decreasing plasma-density gradient and increasing oscillation frequency. Virtually total absorption is observed in the case when the wavelength of the oscillations turns out to be comparable in magnitude to the characteristic dimension of an inhomogeneity of the plasma.

The found experimental values of the reflection coefficients are in fairly good agreement with the values theoretically computed from the formula (3).

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Direction of transfer of energy and quasi-particle number along the spectrum in stationary power-law solutions of the kinetic equations for waves and particles

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We determine the sign of the flux along the spectrum in exact power-law solutions of the kinetic equations for waves and particles. The direction of the fluxes is uniquely determined by the exponents of the distribution and of the dispersion law, and for the activation spectrum also by the sign of the dispersion term; it depends only on the sign of a simple combination of the exponents. In particular, the direction of the energy flux for waves with a decay spectrum depends on whether the index of the distribution is larger or smaller than -1. We obtain similar criteria also in other cases.

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§1. INTRODUCTION

We know at present many examples of power-law distributions for waves[1–4] and also for particles,[5] obtained as exact solutions of the kinetic equations which describe the interaction of a stochastic ensemble of waves or particle collisions. Stationary non-equilibrium isotropic solutions correspond to a constant, nonvanishing flux of the number of quasi-particles or of energy along the spectrum (see, e.g.,[6,7]). The direction of the flux, which is an important characteristic of the distribution, can sometimes be determined from qualitative considerations using the specific nature of the system (e.g., short-wavelength damping in hydrodynamics, and so on).

We obtain in the present paper general results about the direction of the fluxes for waves with a non-decay (§3) and with a decay (§6) spectrum, and also for particles (§4).[1)] We consider separately the case of an activation spectrum which is of interest in connection with plasmon, exciton, electron, and hole distributions in metals and semiconductors (§5). It turns out that when we determine the direction of the flux we can bypass the calculation of the integral which connects the flux with the distribution functions (§2) if we use symmetry transformations which enable us to find the power-law distributions.[1,2,7,9] These calculations are, however, necessary for finding the dimensionless constants in the distribution function (see the Appendixes).

The direction of the flux is determined by simple inequalities concerning the exponent of the distribution. An analysis of these inequalities shows, in particular, that the transfer of energy and of quasi-particle number proceeds, as a rule, in different directions, as was noted by V. E. Zakharov for hydrodynamic types of systems.

§2. CONNECTION BETWEEN FLUXES AND DISTRIBUTION FUNCTIONS

In a uniform medium when there are no external forces the kinetic equation has the form

$$\frac{\partial n(\mathbf{k})}{\partial t}=I_{\rm coll}\{n\}, \tag{2.1}$$

where $I_{\rm coll}\{n\}$ is the collision integral which describes the interaction of the waves or the particles, $\mathbf{k}$ is the wavevector (or momentum), and $n(\mathbf{k})$ is the distribution function. Since the energy is conserved in the elementary interaction process, we have

$$\int d\mathbf{k}\omega(\mathbf{k})\frac{\partial n(\mathbf{k})}{\partial t}=\int d\mathbf{k}\,\omega(\mathbf{k})I_{\rm coll}\{n\}=0,$$

$\omega(\mathbf{k})$ is the frequency (energy) of the waves. Hence, we can introduce the energy flux density in $\mathbf{k}$-space $\mathbf{j}_E=\mathbf{j}_1(\mathbf{k})$[7]:

$$\operatorname{div}\mathbf{j}_1=-\omega(\mathbf{k})I_{\rm coll}\{n\}. \tag{2.2}$$

