Transformation of first sound into second in superfluid helium

V. L. Pokrovskiĭ and I. M. Khalatnikov

L. D. Landau Theoretical Physics Institute, USSR Academy of Sciences
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A Hamiltonian formalism in two-fluid hydrodynamics is developed. On the basis of the formalism the nonlinear processes of Čerenkov emission of second sound by first sound and decay of first sound into two second sounds are investigated. It is shown that for all temperatures $T > 0.9$K, excluding a narrow region near $T = 1.2$K, the Čerenkov-excitation threshold is below the decay threshold. However, the latter process becomes dominant after the decay threshold has been reached. For a certain range of parameters the steady-state regime depends strongly on the geometry and boundary conditions. In particular, in a broad beam of first sound, second-sound waves almost perpendicular to the incident wave are preferentially excited.

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INTRODUCTION

Pushkina and Khokhlov[1] considered the process of parametric excitation of second sound in superfluid helium during decay of first sound. The equations of two-fluid hydrodynamics were used, with the assumption of zero thermal-broadening coefficient. In the present article it will be shown that allowance for the terms depending on the thermal broadening changes the result in two respects. First, it leads to a change of the effective vertex describing the decay, by a quantity of the same order as that taken into account in[1]. Second, Čerenkov second-sound emission processes for which the threshold for their nonlinear excitation is lower than the decay threshold become possible. Moreover, in a broad sound beam an important role is played by decays in which the emitted second sound propagates at an angle to the incident sound.

The nonlinear processes are conveniently investigated using the Hamiltonian formalism. For two-fluid hydrodynamics this formalism was developed by us earlier.[2]

We shall describe briefly the content of the paper. In Sec. 1 an account of the Hamiltonian formalism is given. The normal coordinates corresponding to first and second sound are introduced in Sec. 2. In Sec. 3 the effective vertices (third-order anharmonicities) describing the Čerenkov process and decay process are calculated, and the kinematics of these processes is analyzed. Section 4 is devoted to calculating the threshold first-sound intensities and to investigating the nonlinear phenomena that occur at high intensities. In Sec. 5 decay at an angle to the incident wave is investigated.

1. HAMILTONIAN FORMALISM IN TWO-FLUID HYDRODYNAMICS

The equations of two-fluid hydrodynamics can be written in Hamiltonian form if we take the Hamiltonian to be the energy of the fluid in the stationary coordinate frame:

$$H=\int\left[\frac{\rho \mathbf{v}_s^2}{2}+\mathbf{p}\mathbf{v}_s+\varepsilon(\rho,S,\mathbf{p})\right]dV, \tag{1.1}$$

where ρ is the density, $\mathbf{v}_s$ is the velocity of the superfluid component, and $\mathbf{p}$ is the momentum of unit volume of the liquid in the frame moving with velocity $\mathbf{v}_s$; $\varepsilon(\rho, S, \mathbf{p})$ is the energy of unit volume of the liquid in the same frame, and is determined by the thermodynamic identity

$$d\varepsilon=TdS+\mu d\rho+(\mathbf{v}_n-\mathbf{v}_s)\,d\mathbf{p}. \tag{1.2}$$

The momentum of unit volume of the liquid in the rest frame is

$$\mathbf{j}=\rho\mathbf{v}_s+\mathbf{p}=\rho_s\mathbf{v}_s+\rho_n\mathbf{v}_n. \tag{1.3}$$

The Hamiltonian variables are the three canonically conjugate pairs (ρ, α), (S, β) and the Clebsch variables[3] (f, γ). Besides the quantities ρ and S already defined earlier, we introduce the four new quantities α, β, f and γ, the physical meaning of which is as follows. The quantity α determines the velocity of the superfluid component

$$\mathbf{v}_s=\nabla\alpha. \tag{1.4}$$

The quantities β, γ and f determine the momentum of the relative motion

$$\mathbf{p}=S\nabla\beta+f\nabla\gamma \tag{1.5}$$

The expression (1.1) acquires the meaning of a Hamiltonian after (1.4) and (1.5) are substituted into (1.1). The Hamilton equations have the usual form:

$$\dot\rho=\delta H/\delta\alpha=-\operatorname{div}\mathbf{j}, \tag{1.6}$$

$$\dot\alpha=-\delta H/\delta\rho=-(\mu+\rho\mathbf{v}_s^2/2), \tag{1.7}$$

$$\dot S=\delta H/\delta\beta=-\operatorname{div}(S\mathbf{v}_n), \tag{1.8}$$

$$\dot\beta=-\delta H/\delta S=-T-\mathbf{v}_n\nabla\beta, \tag{1.9}$$

$$\dot f=\delta H/\delta\gamma=-\operatorname{div}(f\mathbf{v}_n), \tag{1.10}$$

$$\dot\gamma=-\delta H/\delta\beta=-\mathbf{v}_n\nabla\gamma. \tag{1.11}$$

Equations (1.6) and (1.8) are the well-known continuity equations for the density and entropy. Equation (1.7) is the equation of the superfluid flow, as one can easily convince oneself by taking the gradient of both sides of the equality. Differentiating Eq. (1.5) with respect to the time and using Eqs. (1.6), (1.7)–(1.11), we obtain the well-known equation of the relative motion[4]:

$$\dot{\mathbf{p}}+\mathbf{p}\,\mathrm{div}\,\mathbf{v}_n+\nabla(\mathbf{p}\mathbf{v}_n)-[\mathbf{v}_n\times\mathrm{rot}\,\mathbf{p}]+S\nabla T=0\,. \qquad (1.12)$$

Combining Eqs. (1.7) and (1.12) we obtain

$$\dot{j}_i+\frac{\partial\Pi_{ik}}{\partial x_k}=0, \qquad \Pi_{ik}=\rho v_{si}v_{sk}+v_{si}p_k+p_i v_{nk}+p\delta_{ik},$$

where the pressure p is determined as follows:

$$p=-\varepsilon+TS+\mu\rho+(\mathbf{v}_n-\mathbf{v}_s,\,\mathbf{p})\,.$$

Thus, we convince ourselves of the correctness of the choice of canonically conjugate variables. The variables β, f and γ are necessary for the description of the three independent components of the vector $\mathbf{p}$.

