (Δk is the "size" of the kernel). For such a scale the amplitude A at the center of the hump ($x = y = 0$) increases up to a magnitude $\tilde{A}$:

$$\int T_{kk'}|A_{k'}|^2 dk' \approx \omega''(\Delta k)^2, \tag{4.3}$$

which follows from the fact that the Hamiltonian (4.1) is conserved. The criterion for randomization (2.7) of the initial equations is then violated and it is no longer possible to assume that the phases of the waves within the limits of each spot with respect to k_z with dimension Δk are random. Bearing in mind, however, the localization of the kernel we can in first approximation neglect the interaction of "spots" with different k_z. When $l\Delta k < 1$ the phases inside each spot are completely correlated and each spot can be described by the dynamic equation for envelopes. In the framework of that equation there is an instability with respect to a spatial modulation in z; the two-dimensional collapse of the filaments therefore changes into a three-dimensional collapse of separate, not mutually correlated, spots.

Of course, in concrete situations the collapse process described above can stop under the action of factors not taken into account in our equation, for instance, a positive non-linear damping.

§5. NON-LINEAR EQUATIONS FOR THE DECAY INSTABILITY IN A NON-UNIFORM MEDIUM

For a study of the decay instability we restrict ourselves to the simple and often encountered case when the pumping is spatially uniform $h(r,t) = h\exp(i\omega_p t)$. The pumping leads to an additional term in Eq. (2.1):

$$ih\exp(i\omega_p t)V_k a^*_{-k}. \tag{5.1}$$

The coherence of the pumping leads to the fact that the description of the system of waves in terms of the $n_{kk'}$ is insufficient and it is necessary to introduce anomalous correlators[7]:

$$\sigma_{kk'} = \langle a_k a_{-k'}\rangle \exp(i\omega_p t). \tag{5.2}$$

If the spatial inhomogeneity is one-dimensional, we have (cf. (2.4))

$$\sigma_{kk'} = \sigma_{k_\perp}(k_z, k_z')\delta(k_\perp - k_\perp'). \tag{5.3}$$

As in §2, we change to the r-representation with respect to k_z and k_z':

$$\sigma_k(z', z) = \sigma_{-k}(z, z') = \frac{1}{(2\pi)^3}\int dk_z dk_z' \sigma_{k_\perp}(k_z, k_z')\exp\{i[k_z^0(z-z') - k_z z + k_z' z']\} \tag{5.4}$$

and using the narrowness of the wave distribution we get instead of (2.5) the set of equations

$$\begin{gathered}\left[\frac{\partial}{\partial t} + 2\gamma_k + v_k\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial z'}\right) + i(\tilde{\omega}_k(z) - \tilde{\omega}_k^*(z'))\right] n_k(z,z') \\ + i[\sigma_{-k}^*(z,z')P_k(z',z') - P_k^*(z,z)\sigma_k(z',z)] = 0, \\ \left[\frac{\partial}{\partial t} + 2\gamma_k + v_k\left(\frac{\partial}{\partial z'} - \frac{\partial}{\partial z}\right) + i(\tilde{\omega}_k(z) + \tilde{\omega}_k(z') - \omega_p)\right]\sigma_k(z',z) \\ + i[P_k(z',z')n_{-k}(z',z) + P_{-k}(z,z)n_k(z,z')] = 0.\end{gathered} \tag{5.5}$$

Here $\tilde{\omega}_k$ and v_k are defined in (2.6)

$$P_k(z,z') = hV_k + \int S_{kk'}\sigma_{k'}(z',z)\,dk', \qquad S_{kk'} = T_{k,-k,k',-k'}, \tag{5.6}$$

and for the sake of simplicity we dropped, as a rule, unimportant terms with the anti-Hermitean parts of S and T and also the diffraction terms proportional to ω''.

