Collective slowing of a relativistic beam of oscillators in a nonlinear dielectric medium

V. B. Krasovitskiĭ

Rostov State University
(Submitted March 25, 1976)
Zh. Eksp. Teor. Fiz. 71, 1358–1362 (October 1976)

The nonlinear theory of the instability of a relativistic beam of oscillators developed here takes into account both the nonlinearity of the equations of motion of the beam and the nonlinear properties of the retarding medium. The maximum field intensity excited by the beam when the anomalous Doppler-effect condition is satisfied is calculated.

PACS numbers: 77.90.+k

If an oscillator moves at a speed v_0 larger than the speed of light $c/n(\omega)$ in a retarding medium [$n(\omega)$ is the index of refraction], its motion is accompanied by the emission of electromagnetic waves in the frequency range in which $c^{-1}v_0 n(\omega)>1$ (see[1]). If we take as the oscillators a beam of charged particles moving along a constant magnetic field H_0, the radiation is coherent and a collective instability occurs, with the maximum increment at the frequency which satisfies the anomalous Doppler-effect condition.[1–3]

The nonlinear stage of the instability of a beam of oscillators in a linear retarding medium without dispersion has been investigated in the nonrelativistic and relativistic cases in two previous papers.[4,5] The nonlinear mechanism which limits the increase of the amplitude of the field is the destruction of phase resonance of the beam with the wave, caused by the longitudinal slowing of the particle while the phase velocity of the wave remains unchanged.

The purpose of the present paper is to take into account not only the nonlinearity of the beam but also the nonlinear properties of the retarding medium, i.e., the dependence of the index of refraction on the amplitude of the wave, which changes the waveguide properties of the medium. It will be shown that, depending on the modulation frequency of the beam, there can be either an increase or a decrease of the phase velocity. In the latter case it is possible for a nonlinear resonance between beam and wave to occur, so that the decrease of the velocity of the particles of the beam is synchronous with a decrease of the phase velocity of the wave.[6] The nonlinear effects are then less important, and the linear stage of the instability is "stretched out" and accompanied by a large increase of the energy density of the oscillations.

1. NONLINEAR DIELECTRIC CONSTANT OF A RESONANCE GASEOUS MEDIUM

When a beam of charged particles moves in a gas whose dielectric constant is sufficiently close to unity, $n^2(\omega)-1\ll 1$, the condition $c^{-1}v_0 n(\omega)-1\ll 1$ is satisfied only in the case of a relativistic beam and in a narrow frequency range. Therefore in describing the medium one can use a two-level approximation, with the assumption that the effective frequency ω of the particle is close to one of the resonance frequencies of the medium, say ω_r: $|\omega-\omega_r|\ll\omega_r$ (cf. Ref. 7).

The propagation of an electromagnetic wave in a dielectric (gaseous) medium is accompanied by the appearance of a ponderomotive force[8]

$$f=-T\nabla N+\frac{1}{2}N\alpha(\mathbf{E})\nabla\mathbf{E}^2+\frac{\alpha(\mathbf{E})}{2c}\frac{\partial}{\partial t}[\mathbf{E}\times\mathbf{H}] \tag{1}$$

(T is the temperature, N is the density of the medium, and E and H are the electric and magnetic fields). The function $\alpha(\mathbf{E})$ gives the connection between the dipole moment d of an individual molecule and the electric field $\mathbf{E}$, $\mathbf{d}=\alpha(\mathbf{E})\mathbf{E}$, and is given by the equation[9,10]

$$\ddot{\mathbf{d}}+\omega_r^2\mathbf{d}=\frac{2d_0}{\hbar}(d_0^2\omega_r^2-\dot{\mathbf{d}}^2-\omega_r^2\mathbf{d}^2)^{1/2}\mathbf{E} \tag{2}$$

(d_0 is a dipole-transition constant characterizing the properties of the medium).

In the case of a traveling wave

$$\tilde{\mathbf{E}}(t,\mathbf{r})=\mathbf{E}(t,\mathbf{r})\exp i(\omega t-kz), \tag{3}$$

with amplitude varying "slowly" with time and the coordinate

$$\dot{\mathbf{E}}\ll|\Delta|\mathbf{E},\quad \nabla\mathbf{E}^2\ll k\mathbf{E}^2,\quad \frac{|\Delta|}{\omega_r}=\frac{|\omega_r^2-\omega^2|}{2\omega_r^2}\ll 1, \tag{4}$$

it follows from Eq. (2) that[7]

$$\alpha(\mathbf{E})=\frac{d_0\varkappa\,\mathrm{sign}\,\Delta}{(1+\varkappa^2\mathbf{E}^2)^{1/2}},\qquad \varkappa=\frac{d_0}{\hbar|\Delta|}, \tag{5}$$

and from the condition $\langle f\rangle=0$ we get[1)]

$$N=N_0\exp\left[\frac{d_0}{2\varkappa T}(\sqrt{1+\varkappa^2\mathbf{E}^2}-1)\,\mathrm{sign}\,\Delta\right] \tag{6}$$

(N_0 is the unperturbed density of the gas; the average is taken over a time of one period, and the total derivative with respect to time in Eq. (1) is dropped).