The Hamiltonian (1.1) can be obtained in the usual way from the Lagrangian formulation of the equations of two-fluid hydrodynamics, proposed by one of the authors.[5]

The Hamiltonian formalism makes it possible to prove easily the following theorem. Suppose that at a certain initial time $t=t_0$ the quantity $\mathrm{curl}(\mathbf{p}/S)$ was equal to zero in all space. Then it also remains equal to zero at all subsequent times. In fact, it follows from (1.8) that the quantity f possesses this property. From this and from Eq. (1.5) we obtain the statement made above. Thus, an analog of the familiar vortex theorem of the hydrodynamics of an ideal fluid exists in two-fluid hydrodynamics. This statement can also be obtained directly from the equations of two-fluid hydrodynamics in their usual form. In fact, it is not difficult to obtain the following equation for $\mathrm{curl}(\mathbf{p}/S)$:

$$\frac{\partial}{\partial t}\left(\mathrm{rot}\,\frac{\mathbf{p}}{S}\right)=\mathrm{rot}\left[\mathbf{v}_n\times\mathrm{rot}\,\frac{\mathbf{p}}{S}\right].$$

2. NORMAL COORDINATES

As usual, it is convenient to treat the nonlinear sound processes in helium in normal coordinates. These can be introduced in the usual way, as the eigenvectors of the linearized Hamilton equations. Flows with small velocities $\mathbf{v}_s$ and momenta $\mathbf{p}$ and small deviations of ρ and S from their equilibrium values are considered. Since we are considering small oscillations against the background of the liquid at rest, we can put the function f equal to zero in any order in v/c, as follows from Eq. (1.5). Therefore, it is sufficient to confine ourselves to just the first four of Eqs. (1.6)–(1.11). In the flows that we are considering the momentum $\mathbf{p}$ has the form

$$\mathbf{p}=S\nabla\beta.$$

The linearized equations of motion in the Fourier representation with respect to the time and coordinates have the form

$$\begin{gathered}-i\omega\rho_{\mathbf{k}}=\mathbf{k}^2(\rho\alpha_{\mathbf{k}}+S\beta_{\mathbf{k}}), \quad -i\omega S_{\mathbf{k}}=\mathbf{k}^2(S^2\beta_{\mathbf{k}}/\rho_n+S\alpha_{\mathbf{k}}),\\ i\omega\alpha_{\mathbf{k}}=\frac{\partial\mu}{\partial\rho}\rho_{\mathbf{k}}+\frac{\partial\mu}{\partial S}S_{\mathbf{k}}, \quad i\omega\beta_{\mathbf{k}}=\frac{\partial\mu}{\partial S}\rho_{\mathbf{k}}+\frac{\partial T}{\partial S}S_{\mathbf{k}}.\end{gathered} \qquad (2.1)$$

The eigenvalues ω of the system (2.1) are equal to $\pm\omega_1=\pm c_1 k$, $\pm\omega_2=\pm c_2 k$, where the first- and second-sound velocities c_1 and c_2 are determined as the positive roots of the equation

$$c^4-c^2\left(\rho\frac{\partial\mu}{\partial\rho}+2S\frac{\partial\mu}{\partial S}+\frac{S^2}{\rho_n}\frac{\partial T}{\partial S}\right)+\frac{\rho_s}{\rho_n}S^2\frac{\partial(\mu,T)}{\partial(\rho,S)}=0.$$

This equation acquires a more familiar form (cf.[4]) when we change to the variable $\sigma=S/\rho$ (in place of S):

$$c^4-c^2\left[\left(\frac{\partial p}{\partial\rho}\right)_\sigma+\frac{\rho_s\sigma^2}{\rho_n}\left(\frac{\partial T}{\partial\sigma}\right)_\rho\right]+\frac{\rho_s\sigma^2}{\rho_n}\left(\frac{\partial T}{\partial\sigma}\right)_\rho\left(\frac{\partial p}{\partial\rho}\right)_T=0.$$

We shall denote the eigenvectors of Eqs. (2.1) by $q_1^\pm$ and $q_2^\pm$. The subscripts indicate the absolute value of the eigenvalue ($\omega=c_{1k}, c_{2k}$) and the superscripts indicate its sign. An arbitrary four-component vector ($\rho_{\mathbf{k}}, S_{\mathbf{k}}, \alpha_{\mathbf{k}}, \beta_{\mathbf{k}}$) can be represented by the linear combination:

$$(\rho_{\mathbf{k}},\ S_{\mathbf{k}},\ \alpha_{\mathbf{k}},\ \beta_{\mathbf{k}})=a_{\mathbf{k}}q_1^{+}+a_{-\mathbf{k}}^{*}q_1^{-}+b_{\mathbf{k}}q_2^{+}+b_{-\mathbf{k}}^{*}q_2^{-}. \qquad (2.2)$$

The quantities $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ are normal coordinates. They are the classical analogs of the annihilation operators for phonons of first and second sound. In the following we shall call $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ the first- and second-sound amplitudes. The normalization of the eigenvectors $q_1^\pm$, $q_2^\pm$ is chosen so that the following conditions are fulfilled for the Poisson brackets:

$$\{a_{\mathbf{k}},\ a_{\mathbf{k}'}^{*}\}=i\delta_{\mathbf{k}\mathbf{k}'}, \quad \{b_{\mathbf{k}},\ b_{\mathbf{k}'}^{*}\}=i\delta_{\mathbf{k}\mathbf{k}'}. \qquad (2.3)$$

All the other Poisson brackets $\{a_{\mathbf{k}}, a'_{\mathbf{k}}\}$, $\{a_{\mathbf{k}}, b^{*}_{\mathbf{k}'}\}$, ..., are equal to zero. In the variables $a_{\mathbf{k}}$, $b_{\mathbf{k}}$, $a^*_{\mathbf{k}}$ and $b^*_{\mathbf{k}}$ Hamilton's equations take the form

$$\dot{a}_{\mathbf{k}}=-i\delta H/\delta a_{\mathbf{k}}^{*}, \quad \dot{b}_{\mathbf{k}}=-i\delta H/\delta b_{\mathbf{k}}^{*}.$$