In the spatially uniform case Eqs. (5.5) and (5.6) change to the basic equations of the S-theory.[7] If the pumping amplitude h changes slowly (as compared to the wavelength k^{-1}) in space, h in (5.6) is $h(z)$. Like (2.5), the equations given here allow a factorized solution of the kind

$$n_k(z,z') = A_k^*(z)A_k(z'), \qquad \sigma_k(z',z) = A_k(z')A_{-k}(z), \tag{5.7}$$

where $A_k(z)$ satisfies the equation

$$\begin{gathered}\partial A_k/\partial t + \gamma_k A_k + v_k\,\partial A/\partial z + i(\tilde{\omega}_k - {}^1\!/_2\omega_p)A_k + iP_k A^*_{-k} = 0, \\ P_k = hV_k + \int S_{kk'}A_{k'}A_{-k'}\,dk'.\end{gathered} \tag{5.8}$$

We emphasize that Eqs. (5.5) obtained in the self-consistent field approximation do not contain randomizing factors and just because of this they admit of the dynamical solution (5.7). If, however, there are external causes for stochastization, for instance, due to initial or boundary conditions, the wave turbulence will be described by solutions of a more general form.

§6. COLLECTIVE OSCILLATIONS IN A SYSTEM OF PARAMETRICALLY EXCITED WAVES[1)]

Spatially uniform stationary solutions of the S-theory Eqs. (5.5) have been studied in detail in a number of papers.[7] It has been shown that the solution is concentrated on a resonance surface

$$2\tilde{\omega}_k = \omega_p. \tag{6.1}$$

The actual form of the distribution n_k on the surface (6.1) depends in an important way on the coefficients V_k, $S_{kk'}$ in the Hamiltonian and on the form of the function γ_k on that surface. In the simplest case when $S_{kk'} = S$, $V_k = V$, $\gamma_k = \gamma$ the wave distribution is isotropic on (6.1) and the integral "amplitude" N and the phase of the pairs Φ are determined from the relations

$$(SN)^2=(hV)^2-\gamma^2, \quad hV\sin\Phi=\gamma, \quad \Phi=\varphi_k+\varphi_{-k}, \qquad (6.2)$$
$$N=\sum_k n_k, \quad a_k=n_k^{1/2}\exp(-i\varphi_k).$$

In an axially symmetric situation and not too far above the threshold waves are excited only on parallel lines corresponding to the maximum of V_k with the characteristics (6.2). The problem naturally arises of the stability of uniform distributions of pairs of waves with respect to spatial modulation of their amplitude and phase. For a study of this problem it is necessary to linearize Eq. (5.8) above the uniform background: $A_k=A_k^0+b_k(r)$ and searching their solution in the form $b_k(r)$, $b_k^*(r) \propto \exp\{i[\varkappa r-\Omega t]\}$, we find the spectrum $\Omega(\varkappa)$ of the collective oscillations. We give the results for two simple, but interesting cases.

1. *Isotropic model*: $S_{kk'}=S$, $T_{kk'}=T$, $V_k=V$. The dependence of the frequency of the collective oscillations on their wavevector is determined by the dispersion equation

$$\frac{\Omega_0^2(\Omega+2i\gamma)}{4\pi^2}\int_0^{2\pi}\frac{d\varphi}{(\varkappa v)^2\cos^2\varphi+\Omega(\Omega+i\gamma)}=1, \qquad (6.3)$$

where

$$\Omega_0^2=4S(2T+S)N^2.$$

At $\varkappa v \ll \Omega_0$ the solution of (6.3) is of the form

$$\Omega(\varkappa)=-i\gamma\pm[\Omega_0^2+(\varkappa v)^2-\gamma^2]^{1/2}. \qquad (6.4)$$

When $\varkappa=0$, (6.4) changes to the expression for the frequency $\Omega(0)$ of the spatially uniform collective oscillations studied experimentally and theoretically in[15,16]. For negative Ω_0^2 an instability develops. The least stable oscillations have $\varkappa=0$, the instability region extends to $\varkappa v=\frac{1}{2}\Omega_0$, and for $\Omega_0^2>0$ the stationary solution is stable for all $\varkappa$.