Similarly, if the number of particles (or waves) is an integral of motion, we can introduce the particle flux density $\mathbf{j}=\mathbf{j}_0$[7]:

$$\operatorname{div}\mathbf{j}_0(\mathbf{k})=-I_{\rm coll}\{n\}. \tag{2.3}$$

We restrict the consideration to the isotropic case when the transition probability, the dispersion law $\omega(k)$, and also the distribution function $n(k)$ are invariant under rotations. Only the radial component of the flux density is then non-vanishing so that $\mathbf{j}=(bk^{d-1})^{-1}J(k)\mathbf{k}/k$, where d is the dimensionality of $\mathbf{k}$-space, $J(k)$ is the total flux, and b is a numerical coefficient ($b=2\pi$ in the two-dimensional case and $b=4\pi$ in the three-dimensional case). We get thus from (2.1) and (2.2) for the fluxes (cf. [5,7])

$$J_i(k)=-b\int^{k}dk\,k^{d-1}\omega^i(k)I_{\rm coll}\{n\},\qquad i=0,1, \tag{2.4}$$

$i=0$ corresponds to the particle flux and $i=1$ to the energy flux. Equation (2.4) enables us to find the flux, if we know the distribution $n(k)$.

We proceed now to find the fluxes for the case of power-law distributions.

§3. SCATTERING OF WAVES

We consider the collision integral which describes wave scattering processes:

$$I_{\rm coll}\{n\}=\int d\tau_k w_k f_k,\qquad d\tau_k=d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3, \tag{3.1}$$
$$w_k=U(\mathbf{k}\mathbf{k}_1|\mathbf{k}_2\mathbf{k}_3)\,\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\,\delta(\omega+\omega_1-\omega_2-\omega_3)$$

is the transition probability and

$$f_k=nn_1n_2n_3(1/n+1/n_1-1/n_2-1/n_3),\ n_1=n(k_1),\ldots \tag{3.2}$$

We assume that the dispersion law $\omega(k)$ and the interaction matrix element are homogeneous functions:

$$\omega(\lambda k)=\lambda^\beta\omega(k),\quad U(\lambda\mathbf{k}\lambda\mathbf{k}_1|\lambda\mathbf{k}_2\lambda\mathbf{k}_3)=\lambda^m U(\mathbf{k}\mathbf{k}_1|\mathbf{k}_2\mathbf{k}_3). \tag{3.3}$$

For a power-law distribution

$$n(k)=A\omega^s, \tag{3.4}$$

$I_{\rm coll}\{n\}$ is also a power-law function:

$$I_{\rm coll}=D(\nu)k^{\beta\nu-d}$$

(if $I_{\rm coll}\{n\}$ agrees with the distribution (3.4)) so that we get for the fluxes from (2.4) (cf. [8]):

$$J_i(k)=-bk^d\omega^i(k)\frac{I_{\rm coll}}{\beta(\nu+i)},\qquad \nu=\nu(s)=3s-1+\frac{m+3d}{\beta},\qquad i=0,1. \tag{3.5}$$

In that case

$$J_0(k)\sim k^{\beta\nu}\sim\omega^{\nu(s)},\ J_1\sim k^{\beta(\nu+1)}\sim\omega^{\nu(s)+1},$$

so that we shall have $J_0=\text{const}$ when $\nu=0$, and $J_1=\text{const}$ when $\nu=-1$.[2)] Those values $s=s_0, s_1$ which lead to $\nu=0$ ($s=s_0$) and $\nu=-1$ ($s=s_1$) correspond to stationary solutions of the kinetic equation ($I_{\rm coll}=0$ for $\nu=0,-1$[7,9]), while the solution with $s=s_0$ corresponds to a constant particle flux ($J_0\neq0$, $J_1=0$) and the solution with $s=s_1$ to a constant energy flux ($J_0=0$, $J_1\neq0$).

Expression (3.5) for J_i with $\nu(s)=-i$ contains an indeterminate expression of the form 0/0. The fluxes are therefore, as noted earlier for the case of particle power-law distributions,[8] proportional to the derivative of $I_{\rm coll}$:

$$J_i\sim\partial I_{\rm coll}/\partial\nu|_{\nu=-i},\ i=0,1.$$

To find the derivative we use the factorized form of the collision integral[7]

$$I_{\rm coll}=\frac{\omega^\nu}{4}\int d\tau_k w_k f_k[\omega^{-\nu}+\omega_1^{-\nu}-\omega_2^{-\nu}-\omega_3^{-\nu}],$$

whence we find, using (3.5)

$$J_i=\frac{b}{4\beta}k^d\int d\tau_k w_k f_k|_{s=s_i}[\omega^i\ln\omega+\omega_1^i\ln\omega_1-\omega_2^i\ln\omega_2-\omega_3^i\ln\omega_3]. \tag{3.6}$$

