Contribution to the theory of the structure of the intermediate state of a superconductor

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The layered structure of the intermediate state of a superconducting plate is investigated in a strongly inclined field close to critical. It is shown that in this case the *ns* boundary has an anomalous shape determined by the surface tension. The transition from the intermediate to the normal state at an arbitrary angle of inclination of the field is also considered.

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The determination of the shape of the boundary between the superconducting and normal regions is the basic problem in the theory of the structure of the intermediate state of a superconductor. It can be usually assumed that the field at the *ns* boundary is equal to the critical value H_c. This makes it possible to find the shape of the boundary and the period of the layered structure in a plate situated in a perpendicular field. [1] The results are easily generalized to include the case of an oblique field. [2,3] But if the external field is close to critical and is strongly inclined to the surface of the plate, the situation changes. To find the shape of the boundary it is then necessary to take into account the surface tension. [4]

It is known that the following thermodynamic condition must be satisfied on the *ns* boundary:

$$H^2=H_c^2+8\pi\alpha/R. \tag{1}$$

Here H is the magnetic field, α is the surface-tension coefficient, and R is the radius of curvature of the boundary, which is assumed to be positive if it is directed into the interior of the n phase. We introduce a Cartesian coordinate system xyz. The z axis is directed along the layers and parallel to the surface of the plate, while the x axis is perpendicular to this surface and the z component of the field is constant both inside the n regions and in the vacuum, and equal to the external-field component $H_\parallel$ parallel to the plate (see[2,3]). The shape of the boundary and the field strength in the xy plane are connected by the condition (1), which we rewrite in the form:

$$H_x^2+H_y^2=H_c^2-H_\parallel^2+H_c^2\Delta/R, \tag{1a}$$

where $\Delta=8\pi\alpha/H_c^2$ is the "thickness" of the *ns* boundary. Although in the macroscopic problem we always have $R\gg\Delta$, in a strongly inclined field close to H_c, both terms in the right-hand side of (1a) can be of the same order.

The period a of the structure and the concentration of the phases are determined from the conditions that the thermodynamic potential of the plate be a minimum (see[1–5]):

$$\Phi=-\frac{1}{2}\mathbf{M}_\perp\mathbf{H}_\perp+\frac{H_\parallel^2-H_c^2}{8\pi}V_s+\alpha S_{ns}+\sigma S_{vs}. \tag{2}$$

Here $H_\perp$ and $M_\perp$ are the perpendicular (to the plate) components of the external field and of the magnetic moment of the current flowing over the s-phase boundary; V_s is the volume of the s phase; S_{ns} and S_{vs} is the area of the *ns* boundaries and of the boundaries of the s phase with the vacuum; σ is the coefficient of the surface tension of the *vs* boundary.

In the Ginzburg-Landau phenomenological theory of superconductivity[6] it is shown that σ is negative (see[6], formulas (48)–(53)) and is connected with the depth δ of penetration of the magnetic field by the relation

$$\sigma=-H_c^2\delta/8\pi$$

in the approximation $\delta/\Delta\ll 1$. In strongly inclined magnetic fields, the energy σS_{vs} cannot be neglected. The results of Dzyaloshinskiĭ[2] and Sharvin[3] indicate that in our case the period will be of the order of the thickness of the plate L and larger. When $a\sim L$ then also $R\sim L$, and if $(H_c^2-H_\parallel^2)L\sim H_c^2\Delta$, then it is impossible to determine the distribution of the magnetic field and the shape of the *ns* boundaries in analytic form; we therefore turn directly to the limiting case $a\gg L$ in a strongly inclined field: $|H_c^2-H_\parallel^2|L\ll H_c^2\Delta$.

The main contribution to the value of the magnetic moment $M_\perp$ at $a\gg L$ is made by the current flowing over the surface of the plate. The current is connected with the magnetic field at the surface by the relation $J_z=cH_y/4\pi$ [5] (c is the speed of light). To calculate the field we can put in general $L=0$. The magnetic complex potential $\psi=\varphi+iA$ then maps conformally the strip on the $x+iy$ plane between the straight lines $y=0$ and $y=a/2$, with a cut along the y axis from $y=0$ to $y=c_s a/2$ (c_s is the s-phase concentration), into a strip in the ψ plane between the straight lines $A=0$ and $A=H_\perp a/2$; this potential is defined by the formula

$$\operatorname{ch}\frac{\pi\psi}{H_\perp a}=\left(\cos\frac{\pi}{2}c_s\right)^{-1}\operatorname{ch}\frac{\pi}{a}(x+iy).$$

Differentiating this expression with respect to y at $x=0$ and $A=0$, we obtain the sought field $H_y=\partial\varphi/\partial y$ on the s-phase boundary:

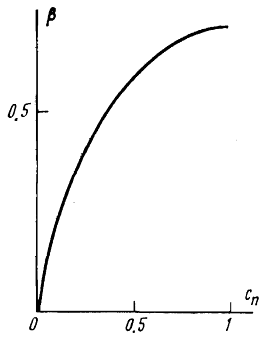

FIG. 1. Plot of the function $\beta(c_n)$; $\max\beta = 2^{-1/2}$ at $c_n = 1$.

$$H_y = H_\perp \sin\frac{\pi}{a}y\left[\left(\cos\frac{\pi}{a}y\right)^2 - \left(\cos\frac{\pi}{2}c_s\right)^2\right]^{-1/2}. \quad (3)$$

We can now determine the magnetic moment per unit volume of the plate, by using the known formula (see[5], formula 29.17), which takes in our case the form

$$M_x = -4\frac{2}{La}\frac{1}{2c}\int_0^{c_s a/2} yJ_z dy.$$

The factor 4 takes into account both sides of the plate and the current flowing on the edges of the plate. Integrating we have after expressing J_z in terms of H_y (3)

$$M_x = H_\perp\frac{a}{2\pi^2}\ln\cos\left(\frac{\pi}{2}c_s\right).$$

Using this value of the moment, we obtain for the density of the thermodynamic potential the expression

$$-\frac{H_\perp^2}{4\pi^2}\frac{a}{L}\ln\cos\left(\frac{\pi}{2}c_s\right) - \frac{H_\sigma^2}{8\pi}c_s + \frac{H_c^2}{4\pi}\frac{\Delta}{a}, \quad (4)$$

where we have introduced the notation $H_\sigma^2 = H_c^2 - H_\parallel^2 + 2H_c^2\delta/L$. The length of the ns boundary, as we shall show, can be assumed equal to L. From the condition that (4) be a minimum, we obtain the period

$$a = \pi^{1/2}\frac{H_c}{H_\perp}\left[-\frac{\Delta L}{\ln\cos(\pi c_s/2)}\right]^{1/2}, \quad (5)$$

and the equation for the normal-phase concentration $c_n = 1 - c_s$ is

$$\operatorname{tg}\frac{\pi}{2}c_n\left(-\ln\sin\left(\frac{\pi}{2}c_n\right)\right)^{1/2} = \beta = \pi^{1/2}\frac{H_\perp H_c}{H_\sigma^2}\left(\frac{\Delta}{L}\right)^{1/2}. \quad (6)$$

The dependence of β on c_n is shown in Fig. 1.

Let us see first what happens at $\beta \ll 1$, when the field $H_\perp$ is weak enough. The concentration of the n phase is small in this case, and, expanding Eqs. (5) and (6) in its terms, we obtain

$$c_n\left(\ln\frac{2}{\pi c_n}\right)^{1/2} = \frac{2}{\pi}\beta,$$
$$a = \pi^{1/2}\frac{H_c}{H_\perp}\left(\frac{\Delta L}{\ln(2/\pi c_n)}\right)^{1/2}, \quad (7)$$

from which we have with logarithmic accuracy the concentration

$$c_n = \frac{2\beta}{\pi\ln^{1/2}(1/\beta)}.$$

When determining the magnetic field (3) we have put $L = 0$. This can be done if the widths a_n and a_s of the normal and superconducting regions are much larger than L. At $\beta \ll 1$ we have $a_s \approx a \gg a_n$, so that it is necessary to have $a_n \gg L$, as is the case when $H_c^2\Delta \gg H_\sigma^2 L$.

We determine next the deviation f of the shape of the ns boundary from linear. To this end we solve Eq. (1a). We replace the curvature R^{-1} in it by its value $R^{-1} = d^2f/dx^2$ at small f (see[7]):

$$\frac{d^2f}{dx^2} = \frac{H^2 - H_c^2 + H_\parallel^2}{\Delta H_c^2}. \quad (8)$$

To find the field we can assume the boundary to be plane, and then the problem reduces to a conformal mapping of the exterior of a semi-infinite band onto a half-plane. We write down the solution in parametric form:

$$H_x + iH_y = H_0/(1-\omega^2)^{1/4},$$
$$x + iy = L\pi^{-1}[\omega(1-\omega^2)^{1/2} + \arcsin\omega]. \quad (9)$$

The point $x = 0$, $y = 0$ was chosen on the ns boundary in the center of the plate.

We determine the value of H_0 by comparing the obtained expression for the field (9) far from the ns boundary with the expression (4) near the point $y = c_s a/2$; we obtain simply $H_0 = 2^{-1/2}H_\sigma$. Substituting finally the value for the field (9) in Eq. (8) and integrating the resultant second-order differential equation, we determine the shape of the boundary:

$$f = -\frac{H_\sigma^2 L^2}{\pi^2 H_c^2\Delta}\left(\omega^2 - \frac{\omega^4}{2}\right) + \frac{\delta}{\Delta L}x^2, \quad x = \frac{L}{\pi}[\omega(1-\omega^2)^{1/2} + \arcsin\omega]. \quad (10)$$

Figure 2 shows the change of the boundary in the case of an oblique field.