We shall represent the Hamiltonian H in the form of a series $H=H_0+H_1+H_2+\dots$ in powers of the deviations $\delta\rho$, δS, α and β of the variables ρ, S, α and β from their equilibrium values (the equilibrium values of the first two variables will be denoted, as before, by ρ and S; the equilibrium values of α and β are assumed to be equal to zero). The terms H_0 and H_1 are trivial. The quadratic Hamiltonian H_2 is diagonalized by the transformation (2.2). With the choice of normalization conditions (2.3) the Hamiltonian H_2 takes the canonical form:

$$H_2=\sum_{\mathbf{k}}(\omega_{1\mathbf{k}}a_{\mathbf{k}}a_{\mathbf{k}}^{*}+\omega_{2\mathbf{k}}b_{\mathbf{k}}b_{\mathbf{k}}^{*}).$$

In the following we shall need the explicit form of the transformation (2.2). In order not to encumber the text with long formulas we shall write out this transformation approximately, neglecting quantities whose ratio to those kept is of the order of c_2^2/c_1^2. We shall assume the ratio c_2^2/c_1^2 to be a small quantity, which is true in the temperature range from 0.8 K to the λ-point. To this accuracy the transformation (2.2) takes the form

$$\rho_{\mathbf{k}}=A(a_{\mathbf{k}}+a_{-\mathbf{k}}^{*})-B\frac{\rho}{S}\frac{u^2}{c_1^2}(b_{\mathbf{k}}+b_{-\mathbf{k}}^{*}), \tag{2.4}$$

$$S_{\mathbf{k}}=A\frac{S}{\rho}(a_{\mathbf{k}}+a_{-\mathbf{k}}^{*})+B(b_{\mathbf{k}}+b_{-\mathbf{k}}^{*}), \tag{2.5}$$

$$ik\alpha_{\mathbf{k}}=A\frac{c_1}{\rho}(a_{\mathbf{k}}-a_{-\mathbf{k}}^{*})-B\frac{c_2}{S}\frac{\rho_n}{\rho_s}(b_{\mathbf{k}}-b_{-\mathbf{k}}^{*}), \tag{2.6}$$

$$ik\beta_{\mathbf{k}}=A\frac{c_1}{S}\frac{u^2}{c_1^2}(a_{\mathbf{k}}-a_{-\mathbf{k}}^{*})+B\frac{c_2\rho_n\rho}{S^2\rho_s}(b_{\mathbf{k}}-b_{-\mathbf{k}}^{*}), \tag{2.7}$$

$$A(\omega)=\left(\frac{\rho\omega}{2c_1^2}\right)^{1/2},\quad B(\omega)=\left(\frac{\omega}{2\,\partial T/\partial S}\right)^{1/2},$$

$$u^2=\frac{S^2}{\rho}\frac{\partial T}{\partial S}+S\frac{\partial\mu}{\partial S}=\frac{S}{\rho}\frac{\partial p}{\partial S}.$$

The quantity u^2 is of order c_2^2, but in formulas (2.4) and (2.7) it is necessary to keep terms $\sim u^2/c_1^2$ since $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ are arbitrary.

3. CALCULATION OF THE THIRD-ORDER ANHARMONICITIES RESPONSIBLE FOR THE DECAY OF FIRST SOUND

The term H_3 of third order of smallness in the Hamiltonian can be obtained from the general formulas (1.1), (1.2):

$$H_3=\int\Big[\frac{1}{6}\frac{\partial^2\mu}{\partial\rho^2}(\delta\rho)^3+\frac{1}{2}\frac{\partial^2\mu}{\partial\rho\,\partial S}(\delta\rho)^2\delta S+\frac{1}{2}\frac{\partial^2\mu}{\partial S^2}\delta\rho(\delta S)^2$$
$$+\frac{1}{6}\frac{\partial^2 T}{\partial S^2}(\delta S)^3+\frac{1}{2}\frac{\partial}{\partial\rho}\left(\frac{1}{\rho_n}\right)S^2\delta\rho(\nabla\beta)^2+\frac{1}{2}\frac{\partial}{\partial S}\left(\frac{1}{\rho_n}\right)S^2\delta S(\nabla\beta)^2$$
$$+\delta S\,\nabla\alpha\cdot\nabla\beta+\frac{S}{\rho_n}\delta S(\nabla\beta)^2+\frac{1}{2}\delta\rho(\nabla\alpha)^2\Big]dV. \tag{3.1}$$

We transform to the Fourier components $\delta\rho_{\mathbf{k}}$, $\delta S_{\mathbf{k}}$, $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$ and then to the sound amplitudes $a_{\mathbf{k}}$, $a_{\mathbf{k}}^*$, $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^*$, in accordance with (2.2) and (2.4)–(2.7). A number of terms of third order in the amplitudes will arise. Those terms which satisfy the momentum and energy (frequency) conservation laws correspond to decays. The general theory guarantees the invariance of these resonance amplitudes under the canonical transformations (cf., e.g., [6]).

Specifically, we shall be interested in the amplitude $U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ for decay of first sound with momentum **k** into two second sounds, and in the amplitude $V_{\mathbf{k}\mathbf{k}'\mathbf{k}_1}$ for emission of second sound by first sound. The first quantity is the coefficient of the product $a_{\mathbf{k}}b_{\mathbf{k}_1}^*b_{\mathbf{k}_2}^*$, where the momenta **k**, $\mathbf{k}_1$ and $\mathbf{k}_2$ satisfy the conditions

$$\mathbf{k}=\mathbf{k}_1+\mathbf{k}_2,\quad c_1k=c_2(k_1+k_2).$$

Since $c_1\gg c_2$, the second-sound momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ are much larger than **k** in magnitude. Therefore, the second-sound phonons that appear as a result of the decay

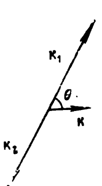

FIG. 1.