2. *The case of axial symmetry*. We study the spectrum $\Omega(\varkappa)$ of the collective oscillations in the case which is important for experiments in ferromagnetics when the amplitude of the waves is non-zero at latitudes $\theta=\theta_0$ and $\theta=\frac{1}{2}\pi-\theta_0$. In cubic ferromagnetics for parallel pumping and for being above criticality by up to 6 to 8 dB $\theta=\frac{1}{2}\pi$ (equator), for transverse pumping the case $\theta=\frac{1}{4}\pi$ is often realized. The function $\Omega(\varkappa)$ looks simplest in the case when $\varkappa$ is parallel to the axis of symmetry. For each axial harmonic with number p we can obtain

$$\tilde\Omega_{p_1}(\varkappa)=i\gamma\pm(\Omega_{p_1}^2-\gamma^2)^{1/2}, \quad \tilde\Omega_{p_2}(\varkappa)=i\gamma\pm(\Omega_{p_2}^2-\gamma^2)^{1/2},$$

where

$$\begin{aligned}\Omega_{p(1,2)}&=(\varkappa v)^2+{}^1/_2(\Delta_{p_1}^2+\Delta_{p_2}^2)\pm\{{}^1/_4(\Delta_{p_1}^2-\Delta_{p_2}^2)^2\\&+(\varkappa v)^2(\Delta_{p_1}^2+\Delta_{p_2}^2+\Delta_p^2)\}^{1/2},\\ \Delta_p^2&=4(T_p^+-T_p^-)(T_p^++T_p^-+S_p)N^2,\end{aligned} \qquad (6.5)$$
$$\Delta_{p_1}^2=[{}^1/_2\omega''\varkappa^2+(2S_p+T_p^++T_p^-)N]^2-(T_p^++T_p^-)^2N^2,$$
$$\Delta_{p_2}^2=[{}^1/_2\omega''\varkappa^2+(T_p^+-T_p^-)N]^2-(T_p^+-T_p^-)^2N^2.$$

Here

$$T_p^\pm=\frac{1}{2\pi}\int e^{ip(\varphi-\varphi')}T_{k_z,\pm k_z}(\varphi-\varphi')\,d\varphi',$$
$$S_p=\frac{1}{2\pi}\int e^{ip(\varphi-\varphi')}S_{k_z,k_z}(\varphi-\varphi')\,d\varphi',$$

$T_p^\pm$, S_p are the Fourier harmonics of the coefficients in Eq. (5.8) which depend only on $\varphi-\varphi'$ because of the axial symmetry. When $\Omega^2(\varkappa)>0$ Eqs. (6.5) correspond to collective oscillations with frequencies $\pm\Delta$ and a damping rate γ equal to the damping rate of the waves. When $\Omega_p^2(\varkappa)<0$, $\mathrm{Im}\tilde\Omega_p>0$, which corresponds to an exponential growth of the collective oscillations, the spatially uniform distribution of the parametric waves is unstable.

We analyze Eqs. (6.5) which we have obtained for the frequencies of the collective oscillations. When $\varkappa=0$

$$\Omega_{p_1}^2=\Delta_{p_1}(0)=4S_p(T_p^++T_p^-+S_p)N^2. \qquad (6.6)$$

This expression for the frequency of spatially uniform oscillations was obtained earlier in[14,15] and confirmed by a series of experiments on ferromagnetics.[15,16] The oscillation with $\Omega_{p1}(0)$ is stable, if $S_p(T_p^++T_p^-+S_p)>0$. The oscillation with Ω_{p2} is indifferently stable. For small $\varkappa v$ we have

$$\Omega_{p_1}^2=\Omega_{p_1}^2(0)+(\varkappa v)^2\frac{2S_p+T_p^+-T_p^-}{S_p}, \quad \Omega_{p_2}^2=(\varkappa v)^2\frac{T_p^--T_p^+}{S_p}. \qquad (6.7)$$

Thus, if $S_p(T_p^+-T_p^-)>0$ the branch of collective oscillations Ω_{p2} becomes unstable at $\varkappa\neq 0$. Putting $\Delta_{p1}^2>0$, $\Delta_p^2>0$, we find that the branch Ω_{p1} is stable for all $\varkappa$ while the instability region of Ω_{p2} is enclosed between $\varkappa=0$ and $\varkappa=\varkappa_0$, where $(\varkappa_0 v)^2=\Delta_p^2$.