Using the explicit expression (3.4) for $n(k)$ for $s=s_i$ we get an expression for the flux which is convenient for what follows:

$$J_i=\frac{bA_i^3}{4\beta}k^d\int d\tau_k w_k(\omega\omega_1\omega_2\omega_3)^{s_i}[\omega^{-s_i}+\omega_1^{-s_i}-\omega_2^{-s_i}-\omega_3^{-s_i}]$$
$$\times[\omega^i\ln\omega+\omega_1^i\ln\omega_1-\omega_2^i\ln\omega_2-\omega_3^i\ln\omega_3]. \tag{3.7}$$

It is clear from (3.7) that $A\propto|J|^{1/3}$ which, by the way, follows both from dimensionality considerations (see[1,2,7]) and from Eqs. (2.2) and (2.3) (cf. [7]). It is clear, however, that (3.7) establishes an exact connection between the flux and the parameter A of the distribution which enables us, in particular, to find the dimensionless numerical coefficient which cannot be determined by dimensionality considerations. The explicit evaluation of the integral in (3.7) requires detailed knowledge of the transition probability which we shall not give in the present paper. It is important that by using the presentation (3.7) we can reach a conclusion about the sign of the flux and thereby find its direction: $J>0$ and $J<0$ correspond to fluxes in the direction of larger and smaller wavenumbers, respectively.

TABLE I.

	J_0	J_1		J_0	J_1
$m > 4\beta - 3d$, $s_1 < -4/3$, $s_0 < -1$	<0	>0	$-3d < m < \beta - 3d$, $-1/3 < s_1 < 0$, $0 < s_0 < 1/3$	<0	<0
$3(\beta - d) < m < 4\beta - 3d$, $-4/3 < s_1 < -1$, $-1 < s_0 < -2/3$	>0	>0	$m < -3d$, $0 < s_1$, $1/3 < s_0$	<0	>0
$\beta - 3d < m < 3(\beta - d)$, $-1 < s_1 < -1/3$, $-2/3 < s_0 < 0$	>0	<0			

Note: The directions of the fluxes are indicated for $\beta > 0$. When $\beta < 0$ the directions of the fluxes are reversed for the same s.

Indeed, the probability w_k is essentially positive and the remaining part of the integrand has a fixed sign so that its sign depends on the value of the index s of the distribution. To prove that statement we use the inequality

$$(\omega_2\omega_3 - \omega\omega_1)(\varphi + \varphi_1 - \varphi_2 - \varphi_3)\operatorname{sign}\varphi'' \geq 0,$$
$$\varphi = \varphi(\omega),\ \varphi_m = \varphi(\omega_m),\ \varphi'' = d^2\varphi/d\omega^2 \neq 0, \tag{3.8}$$
$$\omega + \omega_1 = \omega_2 + \omega_3,$$

which can be obtained by using Szegö's inequality[11] for convex functions. Applying (3.8) to the functions $\varphi(\omega) = \omega^i \ln\omega$ and $\chi(\omega) = \omega^{-s_i}$, $i = 0, 1$, we get

$$(\chi + \chi_1 - \chi_2 - \chi_3)(\varphi + \varphi_1 - \varphi_2 - \varphi_3)\operatorname{sign}\varphi''\chi'' \geq 0, \tag{3.9}$$

whence, if we use the fact that $(\omega\ln\omega)'' > 0$ and $(\ln\omega)'' < 0$, it follows that the integrand in (3.7) has a fixed sign. From (3.9) and (3.7) we get for the direction of the fluxes

$$\operatorname{sign} J_0 = -\operatorname{sign}\beta s_0(s_0+1),\ \operatorname{sign} J_1 = \operatorname{sign}\beta s_1(s_1+1). \tag{3.10}$$