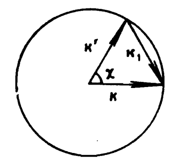

FIG. 2.

have almost opposite momenta. The angle at which the second-sound phonons fly apart, relative to the direction of propagation of the first sound, will be denoted by θ (see Fig. 1). The moduli k_1 and k_2 are approximately equal. Therefore, second-sound phonons with frequencies close to half the first-sound frequency appear: $\omega_2\approx\omega_1/2$ ($\omega_1=c_1k$). The decay amplitude $U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ depends essentially on only two variables ω and θ.

In the following we shall use the notation $U(\omega,\theta)$ in place of $U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$. Using the conservation laws we find, after standard transformations,

$$U(\omega,\theta)=\frac{\rho^{1/2}\omega^{3/2}}{2^{1/2}\rho_s c_1}\left[-1-\frac{2\rho_s}{\rho}\cos^2\theta-\rho_n\rho\left(\frac{\partial\rho_n^{-1}}{\partial\rho}\right)_\sigma+\frac{\rho_s}{\rho^2}\frac{\partial^2 p/\partial\sigma^2}{\partial T/\partial\sigma}\right]. \tag{3.2}$$

Here ρ and σ have been chosen as the independent variables. If we change to the variables p and T, the expression in the square brackets (which will be denoted by $[U]$ in the following) acquires the following form:

$$[U]=-1-\frac{2\rho_s}{\rho}\cos^2\theta-\rho_n\rho c_1^2\frac{\partial\rho_n^{-1}}{\partial p}$$
$$+\rho_n c_1^2\left[\frac{\partial}{\partial T}\left(\frac{\rho_s}{\rho_n}\right)\frac{\partial\sigma/\partial p}{\partial\sigma/\partial T}-\frac{\rho_s}{\rho_n}\frac{\partial}{\partial T}\left(\frac{\partial\sigma/\partial p}{\partial\sigma/\partial T}\right)\right]. \tag{3.3}$$

According to the thermodynamic identity, we have $\partial\sigma/\partial p=\rho^{-2}\partial\rho/\partial T$. Therefore, the terms in the square brackets in (3.3) contain $\partial\rho/\partial T$ and $\partial^2\rho/\partial T^2$.

In[1] the interaction between first and second sounds was obtained directly from the hydrodynamic equations, and it was assumed from the outset that $\partial\rho/\partial T\equiv 0$. As can be seen from the result (3.3), such an assumption is incorrect. Moreover, we have taken into account the possibility of decay into a pair of phonons at an arbitrary angle to the incident sound, and as a result a dependence of the amplitude (3.2) on the angle θ has appeared. The results of[1] correspond to $\theta=0$.

We turn to the calculation of the amplitude $V_{\mathbf{k}\mathbf{k}'\mathbf{k}_1}$ for emission of second sound by first sound. This amplitude is the coefficient of the product $a_{\mathbf{k}}a_{\mathbf{k}'}^*b_{\mathbf{k}_1}^*$, where the momenta **k**, **k′** and $\mathbf{k}_1$ satisfy the conservation laws

$$\mathbf{k}-\mathbf{k}'=\mathbf{k}_1,\quad c_1(k-k')=c_2k_1.$$

Because of the inequality $c_2\ll c_1$ the change in the frequency of the first sound in this process is small. Therefore, the vectors **k** and **k′** lie on the same sphere, of radius ω/c_1. We denote the angle between the vectors **k** and **k′** by χ (see Fig. 2). The second-sound frequency ω_2 is expressed in terms of the first-sound frequency ω_1 and the angle χ:

$$\omega_2=2\omega_1\frac{c_2}{c_1}\sin\frac{\chi}{2}. \tag{3.4}$$

The emission amplitude $V_{\mathbf{k}\mathbf{k}'\mathbf{k}_1}$ depends on two quantities: the frequency ω and angle χ. In the following we shall use the notation $V(\omega, \chi)$. The calculations give

$$V(\omega,\chi)=\frac{\omega^{3/2}}{2\rho^{3/2}}\cdot\frac{(\partial\rho/\partial T)_p}{(\partial\sigma/\partial T)^{1/2}}\left(\frac{c_2}{c_1}\sin\frac{\chi}{2}\right)^{1/2}\left[\rho\left(\frac{\partial}{\partial\rho}\ln\frac{\rho c^2}{(\partial p/\partial\sigma)_\rho}\right)_\sigma+\cos\chi\right], \tag{3.5}$$

where $c^2=(\partial p/\partial\rho)_\sigma$.

The amplitude $U(\omega, \theta)$ depends strongly on the temperature. Using the known experimental data (cf. [7,8]) we can show that the expression (3.3) for $[U]$ vanishes at $T\sim 1.6$ K. In the "roton" region of temperatures ($T>1$ K) the characteristic magnitude of $[U]$ is of the order of 0.2–0.5. At $T\approx 1$ K this quantity increases sharply and then remains ≈ 5. The other factors of $U(\omega, \theta)$ are practically independent of temperature.

The temperature dependence of the amplitude $V(\omega, \chi)$ of the Čerenkov process is principally associated with the factors $(\partial\rho/\partial T)_p$ and $(\partial\sigma/\partial T)^{-1/2}$. It is known[9] that $\rho^{-1}\partial\rho/\partial T$ changes sign at $T\approx 1.15$ K. In the region $T>1.15$ K the quantity $\partial\ln\rho/\partial T$ is positive (Panomalous behavior). The quantity in the square bracket in (3.5), denoted by $[V]$ below, is of the order of 3–10 in the roton temperature region. In the low-temperature region it is expressed in terms of the two dimensionless parameters v and z in the following way:

$$[V]=\frac{3z+3v-6v^2+1/3}{1+3v}+\cos\chi, \tag{3.6}$$

where

$$v=\left(\frac{\partial\ln c_1}{\partial\ln\rho}\right)_T,\qquad z=\frac{\rho^2}{c_1}\frac{\partial^2 c_1}{\partial\rho^2}.$$

The parameters v and z were estimated in the paper[10] by Landau and one of the authors: $v\approx 3$, $z\approx 20$. Therefore, $[V]\sim 1$ in the low-temperature region.