From Eqs. (5) and (6) it follows that the nonlinear dielectric constant $n^2(\omega,\mathbf{E})$ of the gaseous medium is given by

$$n^2(\omega,\mathbf{E})=1+\frac{4\pi\varkappa d_0N_0\,\mathrm{sign}\,\Delta}{(1+\varkappa^2\mathbf{E}^2)^{1/2}}\exp\left[\frac{d_0}{2\varkappa T}(\sqrt{1+\varkappa^2\mathbf{E}^2}-1)\,\mathrm{sign}\,\Delta\right]. \tag{7}$$

2. INSTABILITY OF A RELATIVISTIC BEAM OF OSCILLATORS IN A NONLINEAR GAS

The motion of a relativistic beam of charged particles in a medium described by Eq. (7) along a constant magnetic field $\mathbf{H}_0$ can be described by a system of equations with a self-consistent field:

$$\operatorname{rot}\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t},\quad \operatorname{rot}\mathbf{H}=\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}+\frac{4\pi}{c}e\rho\mathbf{v},$$
$$\frac{\partial\mathbf{p}}{\partial t}+(\mathbf{v}\nabla)\mathbf{p}=e\mathbf{E}+\frac{e}{c}[\mathbf{v}\times\mathbf{H}+\mathbf{H}_0],\tag{8}$$
$$\frac{\partial\rho}{\partial t}+\operatorname{div}\rho\mathbf{v}=0,$$

where $\mathbf{p}=\gamma\mathbf{v}$, $\gamma=(1-\mathbf{v}^2/c^2)^{-1/2}$; $\mathbf{v}$ is the velocity of the beam and ρ its density; $\mathbf{E}$ and $\mathbf{H}$ are the components of the self-consistent field,[2)] and $\mathbf{D}=\hat{n}^2\mathbf{E}$.

We shall look for the solution of this system of equations in the form of running waves with "slowly" changing amplitudes and phases, propagated along the magnetic field $\mathbf{H}_0 \parallel z$:

$$H_y-iH_x=n_0(E_x+iE_y)=n_0E(t)\exp\{i[\Phi-\varphi(t)]\},$$
$$v_x+iv_y=v_\perp(t)\exp\{i[\Phi-\vartheta(t)]\},\tag{9}$$

where $n_0=n(\omega,0)$ and $\Phi=\omega(t-n_0z/c)$. As has been shown previously,[4,5] a solution $\rho=\rho_0$, $v_\parallel=v_\parallel(t)$ exists which satisfies the equation of continuity, in spite of the dependence of the transverse velocity on the coordinate. Taking this into account and introducing the dimensionless variables

$$a=\frac{v_\perp}{c}\gamma,\quad b=\gamma\left(\frac{v_\parallel}{c}n_0-1\right),\quad \varepsilon=\frac{E}{H_0},\quad d\tau=\frac{\omega_H}{\gamma}dt,$$
$$\Omega=\frac{\omega}{\omega_H},\quad \gamma=\left(1-\frac{v_\parallel^2}{c^2}-\frac{v_\perp^2}{c^2}\right)^{1/2},\quad q^2=\frac{2\pi e^2\rho_0}{m\omega_H^2n_0},$$
$$\varkappa_1=\varkappa H_0,\quad \mu=\frac{d_0H_0}{2\varkappa_1T},$$

we can put the equations for the field amplitude and the velocity, and also for the phase difference $\eta=\varphi-\vartheta$ in the form

$$\dot a=-b\varepsilon\cos\eta,\tag{10}$$

$$\dot\varepsilon=-q^2a\cos\eta,\tag{11}$$

$$\dot b=(n_0^2-1)a\varepsilon\cos\eta,\tag{12}$$

$$\dot\eta=1-\Omega b+\frac{\Omega}{2}(1-n_0^{-2})\left[1-\frac{\exp\{\mu(\sqrt{1+\varkappa_1^2\varepsilon^2}-1)\}}{(1+\varkappa_1^2\varepsilon^2)^{1/2}}\right]-\operatorname{tg}\eta\frac{d}{d\tau}\ln\varepsilon a.\tag{13}$$

In substituting the $n^2(\omega,\varepsilon)$ given by Eq. (7) into (13) we have set sign $\Delta=1$, since only in this case do we have $n_0>1$ so that the radiation condition is satisfied.