It is clear from (3.10) that the direction of the fluxes depends strongly on the sign of the degree of homogeneity β of the dispersion law, i.e., on whether the frequency increases or decreases with increasing wavenumber. Moreover, the direction of the fluxes changes when the corresponding exponents s_0 and s_1 pass through the values 0 and -1, while in agreement with (3.7) $J_i = 0$ when $s_i = 0, 1$. When $s = 0, -1$ the collision integral (3.1) is made to vanish by the power-law distribution (3.4) which in these cases corresponds to the equilibrium distributions $n = T/\omega(k)$ $(s = -1)$ and $n = -T/\mu$ $(s = 0)$, where T is the temperature and μ the chemical potential. Equilibrium distributions correspond to zero fluxes. Since the collision integral vanishes for the distributions (3.4) for four values of s ($s = 0$, -1, s_0, and s_1), its derivatives with respect to s in the points $s = s_0$, s_1 which determine the magnitude and the direction of the fluxes will have different signs, depending on the relative positions of the numbers 0, -1, s_0, and s_1. As the exponents s_0 and s_1 are not independent, $s_0 = s_1 + \frac{1}{3}$, $s_1 = -(m + 3d)/3\beta$,[7] there are five variants for the arrangement of the points enumerated (see Table I).

For all presently known wave distributions with a non-decay dispersion law (see[1] for gravitational waves on the surface of a liquid and[2] for Langmuir plasmons) the condition $m > 4\beta - 3d$ $(s_0, s_1 < -1)$ is satisfied and, hence, the particle flux is in the direction of long waves and the energy flux in the direction of short waves. This is not an accident, for two reasons. Firstly, if there are in the system no parameters with the dimensions of length and time, except k^{-1} and $\omega(k)^{-1}$ (i.e., if there occurs total similarity as, for instance, for gravitational waves on the surface of a deep liquid[1,4]), we have $m = 10 - 2d$ as follows from dimensionality considerations.[7] A violation of the inequality $m > 4\beta - 3d$ is thus possible only for large $\beta > \frac{1}{4}(d + 10) \geq 3$ $(d \geq 2)$. For a non-decay dispersion law without activation $\beta < 1$ and therefore necessarily $J_0 < 0$, $J_1 > 0$. Secondly, when $\beta < 1$, $d \geq 2$, $4\beta - 3d < -2$ and the condition $m > 4\beta - 3d$ can be violated only for negative m, $m < -2$. However, in hydrodynamic types of systems the interaction is "frozen in" when the wavelength increases, $\lambda = 2\pi/k \to \infty$, which leads to positive m. The change in the sign of the quantity $4\beta - 3d$ corresponds to too large a β so that also for an activation non-decay dispersion law[3)] when the restriction $\beta < 1$ is lifted the change in the direction of the fluxes also correspond to negative m. Nonetheless, in principle it is not excluded that there is a possibility of an interaction leading in a limited range of wavenumbers to $m < 0$ (see, e.g., $m = 0$ in[2]) which can lead to a change in the direction of the fluxes as compared to the usual $J_0 < 0$, $J_1 > 0$. Moreover, for a decreasing dispersion law $(\beta < 0)$ the direction of the fluxes changes also for $m > 0$.

It is necessary to note that the consideration given here is valid in the case when the collision integral converges for power-law distributions with $s = s_0, s_1$, i.e., if these distributions are local.[1,4,7] If, on the other hand, one of these distributions is local, only the formulae referring to it are correct.

§4. SCATTERING OF PARTICLES

We consider fluxes for non-equilibrium stationary power-law particle distributions which are produced by a Boltzmann type collision integral.[5],4) I_{coll} then has the form (3.1) with the obvious substitutions $\mathbf{k} \to \mathbf{p}$, $\omega(k) \to E(p)$, $f_k \to f_p$, where

$$f_p = n_2 n_3 - n n_1, \tag{4.1}$$

$\mathbf{p}$ is the momentum and $E(p)$ the energy. Under the same assumptions as before we get for the fluxes expressions such as (3.6), whence we get after substituting $n(p) = AE^s$