4. THRESHOLDS FOR PARAMETRIC EXCITATION OF SECOND SOUND

In the propagation of high-intensity first sound in helium, two processes leading to coherent excitation of second sound occur: Čerenkov emission of second sound (I) and the creation of a pair of phonons with almost opposite momenta (II). The analysis of the conservation laws (cf. Sec. 3) shows that the frequencies of the second sounds excited in these processes differ greatly from each other. In the first process second sound with frequency of the order of $2c_2\omega/c_1$ is excited, and in the second process the second-sound frequency is equal to $\omega/2$, where ω is the first-sound frequency.

The threshold intensities for excitation of these processes also differ substantially. To simplify the calculations we shall consider the simplest situation, in which the first sound propagates as a narrow beam. In this case the phonons flying out at an angle to the original beam are not amplified. It can be assumed that the wave vectors of the incident sound and excited sounds lie on one straight line. To calculate the threshold intensity it is necessary to take into account the attenuations $\delta_1(\omega)$ and $\delta_2(\omega)$ of the first and second sounds. We cite the well-known formulas for these quantities[4]:

$$\delta_1(\omega)=\frac{\omega^2}{2\rho c_1^2}\left(\frac{4}{3}\eta+\zeta\right),\qquad \delta_2(\omega)=\frac{\omega^2}{2\rho c_2^2}\frac{\rho_s}{\rho_n}\left(\frac{4}{3}\eta+\frac{\rho_n}{\rho_s}\frac{\varkappa}{T}\frac{\partial T}{\partial\sigma}\right), \tag{4.1}$$

where η, ζ and $\varkappa$ are kinetic coefficients. The threshold first-sound amplitudes a_{I} and a_{II} in processes I and II are determined in the familiar manner (cf., e.g., [8]):

$$|a_{\text{I}}|=\left[\delta_1(\omega)\,\delta_2\left(2\frac{c_2}{c_1}\omega\right)\right]^{1/2}\Big/\,V, \tag{4.2}$$

$$|a_{\text{II}}|=\delta_2(\omega/2)/U. \tag{4.3}$$

The ratio $|a_{\text{I}}|/|a_{\text{II}}|$ is formally of order $(c_2/c_1)^{1/2}$. However, the quantity c_2/c_1 is not very small (~ 0.1), and this is all the more true of its root. Therefore, a more careful numerical analysis is necessary. Using formulas (3.2)–(3.6) and (4.1), we find

$$\left|\frac{a_{\text{I}}}{a_{\text{II}}}\right|=\left(\frac{\delta_1(\omega)}{\delta_2(\omega)}2\frac{c_2}{c_1}\right)^{1/2}\frac{\rho^2}{\rho_s c_1}\frac{(\partial\sigma/\partial T)^{1/2}}{\partial\rho/\partial T}\frac{[U]}{[V]}. \tag{4.4}$$

Experimental data for the ratios $\delta_1/\omega^2 c_1$ and $\delta_2/\omega^2 c_2$ are given in the review.[12] The other experimental data have been gleaned from[7,8]. For an analysis of the expressions for $[U]$ and $[V]$ see Sec. 3. The results of the analysis are as follows. In the region $T\gtrsim 1.4\sim 1.7$ K the ratio $|a_{\text{I}}/a_{\text{II}}|$ is small (~ 0.01–0.1), and at $T\approx 1.6$ K this ratio passes through zero; at $T\approx 1.2$ K the ratio (4.4) is much greater than unity. On further lowering of the temperature it decreases, and for $0.8<T<1$ K becomes equal to ~ 0.5–1. The behavior of the ratio $|a_{\text{I}}/a_{\text{II}}|$ is shown schematically in Fig. 3. The intersection of this curve with the straight line $|a_{\text{I}}/a_{\text{II}}|=1$ separates the region in which the Čerenkov threshold comes before the decay threshold (region I) from the region (II) in which the decay threshold is reached before the Čerenkov threshold. The threshold energy-flux for process I is

$$W_{\text{I}}=c_1\omega|a_{\text{I}}|^2=\frac{16\delta_1(\omega)\,\delta_2(\omega)\,\rho^3 c_2\,\partial\sigma/\partial T}{\omega^2(\partial\rho/\partial T)^2[V]^2}. \tag{4.5}$$

At $T\approx 1.5$ K we find from (4.5)

$$W_{\text{I}}\approx 0.5\cdot 10^{-(14-15)}\omega^2\ \text{W/cm}^2.$$

At $T\approx 0.9$ K,

$$W_{\text{I}}\approx 10^{-(8-9)}\omega^2\ \text{W/cm}^2.$$

Thus, parametric excitation at $T<1$ K and reasonable frequencies is scarcely possible. We note that in the region $T\approx 1.2$ K we can observe the pure decay process.

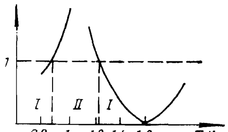

FIG. 3.