Equations (10)–(12) are identical to the analogous equations in our previous paper,[5] in which the retarding medium was assumed linear. In Eq. (13) there is an additional term in the right member, which along with $1-\Omega b$ leads to a shift of the phase difference η between the transverse velocity and the field.

In the linear approximation $\eta=0$ and $b=\Omega^{-1}$ (the anomalous Doppler-effect condition) the amplitude of the field satisfies the condition

$$\ddot\varepsilon-q^2\Omega^{-1}\varepsilon=0\tag{14}$$

and consequently increases exponentially with time:

$$\varepsilon(\tau)=\varepsilon(0)\exp(q\Omega^{-1/2}\tau).$$

We can regard the change of the amplitude as "slow" and use Eqs. (10)–(13) only under the condition $q\Omega^{-1/2}\ll 1$, which we assume is satisfied.

It can be shown that the system (10)–(13) possesses the following integrals of the motion:

$$(n_0^2-1)a^2+b^2=\Omega^{-2},\quad \varepsilon^2=\frac{2q^2}{n_0^2-1}(\Omega^{-1}-b),$$
$$-\varepsilon a\sin\eta=\frac{(1-\Omega b)^2}{2\Omega(n_0^2-1)}+\frac{\Omega}{2q^2}(1-n_0^{-2})$$
$$\times\left\{\frac{\varepsilon^2}{2}-\frac{1}{\mu\varkappa_1^2}\exp\{\mu(\sqrt{1+\varkappa_1^2\varepsilon^2}-1)\}+\frac{1}{\mu\varkappa_1^2}\right\}.\tag{15}$$

According to Eq. (11) the increase of the amplitude is cut off when the phase shift reaches the value $|\eta_m|=\pi/2$. At this instant the amplitude of the field satisfies the equation

$$\frac{2q^3}{\Omega}\varepsilon_m^2\left(\frac{1}{\Omega}-\frac{n_0^2-1}{4q^2}\varepsilon_m^2\right)^{1/2}=(n_0^2-1)\frac{\varepsilon_m^4}{4}+(1-n_0^{-2})$$
$$\times\left\{\frac{\varepsilon_m^2}{2}-\frac{1}{\mu\varkappa_1^2}\exp\{\mu(\sqrt{1+\varkappa_1^2\varepsilon^2}-1)\}+\frac{1}{\mu\varkappa_1^2}\right\}.\tag{16}$$

Let us expand the left and right sides of Eq. (16) in series with respect to the field amplitude, assuming that the inequalities

$$(n_0^2-1)\varepsilon_m^2\ll 4q^2/\Omega,\quad \varkappa_1^2\varepsilon_m^2\ll 1.\tag{17}$$

are satisfied. Dropping small terms of the order of the ratio of the increment to the frequency, we get

$$A\varepsilon_m^4-B\varepsilon_m^2\pm C=0\tag{18}$$
$$A=\frac{\varkappa_1^4}{4}\left(1-\mu+\frac{\mu^2}{3}\right),\quad B=1-\frac{\varkappa_1^2}{2}(\mu-1),\quad C=\frac{1}{n_0^2-1}\frac{8q^3}{\Omega^{3/2}}.$$

The first term can be dropped under the condition $A\varepsilon_m^2\ll|B|$, i.e., for

$$\varepsilon_m^2=C/|B|,\quad AC\ll B^2.\tag{19}$$

Obviously this approximation is always justified for $\mu<1$ when $B>1$.

In the case $\mu>1$, when the inequality opposite to (19) holds, we have

$$\varepsilon_m^2=(C/A)^{1/2},\quad AC\gg B^2.\tag{20}$$

For a linear medium $\varkappa_1=0$ Eq. (19) is the same as the analogous equation of[5]. The presence of the nonlinearity leads to a decrease ($|B|>1$) or to an increase ($|B|<1$) of the maximum value of the energy density of the field. We can get a physical explanation of this effect by analyzing the dependence of the index of refraction (7) of the medium on the field amplitude in the relevant frequency range $\Delta>0$:

$$n^2(\omega,\varepsilon)=1+4\pi N_0\varkappa d_0\left[1+\frac{\varkappa^2E^2}{2}(\mu-1)+\frac{3\varkappa^4E^4}{8}\left(1-\mu+\frac{\mu^2}{3}\right)\right].\tag{21}$$

For $\mu < 1$ the nonlinear dependence of the polarization of the medium on the field amplitude is decisive and leads to a decrease of the function $n^2(\omega, \varepsilon)$ with increasing ε^2. Because of this the phase velocity of the wave increases and there is an additional (as compared with the case of a linear medium) mismatching in phase between the beam and the field. The particles in the beam get out of resonance with the wave more quickly, and the maximum energy density is smaller.