$$J_i = \frac{bA_i^2}{4\beta} p^d \int d\tau_p w_p [(E_2E_3)^{s_i} - (EE_1)^{s_i}][E^i \ln E + E_1^i \ln E_1 - E_2^i \ln E_2 - E_3^i \ln E_3], \quad i = 0, 1, \tag{4.2}$$

where, as before, the s_i are determined by the conditions $\nu(s_0) = 0$, $\nu(s_1) = -1$, and $\nu(s) = 2s - 1 + (m + 3d)/\beta$.[5] From (4.2) we get immediately that the particle flux has a fixed sign, if we transform the integrand for $i = 0$ to the form

$$[(E_2E_3)^{s_0} - (EE_1)^{s_0}] \ln\frac{EE_1}{E_2E_3} = s_0^{-1}(E_2E_3)^{s_0}(1 - x)\ln x,$$
$$x \equiv (EE_1/E_2E_3)^{s_0}$$

and use the inequality $(1 - x)\ln x \leq 0$. Hence

$$\text{sign}\, J_0 = -\text{sign}\, \beta s_0. \tag{4.3}$$

To determine the direction of the energy flux it is sufficient to use inequality (3.8) and the easily verified inequality

$$s(E_2E_3 - EE_1)[(E_2E_3)^s - (EE_1)^s] \geqslant 0,$$

whence we get

$$\text{sign}\, J_1 = \text{sign}\, \beta s_1. \tag{4.4}$$

The different directions of the flux for $\beta s < 0$ and $\beta s > 0$ are a consequence of the fact that $n(p) = \text{const}$ $(s = 0)$ makes the particle collision integral vanish. It follows from (4.4) for a Coulomb distribution of non-relativistic particles ($\beta = 2$, $s_1 = -\frac{5}{4}$ [5]) that the energy flux is in the direction of small velocities, as had been shown earlier by us[5] using the Landau equation and also by Karas' *et al.*[8] We note here that under the conditions when the Born approximation is valid the distribution with an energy flux will be local, if the degree of homogeneity of the matrix element $-5 < m < -3$, and the one with a particle flux, if $-3 < m < -1$.[5] For a given m only one of the distributions is therefore local and the energy flux is in the direction of small, and the particle flux in the direction of large momenta.

In the case of a power-law interaction potential $V(r) = V_0 r^{-\alpha}$ the fluxes will be in the same direction as in the Born approximation for a wide range of the exponents α. Using similarity considerations we find[12,5] in three-dimensional space $m = 2\beta - 4 - 2\beta/\alpha$. Using the fact that $s_1 = -(m + 3d)/2\beta$, $s_0 = s_1 + \frac{1}{2}$ we get from (4.3), (4.4) ($\beta > 0$):

a) $J_0 > 0,\ J_1 < 0$, if $\alpha < 0$ or $\alpha > 2\beta/(\beta + 5)$,
b) $J_0 < 0,\ J_1 < 0$, if $2\beta/(2\beta + 5) < \alpha < 2\beta/(\beta + 5)$,
c) $J_0 < 0,\ J_1 > 0$, if $0 < \alpha < 2\beta/(2\beta + 5)$.

Hence it follows that the cases b) and c) correspond to very narrow ranges of values of the exponent α of the interaction potential both for non-relativistic ($\beta = 2$), and also for ultra-relativistic particles ($\beta = 1$), while only the possibility a) is realistic with $\alpha > 2\beta/(\beta + 5)$.

§5. ACTIVATION DISPERSION LAW

In a number of interesting cases (plasmons, excitons) the dispersion law has the form $\omega(k) = \omega_0 + \delta\omega(k)$. If in this case the additional dispersion $\delta\omega(k)$ is a power function $\delta\omega(k) \sim k^\beta$, there exist, as before, power-law distributions (see, e.g., [2]) corresponding to the energy or the particle flux being constant, because the activation frequency ω_0 drops out of the energy conservation law when scattering takes place.[5)]