5. NONLINEAR PROCESSES IN THE REGION BEYOND THE THRESHOLD

We shall consider the Čerenkov process in the case when the incident-wave amplitude is considerably greater than the threshold amplitude. In this case we can neglect the damping of the waves. We confine ourselves to considering the stationary regime. We denote the slowly varying first-sound amplitudes by $a(x)$ (the incident wave) and $a'(x)$ (the scattered wave) and the slowly varying second-sound amplitude by $b(x)$. In the case under consideration the wave vector of the scattered wave has the opposite direction to that of the incident wave ($\chi=\pi$), and the second-sound frequency is equal to $2c_2\omega/c_1$. The equations for the slowly varying amplitudes have the standard form[13] (cf. also[6]):

$$c_1\frac{da}{dx}=-iVa'b,\quad c_1\frac{da'}{dx}=iV^*ab^*,\quad c_2\frac{db}{dx}=-iV^*aa'^*. \qquad (5.1)$$

These equations have two constants of the motion:

$$|a'|^2+\frac{c_2}{c_1}|b|^2=q^2,\qquad |a|^2-|a'|^2=r^2-q^2.$$

There exists a solution of Eqs. (5.1) with a coherent phase relationship:

$$\arg a-\arg a'-\arg b=\pi/2.$$

In the following, this relation is assumed to be fulfilled. The only unknown quantities are the absolute values, which, as before, we shall denote by a, a' and b. It is convenient to introduce the new variable

$$u=xVr/(c_1c_2)^{1/2} \qquad (5.2)$$

and the unknown function $\tilde{b}=(c_2/c_1)^{1/2}b$. The equations (5.1) take the form

$$r\frac{da}{du}=-a'\tilde{b},\quad r\frac{da'}{du}=-a\tilde{b}^*,\quad r\frac{d\tilde{b}}{du}=aa'^*. \qquad (5.3)$$

The general solution of Eqs. (5.3) is

$$a'=q\,\mathrm{cn}\,(u+u_0),\quad \tilde{b}=q\,\mathrm{sn}\,(u+u_0),\quad a=r\,\mathrm{dn}\,(u+u_0). \qquad (5.4)$$

The modulus k of the elliptic functions is expressed in terms of q and r:

$$k^2=q^2/r^2. \qquad (5.5)$$

The solutions (5.4) depend in an essential way on the boundary conditions. As an example we shall consider the case when the first- and second-sound waves emerge from the system without reflection. In this case,

$$a|_{x=0}=a_0,\quad a'|_{x=l}=b|_{x=0}=0. \qquad (5.6)$$

From (5.2), (5.5) and (5.6) we find $u_0=0$, $r=a_0$ and

$$lVa_0/(c_1c_2)^{1/2}=\mathbf{K}(k). \qquad (5.7)$$

The quantity k is found from Eq. (5.7) and substituted into (5.5) to determine q. A solution of the type considered exists only under the condition

$$Va_0l/(c_1c_2)^{1/2}>\pi/2. \qquad (5.8)$$

Otherwise, a coherent solution with the boundary conditions indicated does not exist. The ratio of the energy flux W_c necessary for realization of the coherent regime to the threshold energy-flux W_{I} turns out to be a quantity of the order of $3\times10^{30}\omega^{-4}l^{-2}$ at a temperature of about 1.5 K. For $l\sim5$ cm and $\omega\sim10^7$ the quantities W_c and W_{I} are of the same order. It should be noted that with these values of the parameters the damping cannot be assumed to be weak. This does not lead to qualitative changes. On the other hand, by varying the temperature, frequency and length, it is possible to select the necessary optimal conditions.

If the condition (5.8) is not fulfilled, coherent second sound can be obtained by mixing two first-sound waves: one (a_0) is admitted from the left ($x=0$) and the other (a_l') from the right ($x=l$). The boundary conditions in this case are

$$a|_{x=0}=a_0,\ a'|_{x=l}=a_l',\ b|_{x=0}=0.$$

The modulus k of the elliptic functions is determined from the equation

$$k^2\,\mathrm{cn}^2\,[Va_0l/(c_1c_2)^{1/2}]=(a_l'/a_0)^2.$$

The constants of the motion are $r=a_0$ and $q=ka_0$.

An entirely different situation arises in the case when the second sound is almost totally reflected at the boundaries while the first sound passes through without reflection. Besides the second-sound wave traveling to the right, there arises a wave of the same intensity, traveling to the left. On interacting with the scattered first-sound wave this gives a secondary scattered first-sound wave, traveling to the right but with a frequency different from the frequency of the incident wave. The interaction of the secondary scattered waves with the second-sound waves leads to the appearance of a whole series of first-sound waves with equally-spaced frequencies

$$\omega_n=\omega(1+2nc_2/c_1),\quad n=0,\pm1,\pm2\ldots,$$

and of a standing second-sound wave with frequency equal to $2c_2\omega/c_1$. The waves traveling to the right correspond to even n and those traveling to the left correspond to odd n. Thus, the spectral composition of the first sound is represented by two combs inserted into each other, corresponding to the different directions of propagation (Fig. 4).

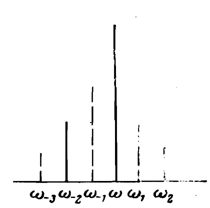

FIG. 4.

A more detailed analysis shows that the geometric resonance of the secondary waves is not exact. The difference in the wave vector of the n-th wave and the sum of the wave vectors of the $(n+1)$-th wave and the corresponding second-sound wave is $\Delta k_n = 2c_2kn/c_1$. Therefore, the amplitudes of all the secondary waves are small. To detect the effect it is advantageous to reduce the ratio c_2/c_1 as far as possible.

At large incident-sound amplitudes, exceeding the threshold (4.3), the nonlinear process of decay into two second sounds with frequencies $\omega/2$ begins. Since at this point the Čerenkov process is already strongly developed, the near-threshold phenomena develop against the background of a spatially nonuniform sound field $a(x)$. Then, with increase of the amplitudes $b_1(x)$ and $b_2(x)$ of the second sound arising as a result of the decay, the waves $a(x)$, $a'(x)$, $b(x)$, $b_1(x)$ and $b_2(x)$ interact with each other in a complicated manner. However, as is easily shown, the amplitudes b_1 and b_2 grow much faster than a' and b. Therefore, at powers substantially exceeding both thresholds, it is the decay process (II) that principally occurs.

We shall show that the process II is the more important one at high powers. We write out the equation for a, with allowance for both processes,

$$da/dx = -iVa'b - iUb_1b_2. \tag{5.9}$$

If process I is unimportant, then $b_1 \sim b_2 \sim (c_1/c_2)^{1/2}a$ (cf. the conservation laws (5.11)). With regard to the amplitudes a' and b, they do not in any case exceed the values they have in conditions when process II is absent. We have already convinced ourselves that in this case $a' \sim a$ and $b \sim (c_1/c_2)^{1/2}a$. The ratio V/U is formally $\sim (c_2/c_1)^{3/2}$. In the region $T \approx 1.6$ K, for numerical reasons this ratio may be not very small. For the statement expressed above to be correct it is sufficient that V/U be much smaller than $(c_1/c_2)^{1/2}$. In this case the second term in the right-hand side of (5.9) is much smaller than the first.