In cubic ferromagnetics there may arise situations when $\varkappa v=0$. For instance, when the waves are excited on the equator and we are interested in a perturbation perpendicular to its plane, or when due to the anisotropic dispersion law the group velocity for the excited waves v vanishes for $\theta=\theta_0$. In that case (6.5) simplifies to the form

$$\Omega_{p_1}(\varkappa)=i\gamma\pm\{[2(T+S)N+{}^1/_2\omega''\varkappa^2]-4T^2N^2-\gamma^2\}^{1/2}. \qquad (6.8)$$

For the sake of simplicity we put $T^+=T^-=T$. The relation is always satisfied, for instance, for waves excited at the equator. The second branch of oscillations Ω_{p2} of (6.5) is then always stable. It is clear from Eq. (6.8) that even when $\Omega_{p1}^2(0)>0$ the occurrence of an instability is possible in the region $\omega''\varkappa^2\sim(T+S)N$, if $\omega''(T+S)<0$.

In[17,18] it was shown theoretically that the development of an instability with $\varkappa=0$ leads to the occurrence of self-oscillations of the integral amplitude of the waves. For a parametric excitation of spin waves in ferromagnetics they appear experimentally as self-oscillations of the magnetization.[18,19]

The sign of ω'' in cubic ferromagnetics depends on the strength of the external field H and can easily produce a situation where the spatially uniform oscillations are stable: $\mathrm{Im}\Omega(0)<0$ and an instability localized in the region $\omega''\varkappa^2\sim SN$. As a result of the development of this instability there may occur a stationary spatially nonuniform pattern $A_k(r)$ with a characteristic scale $(\omega''/SN)^{1/2}$, a depth of the modulation of the order of unity, and an average level of the order of (6.2).[20] An interesting problem arises of finding such stationary states and of studying their stability in the framework of Eqs. (5.8). It is of interest to attempt also to discover this effect experimentally.

In conclusion we consider briefly the problem of the effect of the boundaries of the sample on the stationary amplitude of the waves beyond the threshold for the parametric instability. It is clear that in the bulk of the sample the solution of the S-theory (6.2) is realized, while at the boundary the amplitude differs from (6.2). Of great interest is therefore the problem of the nature of the approach of the solution $A(z)$ to the uniform distribution (6.2). It is clear that the equations obtained are identical with Eqs. (6.5) studied in the problem of the collective oscillations, if we put there $\partial/\partial t=0$. We obtain therefore the dispersion equation for $\varkappa$ directly from (6.5) putting $\Omega(\varkappa)=0$:

$$\delta A\propto e^{i\varkappa z},\quad \Omega^2_{p(1,2)}=0. \tag{6.9}$$

As the $\Omega^2_{p(1,2)}$ as functions of $\varkappa$ are polynomials the merging into the asymptotic behavior of the S-theory (6.2) carries an exponential character. We determine the characteristic scales of the approach to the uniform solution.

A. Excluding the case when the waves propagate parallel to the sample boundary, we can neglect in (6.9) the terms with $\omega''\varkappa^2$ and (6.9) then simplifies to the form

$$(\varkappa v)^2=\Delta_p{}^2=4(T_p{}^+-T_p{}^-)(T_p{}^++T^-+S_p)N^2. \tag{6.10}$$

The approach to the uniform solution occurs only when $\Delta_p^2<0$, i.e., when the mode Ω_{p2} turns out to be stable.

B. If, however, the waves propagate parallel to the boundary, $\varkappa v=0$ and $T^+=T^-$. We obtain the dispersion equation for $\varkappa$ by putting $\Omega(\varkappa)$ equal to zero in Eq. (6.8):

$$[2(T+S)N+{}^1/{}_2\omega''\varkappa^2]^2=4T^2N^2. \tag{6.11}$$

Negative $\varkappa^2$ are a solution of (6.11) only in the case where $S(T+S)>0$, $\omega''(T+S)>0$.

Turning to (6.8) we see that these requirements are the same as the conditions for the stability of the ground state. The exponential asymptotic behavior of the approach to the uniform solution of the S-theory is thus closely connected with the stability of the ground state with respect to collective oscillations: just in the stable case there exist only complex solutions of the equation for $\Omega_p(\varkappa)$. If, however, the ground state is unstable with respect to spatially non-uniform perturbations, there exist, as we noted already above (see also[20]) apart from the uniform also quasi-periodic distributions of the oscillations with an average level $\sim SN$. In that case, real $\varkappa$ obtained from (6.10), (6.11) indicate that we found the simplest solutions of such a kind—"fine domains" in the terminology of[20].