There arise, however, important differences in comparison with the considerations given above. Firstly, the energy flux J_1 which is a parameter of the non-equilibrium distribution is determined solely by the dispersion terms, which corresponds to replacing $\omega(k)$ by $\delta\omega(k)$ in (2.4). The total energy flux $\tilde{J}_1$ is a linear combination of the quasi-particle flux J_0 and J_1, $\tilde{J}_1 = \omega_0 J_0 + J_1$ and is the same as J_1 for the distribution which corresponds to an energy flux when $s = s_1$ and $J_0 = 0$. In contrast to (3.4) we look for the power-law distribution in the form

$$n(k) = A\,|\delta\omega(k)|^s. \tag{3.4'}$$

Secondly, the dispersion term can have either sign (in particular, $\delta\omega > 0$ for Langmuir plasmons, and $\delta\omega < 0$ for optical phonons) which leads to a change in the criterion (3.10) which determines the direction of the energy flux to

$$\text{sign}\, J_1 = \text{sign}\, \beta s_1 (s_1 + 1)\, \delta\omega. \tag{3.10'}$$

We have thus verified that for given parameters β and s_1 the direction of the energy flux is reversed when the sign of $\delta\omega(k)$ is changed. On the other hand, the flux of the number of waves is, as before, given by the criterion (3.10).

Similarly in the case of scattering of particles with a dispersion law $E(p) = E_0 + \delta E(p)$ the direction of the energy flux is given by the condition

$$\text{sign}\, J_1 = \text{sign}\, \beta s_1 \delta E \tag{4.4'}$$

instead of by (4.4). For instance, for holes in semiconductors or metals $\delta E < 0$ and the energy flux is in the direction of larger momenta, if the interaction is the Coulomb interaction, which, however, corresponds to a transfer towards lower energies. We note in this connection that the heating up of carriers and the appearance of large emission currents when tungsten foils are illuminated by a laser[15,8] require a reverse flux direction for their explanation.

We note here that V. M. Kontorovich drew the author's attention to the possibility that the direction of the energy transfer might be changed for quasi-particles with a negative dispersion term in their spectrum.

§6. DECAY AND FUSION OF WAVES

Because of the non-conservation of the number of waves there is for a random ensemble of weakly interacting waves with a decay spectrum $\omega(k) \sim k^\beta$, $\beta > 1$ the possibility of a non-equilibrium stationary distribution only with a constant energy flux.[1-4,6,7] Proceeding as before and using the collision integral in the factorized form[7] we get for the energy flux the expression

$$J_1 = \frac{b}{\beta} k^d \int d\tau_k w_k f_k|_{\nu = -1} [\omega \ln \omega - \omega_1 \ln \omega_1 - \omega_2 \ln \omega_2],$$
$$f_k = n_1 n_2 - n n_1 - n n_2, \quad \nu(s) = 2s - 1 + (m + 2d)/\beta, \tag{6.1}$$

where the decay probability w_k contains the energy conservation law $\omega = \omega_1 + \omega_2$ and where the solution with a constant energy flux corresponds as before to $\nu(s) = -1$ $(s = s_1 = -(m + 2d)/2\beta)$.[7] Hence, for $n = A\omega^s$

$$J_1 = \frac{bA^2}{\beta} k^d \int d\tau_k w_k (\omega\omega_1\omega_2)^s [\omega^{-s} - \omega_1^{-s} - \omega_2^{-s}]$$
$$\times [\omega \ln \omega - \omega_1 \ln \omega_1 - \omega_2 \ln \omega_2], \quad s = s_1. \tag{6.2}$$

Using the energy conservation law we can check that

$$\omega\ln\omega-\omega_1\ln\omega_1-\omega_2\ln\omega_2=\omega_1\ln\frac{\omega}{\omega_1}+\omega_2\ln\frac{\omega}{\omega_2}\geqslant 0,$$

while the function

$$\omega^{-s}-\omega_1^{-s}-\omega_2^{-s}=\omega^{-s}\left\{\frac{\omega_1}{\omega}\left[1-\left(\frac{\omega_1}{\omega}\right)^{-s-1}\right]+\frac{\omega_2}{\omega}\left[1-\left(\frac{\omega_2}{\omega}\right)^{-s-1}\right]\right\}$$