We note that in our solution the field becomes infinite where the ns boundary emerges to the surface. But since the field becomes larger than H_c at a microscopic distance on the order of Δ, the singularity does not influence the solution of the macroscopic problem, inasmuch as the corresponding contribution to the energy can be neglected.

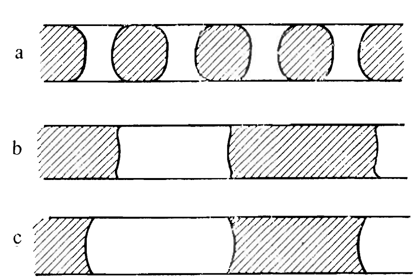

FIG. 2. Change of shape of the boundary in an oblique field: a) ordinary structure at $(H_c^2 - H_\parallel^2)L \gg H_c^2\Delta$; b) anomalous structure at $H_c^2\Delta \gg (H_c^2 - H_\parallel^2)L \gg H_c^2\delta$; c) anomalous structure at $H_\perp \to H_c(1 + 2\delta/L)$, $f = \delta x^2/\Delta L$.

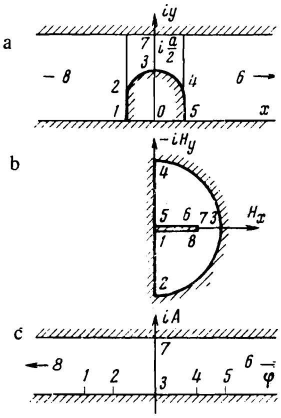

FIG. 3. a) Half-period of layered structure on the plane $z=x+iy$; b) corresponding regions on the plane of the field $H=H_x-iH_y$; c) the same on the field of the magnetic potential $\psi=\varphi+iA$.

We determine now the period and the concentration at $\beta\to 2^{-1/2}$. In this case the s-phase concentration is small, and expanding (5) and (6) in its terms we obtain

$$c_s=\frac{4}{\pi}\left(1-\frac{H_\perp}{H_\perp^c}\right)^{1/2},$$
$$a=\pi\left(\frac{H_c}{H_\sigma}\right)^2\left(\frac{H_\perp^c}{H_\perp^c-H_\perp}\right)^{1/2}\Delta, \tag{11}$$
$$H_\perp^c=(H_\sigma^2/H_c)(L/2\pi\Delta)^{1/2}.$$

If $H_\perp > H_\perp^c$, then the production of s layers is not profitable, since the gain in the volume energy is suppressed by the energy required to produce the ns boundary; this circumstance is pointed out in de Gennes' book.[8] The same book contains an estimate of the field at which a layered structure goes over from the intermediate to the normal state in a perpendicular field. We calculate here the exact value of this field at $(H_c^2-H_\parallel^2)L\gg H_c^2\Delta$. For this purpose we must, in contrast to Landau[1] consider the case $a\gg L$. It turns out that a solution of the problem can be obtained without any assumptions concerning the relations between the quantities a, a_s, and L.

We use again the conformal-mapping method. In an oblique field that is not too strong it is possible, as usual, to neglect the surface tension when the shape of the boundary is determined. Figure 3 shows the half-period of the layered structure on the plane $z=x+iy$ (a) and the corresponding regions on the plane of the field $H=H_x-iH_y$ (b) and on the plane of the magnetic potential $\psi=\varphi+iA$ (c). The points 6 and 8 correspond to infinity on the z plane, where the field is equal to $H_\perp$. The radius of the semicircle (b) is $(H_c^2-H_\parallel^2)^{1/2}$. By mapping the region b into the strip c, we obtain the function $\psi(H)$ in parametric form:

$$\psi=\frac{H_\perp a}{2\pi}\ln\frac{\omega-\omega_6}{\omega+\omega_6},\qquad \omega=\frac{h}{(h^2-h_7^2)^{1/2}},\qquad h=\frac{H_c^2-H_\parallel^2-H^2}{2H(H_c^2-H_\parallel^2)^{1/2}}, \tag{12}$$

where h_7 is the value of h at the point 7 and ω_6 is the value of ω at the point 6. The shape of the boundary is determined from the relation $H=d\psi/dz$, whence $dz=H^{-1}d\psi$. We shall not dwell on the rather cumbersome calculations and write out only the value of the period in the case when $a\gg L$.

$$a=\pi\alpha^{1/4}L^{3/4}/4(H_\perp^c-H_\perp)^{1/2},$$
$$H_\perp^c=(H_c^2-H_\parallel^2)^{1/2}-2^{3/2}H_c(\Delta/\pi L)^{1/2}. \tag{13}$$

We indicate also that the s-layer thickness tends to zero on the surface and to the value $2^{3/2}\pi^{-1/2}(\Delta L)^{1/2}H_c/(H_c^2-H_\parallel^2)^{1/2}$, in the center of the plate as $H_\perp\to H_\perp^c$.

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