We want to thank V. E. Zakharov for discussing a number of problems connected with the present paper.

1)The results of this section were partially published in a preprint by one of the authors.[14]

1N. Bloembergen, Non-linear Optics, Benjamin, New York, 1965.
2V. E. Zakharov, Zh. Eksp. Teor. Fiz. 62, 1745 (1972) [Sov. Phys. JETP 35, 908 (1972)].
3E. Valeo, C. Oberman, and F. W. Perkins, Phys. Rev. Lett. 28, 340 (1972). B. N. Breizman, V. E. Zakharov, and S. L. Musher, Zh. Eksp. Teor. Fiz. 64, 1297 (1973) [Sov. Phys. JETP 37, 658 (1973)].
4A. M. Rubenchik, I. Ya. Rybak, and B. I. Sturman, Zh. Eksp. Teor. Fiz. 67, 1364 (1974) [Sov. Phys. JETP 40, 678 (1975)].
5E. Valeo and W. L. Kruer, Phys. Fluids 16, 675 (1973). V. E. Zakharov, S. L. Musher, and A. M. Rubenchik, Zh. Eksp. Teor. Fiz. 69, 155 (1975) [Sov. Phys. JETP 42, 80 (1976)].
6A. A. Vedenov and L. I. Rudakov, Dokl. Akad. Nauk SSSR 159, 767 (1964) [Sov. Phys. Dokl. 9, 1073 (1965)].
7V. E. Zakharov, V. S. L'vov, and S. S. Starobinets, Usp. Fiz. Nauk 114, 609 (1974) [Sov. Phys. Usp. 17, 896 (1975)].
8V. E. Zakharov, Zh. Eksp. Teor. Fiz. 60, 1714 (1971) [Sov. Phys. JETP 33, 927 (1971)].
9A. M. Rubenchik, Izv. vuzov Radiofizika 17, 1635 (1974) [Radiophys. Qu. Electron. 17, 1249 (1976)].
10S. L. Musher, Thesis, Novosibirsk, 1974
11S. A. Akhmanov, A. P. Sukhorukov, and R. V. Khokhlov, Usp. Fiz. Nauk 93, 19 (1967) [Sov. Phys. Usp. 10, 609 (1968)].
12V. I. Vlasov, V. I. Talanov, and V. A. Petrishev, Izv. vuzov Radiofizika 14, 1353 (1971) [Radiophys. Qu. Electron 14, 1062 (1974)].
13V. E. Zakharov and V. S. Synakh, Zh. Eksp. Teor. Fiz. 68, 940 (1975) [Sov. Phys. JETP 41, 465 (1975)].
14V. S. L'vov, Ustoĭchivost' osnovnogo sostoyaniya i kollektivnye kolebaniya sistemy parametricheski vozbyzhdennykh spinovykh voln (Stability of the ground state and collective oscillations of a system of parametrically excited spin waves) Preprint Inst. Nucl. Phys. 8–73, Novosibirsk, 1973.
15V. V. Zautkin, V. S. L'vov, and S. S. Starobinets, Zh. Eksp. Teor. Fiz. 63, 182 (1972) [Sov. Phys. JETP 36, 96 (1973)].
16V. V. Zaytkin, B. I. Orel, and S. S. Starobinets, Zh. Eksp. Teor. Fiz. 68, 317 (1975) [Sov. Phys. JETP 41, 154 (1975)].
17V. V. Zautkin and S. S. Starobinets, Fiz. Tverd. Tela (Leningrad) 16, 678 (1974) [Sov. Phys. Solid State 16, 446 (1975)].
18V. S. L'vov, S. L. Musher, and S. S. Starobinets, Zh. Eksp. Teor. Fiz. 64, 1074 (1973) [Sov. Phys. JETP 37, 546 (1973)].
19T. S. Hartwick, E. R. Peressini, and M. T. Weiss, J. Appl. Phys. 32, 323S, (1961).
20V. S. L'vov and A. M. Rubinchik, Dinamicheskie domeny pri parametricheskom vozbuzhdenii voln. Ikh neustoĭchivost' i razrushenie (Dynamic domains when waves are parametrically excited. Their instability and disintegration) Preprint Inst. Nucl. Phys. 1–72, Novosibirsk, 1972.

Translated by D. ter Haar