has a sign which is the opposite of that of $s+1$ as $\omega_1, \omega_2 \leqslant \omega$. Hence

$$\operatorname{sign} J_1=-\operatorname{sign}\beta(s_1+1). \qquad (6.3)$$

The energy flux is thus for $s_1<-1$ ($m>2(\beta-d)$) in the direction of short wavelengths, and for $s_1>-1$ ($m<2(\beta-d)$) in the direction of long wavelengths. The change in the sign of the flux for $s=-1$ is connected with the vanishing of I_{coll} for the Rayleigh–Jeans distribution $n=T/\omega$ which corresponds to a zero flux along the spectrum. As $d-\beta$ is as a rule non-negative, the energy flux is for $m>0$ (cf. the discussion in Sec. 3) in the direction of $k=\infty$. As an example we mention capillary waves on deep[1] and shallow[4] water.

The calculation of the constant A in the power-law distribution $n_k=A\omega^s$ requires, in fact, the evaluation of the integral (6.2). It is shown in Appendix I that this integral can in the general case be reduced to a single one. As an example we calculate in Appendix II the constant for the turbulent distribution of capillary waves on shallow water.

In conclusion we note that the approach to the calculations of the fluxes considered here can be transferred also to other systems, for instance, such systems in which power-law distributions are formed which are caused by wave-particle interactions.[13,14]

I consider it a pleasant duty to express my gratitude to V. M. Kontorovich for valuable hints and discussions of this paper.

APPENDIX I

Expression (6.2) for the energy flux in the case of waves with a decay dispersion law can be reduced to a single integral

$$J_1=-\frac{2b}{\beta}A^2\left(\frac{k}{\omega^{1/\beta}}\right)^d\int_0^1 dx\,T(x,1-x)\,[x(1-x)]^s[1-x^{-s}-(1-x)^{-s}]\,x\ln x,$$
$$s=s_1. \qquad (I.1)$$

$$T(x,y)=T(y,x)=\omega^{2-(m+2d)/\beta}(k_1k_2)^{d-1}\frac{dk_1}{d\omega_1}\frac{dk_2}{d\omega_2}U(k|k_1k_2)\chi(k|k_1k_2),$$
$$\omega_1=x\omega,\quad \omega_2=y\omega, \qquad (I.2)$$

$$\chi(k|k_1k_2)=\int d\varkappa_1 d\varkappa_2\,\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2),\quad \varkappa_i\equiv\mathbf{k}_i/k_i. \qquad (I.3)$$

In deriving (I.1) we integrated over angles which reduces to an averaging of the momentum conservation law (I.3) as the matrix element $U(\mathbf{k}|\mathbf{k}_1\mathbf{k}_2)$ can without restriction of generality be considered to be a function only of the wavenumbers. Apart from this we changed to an integration over the frequencies and the integration over ω_2 was performed by using the energy conservation law. The final result was obtained by using the symmetry of $T(x,y)$. The averaging over the angles in (I.3) leads to the result:

$$\chi(k|k_1k_2)=\begin{cases}\Delta^{-1}(k,k_1,k_2), & d=2,\\ 2\pi/kk_1k_2, & d=3,\end{cases} \qquad (I.4)$$

if k, k_1, k_3 satisfy the triangle inequality, and $\chi=0$ in the opposite case. Equation (I.1) enables us to find the constant A for the distribution (3.4).

We note that the fluxes can for the case of a non-decay dispersion law and also in the case of scattering of waves be reduced to a form analogous to (I.1) (with an integration over two frequencies), but the averaging over the angles involves in that case also the matrix element.