The situation is different when $\mu > 1$ and the nonlinearity caused by electrostriction is the dominant effect. Molecules of the gas are pulled into the region of high field intensity, so that the index of refraction there is increased. The phase velocity decreases along with the deceleration of the beam, so that synchronism between the beam and the wave is preserved longer than in the case of a linear medium.[5] This effect is particularly pronounced if the condition $(\varkappa^2/2)H_0^2(\mu - 1) \approx 1$ holds; in this case there is nonlinear resonance between the beam and the field (analogous to that considered in[6] for a plasma wave). According to Eq. (20) the maximum energy density of the field is proportional to $(q^3/\Omega^{3/2})^{1/2}$ instead of the $q^3/\Omega^{3/2}$ given by Eq. (19) for the other case ($q/\Omega^{1/2} \ll 1$ is a small parameter).

After the energy density of the field has reached its maximum value, as given by Eqs. (18)–(20), and the longitudinal velocity of the beam has reached its minimum value, the inverse process begins—that of acceleration of the beam in the field of the wave, back to its original energy. The process of exchange of energy between beam and field thereafter repeats periodically, so that the system of the beam and the retarding medium exhibits oscillations with a period of the order of the reciprocal of the increment given by the linear theory.[4,5]

In conclusion we note the possibility of self-focusing of an electromagnetic wave[11–13] in a nonlinear resonance medium described by Eq. (7). In the frequency range $\Delta > 0$ the condition $\partial n^2/\partial \varepsilon^2 > 0$ for the occurrence of this effect is satisfied for $\mu > 1$. The increase of the index of refraction occurs owing to the electrostrictive increase of the particle density of the medium in the region where the wave is propagating.[11] The possibility of a resonance self-focusing in the case $\Delta < 0$ (with electrostriction not considered) was indicated in[7]. Calculations taking the nonlinear dielectric constant (7) into account can be carried out by the methods given in[7].

1) For $\varkappa^2 E^2 \ll 1$ this formula goes over into the expression given on p. 1569 of Ref. 11.

2) This model will be consistent if we assume that besides the gas there is a plasma with density ρ_p: $\rho \ll \rho_p \ll N_0$, containing electrons which produce a return current which compensates for the static magnetic field of the beam.

1. V. L. Ginzburg, Usp. Fiz. Nauk 69, 537 (1959) [Soviet Phys. Usp. 2, 874 (1969)].
2. V. V. Zheleznyakov, Izv. vuzov, Radiofizika, 2, 14, (1959).
3. A. I. Akhiezer, I. A. Akhiezer, R. V. Polovin, A. G. Sitenko, and K. N. Stepanov, Élektrodinamika plazmy (Plasma electrodynamics), Nauka, 1974.
4. V. B. Krasovitskiĭ and V. I. Kurilko, Zh. Eksp. Teor. Fiz. 49, 1831 (1965) [Sov. Phys. JETP 22, 1252 (1966)].
5. V. B. Krasovitskiĭ, Zh. Eksp. Teor. Fiz. **66**, 154 (1974) [Sov. Phys. JETP 39, 71 (1974)].
6. V. B. Krasovitskiĭ, Pis'ma Zh. Eksp. Teor. Fiz. **15**, 341 (1965) [JETP Lett. 15, 244 (1965)].
7. V. B. Krasovitskiĭ and V. I. Kurilko, Zh. Tekh. Fiz. **36**, 401 (1966) [Sov. Phys. Tech. Phys. **11**, 293 (1966)].
8. L. D. Landau and E. M. Lifshitz, Élektrodinamika sploshnykh sred (Electrodynamics of continuous media), Fizmatgiz, 1959, pp. 95, 306. English translation: Reading, Mass., Addison-Wesley, 1960, pp. 68, 241f.
9. V. M. Fain, Zh. Eksp. Teor. Fiz. 33, 945 (1957) [Sov. Phys. JETP 6, 726 (1958)].
10. A. N. Oraevskiĭ, Radiotekhnika i électronnika 4, 718 (1957) [Radio Engineering and Electronics 4, No. 4, 228 (1959)].
11. G. A. Askaryan, Zh. Eksp. Teor. Fiz. **42**, 1567 (1962) [Sov. Phys. JETP **15**, 1088 (1962)].
12. V. I. Talanov, Izv. vuz., Radiofizika, 7, 564 (1964).
13. A. G. Litvak, Pis'ma Zh. Eksp. Teor. Fiz. **4**, 341 (1966) [JETP Lett. 4, 230 (1966)].

Translated by W. H. Furry