The equations describing process II have the form

$$c_1\frac{da}{dx} = -iUb_1b_2, \quad c_2\frac{db_1}{dx} = -iUab_2^*, \quad c_2\frac{db_2}{dx} = iUab_1^* \tag{5.10}$$

The equations (5.10) have two constants of the motion:

$$|b_1|^2 + |b_2|^2 = q^2, \quad c_1|a|^2 + c_2|b_1|^2 = c_1r^2. \tag{5.11}$$

The phases a, b_1 and b_2 are connected by the relation

$$\arg a - \arg b_1 - \arg b_2 = \pi/2.$$

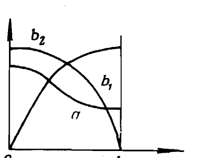

FIG. 5.

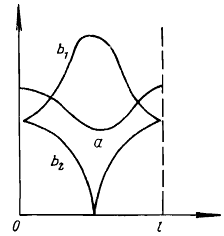

FIG. 6.

The general solution for the moduli $|a|$, $|b_1|$ and $|b_2|$ has the form

$$a = r\,\mathrm{dn}\,(u+u_0), \quad b_1 = q\,\mathrm{sn}\,(u+u_0), \quad b_2 = q\,\mathrm{cn}\,(u+u_0)$$
$$u = xUr/c_2, \quad k^2 = c_2q^2/c_1r^2. \tag{5.12}$$

The constants of the motion, q and r, are determined by the boundary conditions. In[1] the solution of the system (5.10) was found with the boundary conditions

$$a|_{x=0} = a_0, \quad b_1|_{x=0} = b_{10}, \quad b_2|_{x=l} = 0,$$

usually used in nonlinear optics.[13] In the present problem the more natural conditions are those corresponding to complete transmission of the waves:

$$a|_{x=0} = a_0, \quad b_1|_{x=0} = b_2|_{x=l} = 0$$

or to reflection of the second-sound waves at the boundaries:

$$(|b_1|^2 - |b_2|^2)|_{x=0} = (|b_1|^2 - |b_2|^2)|_{x=l} = 0$$

In the former case, $u_0 = 0$ and $r = a_0$. The modulus k of the elliptic functions is found from the equation

$$\mathrm{K}(k) = Ua_0l/c_2, \tag{5.13}$$

and the constant of the motion q^2 is determined from Eq. (5.12). A solution of (5.13) exists only under the condition $Ua_0l/c_2 > \pi/2$. The distribution of the first- and second-sound amplitudes is shown schematically in Fig. 5.

In the case of reflection of the second sound the equation determining the modulus k has the form

$$\mathrm{sn}\frac{Ua_0l}{c_2(4-2k^2)^{1/2}} = (2-k^2)^{-1/2}.$$

The parameter u_0 is found from the equation $\mathrm{sn}u_0 = 1/\sqrt{2}$. In this case,

$$r = a_0\left(\frac{2}{2-k^2}\right)^{1/2}.$$

The distribution of amplitudes in this case is shown in Fig. 6. The criterion for the existence of a solution in this case is the same as in the previous case.

6. DECAY AT AN ANGLE TO THE INCIDENT WAVE

In a broad incident beam, processes of Čerenkov emission and decay in which the emitted waves are at an arbitrary angle to the incident wave are possible.

Here we shall consider only decay processes. The power thresholds at which second-sound waves appear at an angle θ depend on the angle. The minimum threshold is that for decay at angle $\theta=0$. At powers substantially greater than the threshold values the equations for the stationary process can be written in the following form:

$$c_1\frac{da}{dx}=-i\sum_{\mathbf{p}}U_{\mathbf{p}}b_{1\mathbf{p}}b_{2\mathbf{p}},$$

$$c_2\cos\theta_{\mathbf{p}}\frac{db_{1\mathbf{p}}}{dx}=-iU_{\mathbf{p}}ab_{2\mathbf{p}}^*,\quad c_2\cos\theta_{\mathbf{p}}\frac{db_{2\mathbf{p}}}{dx}=iU_{\mathbf{p}}ab_{1\mathbf{p}}^*, \qquad (6.1)$$

where $\mathbf{p}$ is the wave vector of one of the second-sound waves, $\theta_{\mathbf{p}}$ is the angle between $\mathbf{p}$ and the direction of the incident sound, and $U_{\mathbf{p}}\equiv U(\omega,\theta_{\mathbf{p}})$.

The constants of the motion of the system (6.1) are the quantities

$$|b_{1\mathbf{p}}|^2+|b_{2\mathbf{p}}|^2=q_{\mathbf{p}}^2,$$

$$2c_1|a|^2+\sum_{\mathbf{p}}c_2\cos\theta_{\mathbf{p}}(|b_{1\mathbf{p}}|^2-|b_{2\mathbf{p}}|^2)=I.$$

The relation between the phases a, $b_{1\mathbf{p}}$ and $b_{2\mathbf{p}}$ is easily established:

$$\arg a-\arg b_{1\mathbf{p}}-\arg b_{2\mathbf{p}}=\pi/2.$$

It is convenient to change to the variables $\varphi_{\mathbf{p}}$:

$$|b_{1\mathbf{p}}|=q_{\mathbf{p}}\sin\varphi_{\mathbf{p}},\quad |b_{2\mathbf{p}}|=q_{\mathbf{p}}\cos\varphi_{\mathbf{p}},$$

The equations (6.1) take the form

$$c_1\frac{da}{dx}=-\frac{1}{2}\sum_{\mathbf{p}}U_{\mathbf{p}}q_{\mathbf{p}}^2\sin 2\varphi_{\mathbf{p}}, \qquad (6.2)$$

$$\frac{d\varphi_{\mathbf{p}}}{dx}=\frac{U_{\mathbf{p}}a}{c_2\cos\theta_{\mathbf{p}}}. \qquad (6.3)$$

The system of equations (6.2), (6.3) can be solved in the case of weak conversion, when a is almost independent of x.