APPENDIX II

We find the constant in the distribution for weak turbulence of capillary waves on shallow water.[4] In that case $d=2$, $\beta=2$, $s=-2$, $m=4$, and the matrix element is

$$U(k|k_1k_2)=U_0k^4,\quad U_0=\frac{1}{32\pi}\left(\frac{\sigma}{\rho h}\right)^{1/2},\quad k>k_1,\ k_2, \qquad (II.1)$$

where σ is the surface tension coefficient, ρ the density, and h the depth of the liquid. The dispersion law is

$$\omega(k)=\gamma k^2,\quad \gamma=(\sigma h/\rho)^{1/2}. \qquad (II.2)$$

Using (I.4), (II.1), and (II.2) we find

$$T(x,y)=\frac{U_0}{2\gamma^3}(xy)^{-1/2},$$

so that

$$J_1=\frac{2\pi U_0}{\gamma^4}A^2I,\quad I=-\int_0^1\frac{dx}{x^2}\left(\frac{x}{1-x}\right)^{3/2}\ln x=2\pi. \qquad (II.3)$$

One can easily evaluate the last integral through the substitution $x^{-1}=1+z^2$ with subsequent integration by parts. The constant A in the distribution $n=A\omega^{-2}$ is thus equal to

$$A=2(2h/\pi)^{1/2}(\sigma h/\rho)^{3/4}J_1^{1/2}. \qquad (II.4)$$

[1]The directions of the fluxes (and the dimensionless constants in the distributions) have previously been found for acoustic turbulence[3] and for the scattering of particles with a quadratic spectrum in the case of the Landau collision integral[5] and in the case of the Boltzmann collision integral in the Born approximation.[8] An explicit calculation of the collision integrals for power-law distributions was then carried out; this is possible only in the simplest cases.

[2]These distributions are related to the spectrum of the Kolmogorov turbulence of an incompressible liquid[10] (for details see[1-7]).

[3]We have here in view the positiveness of the power-law term in the activation spectrum; for details see § 5.

[4]Explicit expressions for the fluxes for the interaction of particles were obtained earlier[8] on the basis of a direct evaluation of the collision integral for a matrix element which depended solely on the transferred momentum.

[5]The activation spectrum for small dispersions $|\delta\omega(k)|\ll\omega_0$ is, clearly, a non-decay spectrum.

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Excitation of ordinary waves in a plasma with a diffuse boundary under anomalous skin-effect conditions

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We study the absorption of an electromagnetic wave by a plasma with a diffuse boundary. We assume that the decrease in the particle density is exponential and that the magnetic field is parallel to the boundary of the plasma. We show that the collisionless absorption of waves with the electric vector directed along the magnetic field is connected with the excitation of ordinary waves in the plasma for those densities for which they can exist in a homogeneous plasma. We study the lineshapes of the electron and ion absorption resonances, especially in the effective collision frequency approximation. We obtain expressions for the limits of the existence of the ordinary cyclotron waves in the plasma and we solve the dispersion equation for a high-pressure uniform plasma.

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§1. INTRODUCTION

P. L. Kapitza[1] was the first to state the problem of the anomalous skin-effect in a plasma with a diffuse boundary in connection with a study of a high-frequency discharge in a plasma at high pressures. Liberman, Meĭerovich, and Pitaevskiĭ[2] constructed a theory of the skin-effect in a semi-infinite non-uniform plasma, and obtained an integro-differential equation for the electromagnetic field in the plasma for an arbitrary relation between the electron mean free path, the penetration depth of the field in the plasma, and the size of the transition region at the boundary. This equation has been solved for the case of an exponential decrease of the electron density outside the plasma under conditions of an extremely anomalous skin-effect[2] and for an arbitrary degree of anomalicity.[3] In[3] a plasma with a magnetic field directed parallel to the density gradient for any degree of anomalicity of the skin-effect was also studied. Dikman and Meĭerovich[4] considered the extremely anomalous skin-effect for the case where the magnetic field was strictly parallel to the boundary and obtained solutions for an exponential and for a power-law decrease in the electron density. We study in the present paper the absorption of electromagnetic waves in the case of arbitrary anomaly of the skin-effect when the electric field of the incident wave is parallel to the constant magnetic field which lies in the plane of the plasma boundary and we also analyze a mechanism for collisionless absorption which consists in the transformation of the incident wave into ordinary cyclotron waves.

We shall assume that the size of the transition zone at the boundary of the plasma is small compared to the characteristic dimensions of the plasma, but large compared to the penetration depth of an electromagnetic