Putting $a=a_0$ in (6.3), we find

$$\varphi_{\mathbf{p}}=xU_{\mathbf{p}}a_0/c_2\cos\theta_{\mathbf{p}}+\varphi_{\mathbf{p}0}.$$

For simplicity we use the boundary conditions for total transmission:

$$b_{1\mathbf{p}}|_{x=0}=b_{2\mathbf{p}}|_{x=l}=0.$$

Then $\varphi_{\mathbf{p}0}=0$ and

$$U_{\mathbf{p}}a_0l/c_2\cos\theta_{\mathbf{p}}=\pi/2. \qquad (6.4)$$

The equation (6.4), in analogy with the inequality (5.8), determines the threshold for coherent excitation of waves traveling at the given angle. This threshold is smaller the smaller is $\cos\theta_{\mathbf{p}}$. The limitations on the size of $\cos\theta_{\mathbf{p}}$ are determined by geometrical factors. Thus, in a broad beam it is primarily second-sound waves propagating at right angles to the incident wave that are coherently excited.

In the developed nonlinear regime, Eqs. (6.2), (6.3) can only be solved numerically.

CONCLUSION

The Hamiltonian formalism of[2] has made it possible to analyze nonlinear processes of conversion of first sound into second sound in detail. These processes are the Čerenkov emission of second sound by first sound (process I) and decay of first sound into two second sounds (process II). According to our estimates, the threshold for parametric excitation of process I is the lower one for $T\sim 1.3$–1.8 K. However, near $T\approx 1.2$ K there is a narrow "window" for process II. In this narrow temperature range the threshold for II is found to be the lower of the two.

Above the I threshold second sound with frequency $2c_2\omega/c_1$, where ω is the frequency of the incident sound, is excited. The stationary process is coherent for sufficiently large amplitudes, satisfying the inequality (5.8). Above the second threshold the role of process II grows sharply and at sufficiently high powers this process becomes dominant. In process II, second sound with frequency $\omega/2$ is emitted. We shall discuss the role of absorption. In the region of powers substantially above threshold, Eqs. (5.1), (5.10), in which terms corresponding to damping have been discarded, are true locally with great accuracy. However, over large distances, damping effects can accumulate and can lead to a substantial change in the picture. The solutions that we have given are valid under the assumption that the damping length is substantially greater than the size of the system. Otherwise, our solutions are qualitatively correct over an interval of the order of the damping length.

In weak solutions of ^{3}He in ^{4}He the phenomena of conversion of first sound into second are much stronger than in pure ^{4}He, since the Čerenkov vertex contains the quantity $\partial\ln\rho/\partial\ln c$ (c is the concentration), which is not small, in place of the small quantity $\partial\ln\rho/\partial\ln T\sim 10^{-3}$–$10^{-4}$ in pure ^{4}He.

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Spontaneous symmetry-breaking in a gas of nonequilibrium phonons

S. A. Bulgadaev and I. B. Levinson

L. D. Landau Theoretical Physics Institute, USSR Academy of Sciences
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Irradiation of a dielectric by nonresonance infrared light leads to excitation of short-wavelength phonon modes (two-phonon absorption). It is shown that there exists an intensity threshold above which spontaneous lowering of the symmetry occurs in the gas of nonequilibrium short-wavelength phonons—the stable state of the gas is one in which the phonon distribution function is of lower symmetry than the crystal.

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1. THE MODEL AND KINETIC EQUATION

An isotropic model of a crystal with a center of inversion is considered; in the crystal there are two acoustic branches (a transverse (TA) and a longitudinal (LA) branch) and several optical branches (O) (see Fig. 1). The crystal is at a low temperature $T \ll \omega_D$, where ω_D is the Debye frequency.

The frequency ν of the incident light does not coincide with any of the limiting ($q=0$) frequencies ω_0 of the optical phonons active in infrared absorption. In this case the absorption is associated with the creation of a pair of short-wavelength phonons (usually acoustic) and proceeds according to the scheme

$$\nu \to TA + LA; \tag{1.1}$$

the frequencies of the phonons created are of the order of ω_D.

The LA phonons created are rapidly thermalized in spontaneous decay processes, and therefore their occupation numbers can be assumed equal to zero. Spontaneous decay of the TA phonons is impossible.[1] Therefore, they can be destroyed either by scattering by defects with the conversion TA → LA[2] or by interaction between nonequilibrium TA phonons. In lowest order in the anharmonicity the latter corresponds to the coalescence process

$$TA + TA \to O. \tag{1.2}$$

O-branch phonons are also rapidly thermalized and if we assume their occupation numbers to be equal to zero the coalescence of two TA phonons is equivalent to their destruction.

The kinetic equation for the occupation numbers of the TA phonons can be written in the following form:

$$\dot{N}(\mathbf{q}) = D(\mathbf{q}) + G(\mathbf{q}). \tag{1.3}$$

The term G describes the generation of TA phonons by the light and the term D describes the destruction of these phonons.

We shall consider first the generation term G, assuming, as in[3], that the spectral intensity of the exciting light is given by a Lorentzian form factor

$$\varphi(\nu) = \frac{(\Delta\nu/2)^2}{(\nu-\nu_0)^2 + (\Delta\nu/2)^2}, \quad \varphi(\nu_0) = 1 \tag{1.4}$$

with central frequency ν_0 and width $\Delta\nu$. We then have[6]

$$G(\mathbf{q}) = \lambda\varphi(q)\,[N(\mathbf{q})+1], \tag{1.5}$$

where

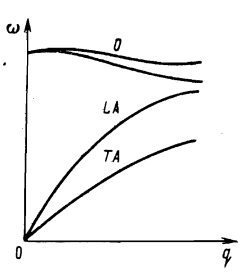

FIG. 1.